CS-XXX: Graduate Programming Languages

Lecture 6 — Little Trusted Languages; Equivalence

Dan Grossman
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Looking back, looking forward

This is the last lecture using IMP (hooray!). Done:

- Abstract syntax
- Operational semantics (large-step and small-step)
- Semantic properties of (sets of) programs
- "Pseudo-denotational" semantics

Now:

- Packet-filter languages and other examples
- Equivalence of programs in a semantics
- Equivalence of different semantics

Next lecture: Local variables, lambda-calculus

Packet Filters

A very simple view of packet filters:

- Some bits come in off the wire
- Some application(s) want the "packet" and some do not (e.g., port number)
- For safety, only the O/S can access the wire
- For extensibility, the applications accept/reject packets

Conventional solution goes to user-space for every packet and app that wants (any) packets

Faster solution: Run app-written filters in kernel-space

What we need

Now the O/S writer is defining the packet-filter language!

Properties we wish of (untrusted) filters:

1. Do not corrupt kernel data structures
2. Terminate (within a time bound)
3. Run fast (the whole point)

Should we download some C/assembly code? (Get 1 of 3)

Should we make up a language and "hope" it has these properties?

Language-based approaches

1. Interpret a language

   + clean operational semantics, + portable, - may be slow (+ filter-specific optimizations), - unusual interface

2. Translate a language into C/assembly

   + clean denotational semantics, + employ existing optimizers, - upfront cost, - unusual interface

3. Require a conservative subset of C/assembly

   + normal interface, - too conservative w/o help

IMP has taught us about (1) and (2) — we'll get to (3)

A General Pattern

Packet filters move the code to the data rather than data to the code

General reasons: performance, security, other?

Other examples:

- Query languages
- Active networks
- Client-side web scripts (Javascript)

Equivalence motivation

- Program equivalence (we change the program):
  - code optimizer
  - code maintainer
- Semantics equivalence (we change the language):
  - interpreter optimizer
  - language designer
    - (prove properties for equivalent semantics with easier proof)

Note: Proofs may seem easy with the right semantics and lemmas

- (almost never start off with right semantics and lemmas)

Note: Small-step operational semantics often has harder proofs, but models more intesting things

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- Equivalence plus complexity bounds
  - Is $O(2^{n^n})$ really equivalent to $O(n)$?
  - Is "runs within 10ms of each other" important?

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In PL, equivalence most often means total I/O equivalence

Program Example: Strength Reduction

Motivation: Strength reduction

- A common compiler optimization due to architecture issues

Theorem: $H \ ; e * 2 \Downarrow c$ if and only if $H \ ; e + e \Downarrow c$

Proof sketch:

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Proof sketch:

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- Invert the assumed derivation, use hypotheses plus a little math to derive what we need
- Hmm, doesn't use induction. That's because this theorem isn't very useful...

Program Example: Nested Strength Reduction

Theorem: If e' has a subexpression of the form $e * 2$,
then $H\ ;\ e' \Downarrow c'$ if and only if $H\ ;\ e'' \Downarrow c'$
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First some useful metanotation:

$$C ::= [\cdot] \mid C + e \mid e + C \mid C * e \mid e * C$$

$C[e]$ is "C with e in the hole" (inductive definition of "stapling")

Crisper statement of theorem:
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Proof sketch: By induction on structure ("syntax height") of C

- The base case ($C = [\cdot]$) follows from our previous proof
- The rest is a long, tedious, (and instructive!) induction

Proof reuse

As we cannot emphasize enough, proving is just like programming

The proof of nested strength reduction had nothing to do with $e * 2$ and $e + e$ except in the base case where we used our previous theorem

A much more useful theorem would parameterize over the base case so that we could get the "nested X" theorem for any appropriate X:

If ($H \; ; \, e_1 \Downarrow c$ if and only if $H \; ; \, e_2 \Downarrow c$),
then ($H \; ; \, C[e_1] \Downarrow c'$ if and only if $H \; ; \, C[e_2] \Downarrow c'$)

The proof is identical except the base case is "by assumption"

Small-step program equivalence

These sort of proofs also work with small-step semantics (e.g., our IMP statements), but tend to be more cumbersome, even to state.

