CS-XXX: Graduate Programming Languages

Lecture 3 — Operational Semantics

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Where we are

- Done: OCaml tutorial, "IMP" syntax, structural induction
- Now: Operational semantics for our little "IMP" language
  - Most of what you need for Homework 1
  - (But Problem 4 requires proofs over semantics)

Review

IMP's abstract syntax is defined inductively:

$$
\begin{array}{rcl}
s & ::= & \textbf{skip} \mid x := e \mid s; s \mid \textbf{if}\ e\ s\ s \mid \textbf{while}\ e\ s \\
e & ::= & c \mid x \mid e + e \mid e * e \\
(c & \in & \{\ldots, -2, -1, 0, 1, 2, \ldots\}) \\
(x & \in & \{\mathrm{x}_1, \mathrm{x}_2, \ldots, \mathrm{y}_1, \mathrm{y}_2, \ldots, \mathrm{z}_1, \mathrm{z}_2, \ldots, \ldots\})
\end{array}
$$

We haven't yet said what programs *mean*! (Syntax is boring)

Encode our "social understanding" about variables and control flow

Outline

- Semantics for expressions
  1. Informal idea; the need for *heaps*
  2. Definition of heaps
  3. The evaluation *judgment* (a relation form)
  4. The evaluation *inference rules* (the relation definition)
  5. Using inference rules
     - *Derivation trees* as interpreters
     - Or as *proofs* about expressions
  6. *Metatheory*: Proofs about the semantics
- Then semantics for statements
  - ...

Informal idea

Given e, what c does e evaluate to?

$$\mathbf{1} + \mathbf{2} \qquad\qquad \mathtt{x} + \mathbf{2}$$

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It depends on the values of variables (of course)

Use a heap H for a total function from variables to constants

- Could use partial functions, but then $\exists$ H and e for which there is no c

We'll define a *relation* over triples of H, e, and c

- Will turn out to be *function* if we view H and e as inputs and c as output
- With our metalanguage, easier to define a relation and then prove it is a function (if, in fact, it is)

Heaps

$$H ::= \cdot \mid H, x \mapsto c$$

A lookup-function for heaps:

$$H(x) = \begin{cases} c & \text{if} \quad H = H', x \mapsto c \\ H'(x) & \text{if} \quad H = H', y \mapsto c' \text{ and } y \neq x \\ 0 & \text{if} \quad H = \cdot \end{cases}$$

- Last case avoids “errors” (makes function *total*)

“What heap to use” will arise in the semantics of statements

- For expression evaluation, “we are given an H”

The judgment

We will write: $\boxed{H \; ; e \Downarrow c}$

to mean, "e evaluates to c under heap H"

It is just a relation on triples of the form (H, e, c)

We just made up metasyntax $H \; ; e \Downarrow c$ to follow PL convention and to distinguish it from other relations

We can write: $., x \mapsto 3 \; ; x + y \Downarrow 3$, which will turn out to be *true*
(this triple will be in the relation we define)

Or: $., x \mapsto 3 \; ; x + y \Downarrow 6$, which will turn out to be *false*
(this triple will not be in the relation we define)

Inference rules

$$\frac{\text{CONST}}{H\ ;\ c \Downarrow c} \qquad \frac{\text{VAR}}{H\ ;\ x \Downarrow H(x)}$$

$$\text{ADD}\ \frac{H\ ;\ e_1 \Downarrow c_1 \qquad H\ ;\ e_2 \Downarrow c_2}{H\ ;\ e_1 + e_2 \Downarrow c_1 + c_2} \qquad \text{MULT}\ \frac{H\ ;\ e_1 \Downarrow c_1 \qquad H\ ;\ e_2 \Downarrow c_2}{H\ ;\ e_1 * e_2 \Downarrow c_1 * c_2}$$

Top: *hypotheses*
Bottom: *conclusion* (read first)

By definition, if all hypotheses hold, then the conclusion holds

Each rule is a *schema* you "instantiate consistently"

- So rules "work" "for all" H, c, e_1, etc.
- But "each" e_1 has to be the "same" expression

