Notes: You must work on the homework by yourself. Each problem is worth 10 points.

1. Consider the class of functions that have a single round interactive proof: the prover sends a single message, which the randomized verifier uses to verify that the value of the function is 1. Show that the set of functions verifiable with such proofs is contained in $\text{NP}^{\text{SAT}}$. HINT: Use the same method as the proof that shows that $\text{BPP} \subseteq \text{NP}^{\text{SAT}}$.

2. Show that if $c(c + d) < 2$, then $\text{SAT}$ cannot be computed by a Turing machine that uses $n^c$ time and $n^d$ space.

3. Recall that we defined an expander to be a constant degree graph for which for every subset $S$ of at most $n/2$ vertices, $|\Gamma(S)| \geq (1 + \Omega(1))|S|$. We also defined the edge expansion of a constant degree graph to be

$$h(G) = \min_{S, |S| \leq n/2} \frac{\# \text{ edges coming out of } S}{|S|}.$$ 

Prove that the graph is an expander if and only if $h(G) = \Omega(1)$.

4. Suppose you are given a $d$-regular expander graph such that all eigenvalues of the normalized adjacency matrix except the largest one are at most $\lambda$ in magnitude. Show that if $S, T$ are two sets of vertices in the graph, and $E$ is the number of edges going from $S$ to $T$, then

$$|E - \frac{d|S||T|}{n}| < \lambda d \sqrt{|S||T|}.$$ 

HINT: Use the fact that if $1_S, 1_T$ are the corresponding indicator vectors, and $A$ is the adjacency matrix, then $E = 1_S^T A 1_T$, and the Cauchy-Schwartz inequality.