Example: The statement-sequence operator is associative. That is,

(a) For all n, if $H\ ;\ s_1; (s_2; s_3)\ \rightarrow^n\ H'\ ;\ \mathbf{skip}$ then there exist H'' and n' such that $H\ ;\ (s_1; s_2); s_3\ \rightarrow^{n'}\ H''\ ;\ \mathbf{skip}$ and $H''(ans) = H'(ans)$.

(b) If for all n there exist H' and s' such that $H\ ;\ s_1; (s_2; s_3)\ \rightarrow^n\ H'\ ;\ s'$, then for all n there exist H'' and s'' such that $H\ ;\ (s_1; s_2); s_3\ \rightarrow^n\ H''\ ;\ s''$.

(Proof needs a much stronger induction hypothesis.)

One way to avoid it: Prove large-step and small-step *semantics* equivalent, then prove program equivalences in whichever is easier.

Language Equivalence Example

IMP w/o multiply large-step:

$$\text{CONST}\ \frac{}{H\ ;\ c \Downarrow c} \qquad \text{VAR}\ \frac{}{H\ ;\ x \Downarrow H(x)} \qquad \text{ADD}\ \frac{H\ ;\ e_1 \Downarrow c_1 \qquad H\ ;\ e_2 \Downarrow c_2}{H\ ;\ e_1 + e_2 \Downarrow c_1 + c_2}$$

IMP w/o multiply small-step:

$$\text{SVAR}\ \frac{}{H;\ x \to H(x)} \qquad \text{SADD}\ \frac{}{H;\ c_1 + c_2 \to c_1 + c_2}$$

$$\text{SLEFT}\ \frac{H;\ e_1 \to e_1'}{H;\ e_1 + e_2 \to e_1' + e_2} \qquad \text{SRIGHT}\ \frac{H;\ e_2 \to e_2'}{H;\ e_1 + e_2 \to e_1 + e_2'}$$

Theorem: Semantics are equivalent: $H\ ;\ e \Downarrow c$ if and only if $H;\ e \to^* c$

Proof: We prove the two directions separately...

Proof, part 1

First assume $H\ ;\ e \Downarrow c$ and show $\exists n.\ H;\ e \rightarrow^{n} c$

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First assume $H\ ; e \Downarrow c$ and show $\exists n.\ H; e \rightarrow^n c$

Lemma (prove it!): If $H; e \rightarrow^n e'$, then $H; e_1 + e \rightarrow^n e_1 + e'$ and $H; e + e_2 \rightarrow^n e' + e_2$.

- Proof by induction on n
- Inductive case uses SLEFT and SRIGHT

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Given the lemma, prove by induction on derivation of $H\ ;\ e \Downarrow c$

Proof, part 1

First assume $H \; ; e \Downarrow c$ and show $\exists n.\ H; e \rightarrow^{n} c$

Lemma (prove it!): If $H; e \rightarrow^{n} e'$, then $H; e_1 + e \rightarrow^{n} e_1 + e'$ and $H; e + e_2 \rightarrow^{n} e' + e_2$.

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Given the lemma, prove by induction on derivation of $H \; ; e \Downarrow c$

- CONST: Derivation with CONST implies $e = c$, and we can derive $H; c \rightarrow^{0} c$

Proof, part 1

First assume $H\ ;\ e \Downarrow c$ and show $\exists n.\ H;\ e \rightarrow^{n} c$

Lemma (prove it!): If $H;\ e \rightarrow^{n} e'$, then $H;\ e_1 + e \rightarrow^{n} e_1 + e'$ and $H;\ e + e_2 \rightarrow^{n} e' + e_2$.