Instantiating rules

Example instantiation:

$$\frac{\cdot, \mathrm{y} \mapsto \mathbf{4} \; ; \; \mathbf{3} + \mathrm{y} \Downarrow \mathbf{7} \qquad \cdot, \mathrm{y} \mapsto \mathbf{4} \; ; \; \mathbf{5} \Downarrow \mathbf{5}}{\cdot, \mathrm{y} \mapsto \mathbf{4} \; ; \; (\mathbf{3} + \mathrm{y}) + \mathbf{5} \Downarrow \mathbf{12}}$$

Instantiates:

$$\text{ADD}\;\frac{H \; ; \; e_1 \Downarrow c_1 \qquad H \; ; \; e_2 \Downarrow c_2}{H \; ; \; e_1 + e_2 \Downarrow c_1 + c_2}$$

with

$H = \cdot, \mathrm{y} \mapsto \mathbf{4}$

$e_1 = (\mathbf{3} + \mathrm{y})$

$c_1 = \mathbf{7}$

$e_2 = \mathbf{5}$

$c_2 = \mathbf{5}$

Derivations

A *(complete) derivation* is a tree of instantiations with *axioms* at the leaves

Example:

$$\dfrac{\dfrac{\overline{\cdot, \mathrm{y}\mapsto\mathbf{4} \; ; \; \mathbf{3} \Downarrow \mathbf{3}} \qquad \overline{\cdot, \mathrm{y}\mapsto\mathbf{4} \; ; \; \mathrm{y} \Downarrow \mathbf{4}}}{\cdot, \mathrm{y}\mapsto\mathbf{4} \; ; \; \mathbf{3} + \mathrm{y} \Downarrow \mathbf{7}} \qquad \overline{\cdot, \mathrm{y}\mapsto\mathbf{4} \; ; \; \mathbf{5} \Downarrow \mathbf{5}}}{\cdot, \mathrm{y}\mapsto\mathbf{4} \; ; \; (\mathbf{3} + \mathrm{y}) + \mathbf{5} \Downarrow \mathbf{12}}$$

By definition, $\boldsymbol{H} \; ; \; \boldsymbol{e} \Downarrow \boldsymbol{c}$ if there exists a derivation with $\boldsymbol{H} \; ; \; \boldsymbol{e} \Downarrow \boldsymbol{c}$ at the root

Back to relations

So what relation do our inference rules define?

- Start with empty relation (no triples) R_0
- Let R_i be R_{i-1} union all $H\ ;\ e \Downarrow c$ such that we can instantiate some inference rule to have conclusion $H\ ;\ e \Downarrow c$ and all hypotheses in R_{i-1}
  - So R_i is all triples at the bottom of height-j complete derivations for $j \leq i$
- R_∞ is the relation we defined
  - All triples at the bottom of complete derivations

For the math folks: R_∞ is the smallest relation closed under the inference rules

What are these things?

We can view the inference rules as defining an *interpreter*

- Complete derivation shows recursive calls to the "evaluate expression" function
  - Recursive calls from conclusion to hypotheses
  - *Syntax-directed* means the interpreter need not "search"

- See OCaml code in Homework 1

Or we can view the inference rules as defining a *proof system*

- Complete derivation proves facts from other facts starting with axioms
  - Facts established from hypotheses to conclusions

Some theorems

- Progress: For all H and e, there exists a c such that $H\ ;\ e \Downarrow c$
- Determinacy: For all H and e, there is at most one c such that $H\ ;\ e \Downarrow c$

We rigged it that way...
what would division, undefined-variables, or `gettime()` do?

Proofs are by induction on the the structure (i.e., height) of the expression e

On to statements

A statement does not produce a constant

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We could define $H_1\ ;\ s \Downarrow H_2$

- Would be a partial function from H_1 and s to H_2
- Works fine; could be a homework problem

On to statements

A statement does not produce a constant

It produces a new, possibly-different heap.

- If it terminates

We could define $H_1 \; ; s \Downarrow H_2$

- Would be a partial function from H_1 and s to H_2
- Works fine; could be a homework problem

Instead we'll define a "small-step" semantics and then "iterate" to "run the program"

$$\boxed{H_1 \; ; s_1 \rightarrow H_2 \; ; s_2}$$

Statement semantics

$$\boxed{H_1 \; ; s_1 \rightarrow H_2 \; ; s_2}$$

ASSIGN

$$\frac{H \; ; e \Downarrow c}{H \; ; x := e \rightarrow H, x \mapsto c \; ; \textsf{skip}}$$

SEQ1

$$\frac{}{H \; ; \textsf{skip}; s \rightarrow H \; ; s}$$

SEQ2

$$\frac{H \; ; s_1 \rightarrow H' \; ; s_1'}{H \; ; s_1; s_2 \rightarrow H' \; ; s_1'; s_2}$$

IF1

$$\frac{H \; ; e \Downarrow c \qquad c > 0}{H \; ; \textsf{if } e \; s_1 \; s_2 \rightarrow H \; ; s_1}$$

IF2

$$\frac{H \; ; e \Downarrow c \qquad c \leq 0}{H \; ; \textsf{if } e \; s_1 \; s_2 \rightarrow H \; ; s_2}$$

Statement semantics cont'd

What about **while** e s (do s and loop if $e > 0$)?

Statement semantics cont'd

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$$\frac{\textsc{while}}{H \;;\; \textbf{while}\ e\ s \rightarrow H \;;\; \textbf{if}\ e\ (s; \textbf{while}\ e\ s)\ \textbf{skip}}$$

Many other equivalent definitions possible

Program semantics

Defined $H\ ;\ s \rightarrow H'\ ;\ s'$, but what does "$s$" mean/do?

Our machine iterates: $H_1;s_1 \rightarrow H_2;s_2 \rightarrow H_3;s_3 \ldots$,
with each step justified by a complete derivation using our single-step statement semantics

Let $H_1\ ;\ s_1\ \rightarrow^n\ H_2\ ;\ s_2$ mean "becomes after n steps"

Let $H_1\ ;\ s_1 \rightarrow^* H_2\ ;\ s_2$ mean "becomes after 0 or more steps"

Pick a special "answer" variable ans

The program s produces c if $\cdot\ ;\ s \rightarrow^* H\ ;$ **skip** and $H(\mathtt{ans}) = c$

Does every s produce a c?

Example program execution

$$\mathrm{x} := \mathbf{3}; (\mathrm{y} := \mathbf{1}; \textbf{while } \mathrm{x}\ (\mathrm{y} := \mathrm{y} * \mathrm{x}; \mathrm{x} := \mathrm{x} - \mathbf{1}))$$

Let's write some of the state sequence. You can justify each step with a full derivation. Let $s = (\mathrm{y} := \mathrm{y} * \mathrm{x}; \mathrm{x} := \mathrm{x} - \mathbf{1})$.

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$$\begin{aligned} & \cdot; \mathrm{x} := 3; \mathrm{y} := 1; \textsf{while}\ \mathrm{x}\ s \\ \rightarrow\quad & \cdot, \mathrm{x} \mapsto 3; \textsf{skip}; \mathrm{y} := 1; \textsf{while}\ \mathrm{x}\ s \end{aligned}$$

Example program execution

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\to & \cdot, \mathrm{x} \mapsto 3; \textbf{skip}; \mathrm{y} := 1; \textbf{while } \mathrm{x}\ s \\
\to & \cdot, \mathrm{x} \mapsto 3; \mathrm{y} := 1; \textbf{while } \mathrm{x}\ s
\end{array}
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\rightarrow & \cdot, \mathrm{x} \mapsto \mathbf{3}; \textsf{\textbf{skip}}; \mathrm{y} := \mathbf{1}; \textsf{\textbf{while}}\ \mathrm{x}\ s \\
\rightarrow & \cdot, \mathrm{x} \mapsto \mathbf{3}; \mathrm{y} := \mathbf{1}; \textsf{\textbf{while}}\ \mathrm{x}\ s \\
\rightarrow^{2} & \cdot, \mathrm{x} \mapsto \mathbf{3}, \mathrm{y} \mapsto \mathbf{1}; \textsf{\textbf{while}}\ \mathrm{x}\ s
\end{array}$$