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Given the lemma, prove by induction on derivation of $H\ ;\ e \Downarrow c$

- CONST: Derivation with CONST implies $e = c$, and we can derive $H;\ c \rightarrow^{0} c$
- VAR: Derivation with VAR implies $e = x$ for some x where $H(x) = c$, so derive $H;\ e \rightarrow^{1} c$ with SVAR

Proof, part 1

First assume $H\ ; e \Downarrow c$ and show $\exists n.\ H; e \rightarrow^n c$

Lemma (prove it!): If $H; e \rightarrow^n e'$, then $H; e_1 + e \rightarrow^n e_1 + e'$ and $H; e + e_2 \rightarrow^n e' + e_2$.

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Given the lemma, prove by induction on derivation of $H\ ; e \Downarrow c$

- CONST: Derivation with CONST implies $e = c$, and we can derive $H; c \rightarrow^0 c$
- VAR: Derivation with VAR implies $e = x$ for some x where $H(x) = c$, so derive $H; e \rightarrow^1 c$ with SVAR
- ADD: ...

Part 1, continued

First assume $H\ ;\ e \Downarrow c$ and show $\exists n.\ H;\ e \rightarrow^n c$

Lemma (prove it!): If $H;\ e \rightarrow^n e'$, then $H;\ e_1 + e \rightarrow^n e_1 + e'$ and $H;\ e + e_2 \rightarrow^n e' + e_2$.

Given the lemma, prove by induction on derivation of $H\ ;\ e \Downarrow c$

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- ADD: Derivation with ADD implies $e = e_1 + e_2$, $c = c_1 + c_2$, $H\ ;\ e_1 \Downarrow c_1$, and $H\ ;\ e_2 \Downarrow c_2$ for some e_1, e_2, c_1, c_2.

Part 1, continued

First assume $H\ ;\ e \Downarrow c$ and show $\exists n.\ H; e \rightarrow^{n} c$

Lemma (prove it!): If $H; e \rightarrow^{n} e'$, then $H; e_1 + e \rightarrow^{n} e_1 + e'$ and $H; e + e_2 \rightarrow^{n} e' + e_2$.

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First assume $H\;;\;e \Downarrow c$ and show $\exists n.\; H;\,e \rightarrow^n c$

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  By induction (twice), $\exists n_1, n_2$. $H;\,e_1 \rightarrow^{n_1} c_1$ and $H;\,e_2 \rightarrow^{n_2} c_2$.
  So by our lemma $H;\,e_1 + e_2 \rightarrow^{n_1} c_1 + e_2$ and $H;\,c_1 + e_2 \rightarrow^{n_2} c_1 + c_2$.

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First assume $H\;;e \Downarrow c$ and show $\exists n.\; H; e \rightarrow^{n} c$

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  By induction (twice), $\exists n_1, n_2$. $H; e_1 \rightarrow^{n_1} c_1$ and $H; e_2 \rightarrow^{n_2} c_2$.
  So by our lemma $H; e_1 + e_2 \rightarrow^{n_1} c_1 + e_2$ and $H; c_1 + e_2 \rightarrow^{n_2} c_1 + c_2$.
  By SADD $H; c_1 + c_2 \rightarrow c_1 + c_2$.

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- ADD: Derivation with ADD implies $e = e_1 + e_2$, $c = c_1 + c_2$, $H\ ;\ e_1 \Downarrow c_1$, and $H\ ;\ e_2 \Downarrow c_2$ for some e_1, e_2, c_1, c_2.
  By induction (twice), $\exists n_1, n_2$. $H;\ e_1 \rightarrow^{n_1} c_1$ and $H;\ e_2 \rightarrow^{n_2} c_2$.
  So by our lemma $H;\ e_1 + e_2 \rightarrow^{n_1} c_1 + e_2$ and $H;\ c_1 + e_2 \rightarrow^{n_2} c_1 + c_2$.
  By SADD $H;\ c_1 + c_2 \rightarrow c_1 + c_2$.
  So $H;\ e_1 + e_2 \rightarrow^{n_1 + n_2 + 1} c$.

Proof, part 2

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- $n = 0$: e is c and CONST lets us derive $H\ ; c \Downarrow c$
- $n > 0$: (Clever: break into *first* step and remaining ones) $\exists e'.\ H; e \rightarrow e'$ and $H; e' \rightarrow^{n-1} c$.

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  By induction $H\ ; e' \Downarrow c$.
  So this lemma suffices: If $H; e \rightarrow e'$ and $H\ ; e' \Downarrow c$, then $H\ ; e \Downarrow c$.

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Now assume $\exists n.\ H; e \to^n c$ and show $H\ ; e \Downarrow c$.

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- $n > 0$: (Clever: break into *first* step and remaining ones)
  $\exists e'.\ H; e \to e'$ and $H; e' \to^{n-1} c$.
  By induction $H\ ; e' \Downarrow c$.
  So this lemma suffices: If $H; e \to e'$ and $H\ ; e' \Downarrow c$, then $H\ ; e \Downarrow c$.

Prove the lemma by induction on derivation of $H; e \to e'$:

- SVAR: ...
- SADD: ...
- SLEFT: ...
- SRIGHT: ...

Part 2, key lemma

Lemma: If $H; e \rightarrow e'$ and $H ; e' \Downarrow c$, then $H ; e \Downarrow c$.

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- SVAR: Derivation with SVAR implies e is some x and $e' = H(x) = c$, so derive, by VAR, $H \ ; x \Downarrow H(x)$.

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- SADD: Derivation with SADD implies e is some $c_1 + c_2$ and $e' = c_1{+}c_2 = c$, so derive, by ADD and two CONST, $H \,;\, c_1 + c_2 \Downarrow c_1{+}c_2$.

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- SLEFT: Derivation with SLEFT implies $e = e_1 + e_2$ and $e' = e'_1 + e_2$ and $H; e_1 \rightarrow e'_1$ for some e_1, e_2, e'_1.

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Lemma: If $H; e \to e'$ and $H\ ; e' \Downarrow c$, then $H\ ; e \Downarrow c$.

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- SLEFT: Derivation with SLEFT implies $e = e_1 + e_2$ and $e' = e'_1 + e_2$ and $H; e_1 \to e'_1$ for some e_1, e_2, e'_1. Since $e' = e'_1 + e_2$ inverting assumption $H\ ; e' \Downarrow c$ gives $H\ ; e'_1 \Downarrow c_1$, $H\ ; e_2 \Downarrow c_2$ and $c = c_1 + c_2$.

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Lemma: If $H; e \rightarrow e'$ and $H \; ; e' \Downarrow c$, then $H \; ; e \Downarrow c$.

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- SLEFT: Derivation with SLEFT implies $e = e_1 + e_2$ and $e' = e'_1 + e_2$ and $H; e_1 \rightarrow e'_1$ for some e_1, e_2, e'_1.
  Since $e' = e'_1 + e_2$ inverting assumption $H \; ; e' \Downarrow c$ gives $H \; ; e'_1 \Downarrow c_1$, $H \; ; e_2 \Downarrow c_2$ and $c = c_1 + c_2$.
  Applying the induction hypothesis to $H; e_1 \rightarrow e'_1$ and $H \; ; e'_1 \Downarrow c_1$ gives $H \; ; e_1 \Downarrow c_1$.

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Lemma: If $H; e \rightarrow e'$ and $H\ ; e' \Downarrow c$, then $H\ ; e \Downarrow c$.