Example program execution

$$\mathrm{x} := 3; (\mathrm{y} := 1; \textbf{while } \mathrm{x}\ (\mathrm{y} := \mathrm{y} * \mathrm{x}; \mathrm{x} := \mathrm{x} - 1))$$

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 & \cdot; \mathrm{x} := 3; \mathrm{y} := 1; \textbf{while } \mathrm{x}\ s \\
\rightarrow & \cdot, \mathrm{x} \mapsto 3; \textbf{skip}; \mathrm{y} := 1; \textbf{while } \mathrm{x}\ s \\
\rightarrow & \cdot, \mathrm{x} \mapsto 3; \mathrm{y} := 1; \textbf{while } \mathrm{x}\ s \\
\rightarrow^2 & \cdot, \mathrm{x} \mapsto 3, \mathrm{y} \mapsto 1; \textbf{while } \mathrm{x}\ s \\
\rightarrow & \cdot, \mathrm{x} \mapsto 3, \mathrm{y} \mapsto 1; \textbf{if } \mathrm{x}\ (s; \textbf{while } \mathrm{x}\ s)\ \textbf{skip}
\end{array}
$$

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\to & \cdot, \mathrm{x} \mapsto 3, \mathrm{y} \mapsto 1; \mathrm{y} := \mathrm{y} * \mathrm{x}; \mathrm{x} := \mathrm{x} - 1; \textbf{while}\ \mathrm{x}\ s
\end{array}
$$

Continued...

$$\rightarrow^{2} \quad \cdot, \mathrm{x} \mapsto \mathbf{3}, \mathrm{y} \mapsto \mathbf{1}, \mathrm{y} \mapsto \mathbf{3}; \mathrm{x} := \mathrm{x} - \mathbf{1}; \textbf{while}\ \mathrm{x}\ s$$

Continued...

$$\rightarrow^{2} \quad \cdot, \mathrm{x} \mapsto \mathbf{3}, \mathrm{y} \mapsto \mathbf{1}, \mathrm{y} \mapsto \mathbf{3}; \mathrm{x} := \mathrm{x}-\mathbf{1}; \textsf{\textbf{while}}\ \mathrm{x}\ s$$

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Continued...

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Continued...

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$$\ldots$$

Continued...

$$\rightarrow^2 \quad \cdot, \mathrm{x} \mapsto 3, \mathrm{y} \mapsto 1, \mathrm{y} \mapsto 3; \mathrm{x} := \mathrm{x}-1; \textbf{while } \mathrm{x}\ s$$

$$\rightarrow^2 \quad \cdot, \mathrm{x} \mapsto 3, \mathrm{y} \mapsto 1, \mathrm{y} \mapsto 3, \mathrm{x} \mapsto 2; \textbf{while } \mathrm{x}\ s$$

$$\rightarrow \quad \ldots, \mathrm{y} \mapsto 3, \mathrm{x} \mapsto 2; \textbf{if } \mathrm{x}\ (s; \textbf{while } \mathrm{x}\ s)\ \textbf{skip}$$

$$\ldots$$

$$\rightarrow \quad \ldots, \mathrm{y} \mapsto 6, \mathrm{x} \mapsto 0; \textbf{skip}$$

Where we are

Defined $H ; e \Downarrow c$ and $H ; s \rightarrow H' ; s'$ and extended the latter to give s a meaning

- The way we did expressions is "large-step operational semantics"
- The way we did statements is "small-step operational semantics"
- So now you have seen both

Definition by interpretation: program means what an interpreter (written in a metalanguage) says it means

- Interpreter represents a (very) abstract machine that runs code

Large-step does not distinguish errors and divergence

- But we defined IMP to have no errors
- And expressions never diverge

Establishing Properties

We can prove a property of a terminating program by "running" it

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Example: **while** 1 **skip**

By induction on n, but requires a *stronger induction hypothesis*

More General Proofs

We can prove properties of executing all programs (satisfying another property)

Example: If H and s have no negative constants and $H\ ;\ s \rightarrow^* H'\ ;\ s'$, then H' and s' have no negative constants.

Example: If for all H, we know s_1 and s_2 terminate, then for all H, we know $H;(s_1; s_2)$ terminates.