Prove the lemma by induction on derivation of $H; e \rightarrow e'$:

- SVAR: Derivation with SVAR implies e is some x and $e' = H(x) = c$, so derive, by VAR, $H\ ; x \Downarrow H(x)$.
- SADD: Derivation with SADD implies e is some $c_1 + c_2$ and $e' = c_1 + c_2 = c$, so derive, by ADD and two CONST, $H\ ; c_1 + c_2 \Downarrow c_1 + c_2$.
- SLEFT: Derivation with SLEFT implies $e = e_1 + e_2$ and $e' = e_1' + e_2$ and $H; e_1 \rightarrow e_1'$ for some e_1, e_2, e_1'.
  Since $e' = e_1' + e_2$ inverting assumption $H\ ; e' \Downarrow c$ gives $H\ ; e_1' \Downarrow c_1$, $H\ ; e_2 \Downarrow c_2$ and $c = c_1 + c_2$.
  Applying the induction hypothesis to $H; e_1 \rightarrow e_1'$ and $H\ ; e_1' \Downarrow c_1$ gives $H\ ; e_1 \Downarrow c_1$.
  So use ADD, $H\ ; e_1 \Downarrow c_1$, and $H\ ; e_2 \Downarrow c_2$ to derive $H\ ; e_1 + e_2 \Downarrow c_1 + c_2$.

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Lemma: If $H; e \rightarrow e'$ and $H \; ; e' \Downarrow c$, then $H \; ; e \Downarrow c$.

Prove the lemma by induction on derivation of $H; e \rightarrow e'$:

- SVAR: Derivation with SVAR implies e is some x and $e' = H(x) = c$, so derive, by VAR, $H \; ; x \Downarrow H(x)$.
- SADD: Derivation with SADD implies e is some $c_1 + c_2$ and $e' = c_1{+}c_2 = c$, so derive, by ADD and two CONST, $H \; ; c_1 + c_2 \Downarrow c_1{+}c_2$.
- SLEFT: Derivation with SLEFT implies $e = e_1 + e_2$ and $e' = e_1' + e_2$ and $H; e_1 \rightarrow e_1'$ for some e_1, e_2, e_1'. Since $e' = e_1' + e_2$ inverting assumption $H \; ; e' \Downarrow c$ gives $H \; ; e_1' \Downarrow c_1$, $H \; ; e_2 \Downarrow c_2$ and $c = c_1{+}c_2$. Applying the induction hypothesis to $H; e_1 \rightarrow e_1'$ and $H \; ; e_1' \Downarrow c_1$ gives $H \; ; e_1 \Downarrow c_1$. So use ADD, $H \; ; e_1 \Downarrow c_1$, and $H \; ; e_2 \Downarrow c_2$ to derive $H \; ; e_1 + e_2 \Downarrow c_1{+}c_2$.
- SRIGHT: Analogous to SLEFT

The cool part, redux

Step through the SLEFT case more visually:

By assumption, we must have derivations that look like this:

$$\frac{H;\, e_1 \rightarrow e_1'}{H;\, e_1 + e_2 \rightarrow e_1' + e_2} \qquad \frac{H\ ;\, e_1' \Downarrow c_1 \qquad H\ ;\, e_2 \Downarrow c_2}{H\ ;\, e_1' + e_2 \Downarrow c_1 + c_2}$$

Grab the hypothesis from the left and the left hypothesis from the right and use induction to get $H\ ;\, e_1 \Downarrow c_1$.

Now go grab the one hypothesis we haven't used yet and combine it with our inductive result to derive our answer:

$$\frac{H\ ;\, e_1 \Downarrow c_1 \qquad H\ ;\, e_2 \Downarrow c_2}{H\ ;\, e_1 + e_2 \Downarrow c_1 + c_2}$$

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Theorem: The small-step semantics is deterministic:
if $H; e \rightarrow^* c_1$ and $H; e \rightarrow^* c_2$, then $c_1 = c_2$

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- Given $(((1 + 2) + (3 + 4)) + (5 + 6)) + (7 + 8)$ there are many execution sequences, which all produce 36 but with different intermediate expressions

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Proof:

- Large-step evaluation is deterministic (easy induction proof)
- Small-step and and large-step are equivalent (just proved that)
- So small-step is deterministic
- Convince yourself a deterministic and a nondeterministic semantics cannot be equivalent

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