In class we sketched several proofs, but proof sketches invariably skip steps and have small errors. Here are the proofs more carefully laid out, as one might do on a homework assignment. Please report any corrections.

**Theorem:** \( H ; e \ast 2 \Downarrow c \) if and only if \( H ; e + e \Downarrow c \).

**Proof:** (Does not use induction)

- First assume \( H ; e \ast 2 \Downarrow c \) and show \( H ; e + e \Downarrow c \). Any derivation of \( H ; e \ast 2 \Downarrow c \) must end with the **MULT** rule, which means there must exist derivations of \( H ; e \Downarrow c' \) and \( H ; 2 \Downarrow 2 \), and \( c \) must be \( 2c' \). That is, there must be a derivation that looks like this:

  \[
  \begin{array}{c}
  \vdots \\
  \hline
  H ; e \Downarrow c' \\
  H \Downarrow 2 \\
  \hline
  H ; e \ast 2 \Downarrow 2c'
  \end{array}
  \]

  So given that there exists a derivation of \( H ; e \Downarrow c' \), we can use **ADD** to derive:

  \[
  \begin{array}{c}
  H ; e \Downarrow c' \\
  \hline
  H ; e + e \Downarrow c' + c'
  \end{array}
  \]

  Math provides \( c' + c' = 2c' \), so the conclusion of this derivation is what we need.

- Now assume \( H ; e + e \Downarrow c \) and show \( H ; e \ast 2 \Downarrow c \). Any derivation of \( H ; e + e \Downarrow c \) must end with the **ADD** rule, which means there exists a derivation that looks like this (where \( c = c_1 + c_2 \)):

  \[
  \begin{array}{c}
  \vdots \\
  \hline
  H ; e \Downarrow c_1 \\
  H ; e \Downarrow c_2 \\
  \hline
  H ; e + e \Downarrow c_1 + c_2
  \end{array}
  \]

  In fact, we earlier proved determinacy (there is at most one \( c \) such that \( H ; e \Downarrow c \)), so the derivation must have this form (where \( c = c_1 + c_2 \)):

  \[
  \begin{array}{c}
  \vdots \\
  \hline
  H ; e \Downarrow c_1 \\
  H ; e \Downarrow c_1 \\
  \hline
  H ; e + e \Downarrow c_1 + c_1
  \end{array}
  \]

  So given that there exists a derivation of \( H ; e \Downarrow c_1 \), we can use **MULT** to derive:

  \[
  \begin{array}{c}
  H ; e \Downarrow c_1 \\
  \hline
  H \Downarrow 2 \\
  \hline
  H ; e \ast 2 \Downarrow 2c_1
  \end{array}
  \]

  Math provides \( c_1 + c_1 = 2c_1 \), so the conclusion of this derivation is what we need.
Theorem: $H ; C[e * 2] \Downarrow c$ if and only if $H ; C[e + e] \Downarrow c$.

Proof: By induction on (the height of) the structure of $C$:

- If the height is 0, then $C$ is $[\cdot]$, so $C[e * 2] = e * 2$ and $C[e + e] = e + e$. So the previous theorem is exactly what we need.

- If the height is greater than 0, then $C$ has one of four forms:

  - If $C$ is $C' + e'$ for some $C'$ and $e'$, then $C[e * 2]$ is $C'[e * 2] + e'$ and $C[e + e]$ is $C'[e + e] + e'$. Since $C'$ is shorter than $C$, induction ensures that for any constant $c'$, $H ; C'[e * 2] \Downarrow c'$ if and only if $H ; C'[e + e] \Downarrow c'$.

    Assume $H ; C'[e * 2] + e' \Downarrow c$ and show $H ; C'[e + e] + e' \Downarrow c$. Any derivation of $H ; C'[e * 2] + e' \Downarrow c$ must end with ADD, i.e., it looks like this (where $c = c' + c''$):

    \[
    \vdots \quad H ; C'[e * 2] \Downarrow e' \quad H ; e' \Downarrow c' \quad H ; C'[e + e] + e' \Downarrow c
    \]

    As argued above, the existence of a derivation of $H ; C'[e * 2] \Downarrow c'$ ensures the existence of a derivation of $H ; C'[e + e] \Downarrow c'$. So using ADD and the existence of a derivation of $H ; e' \Downarrow c''$, we can derive:

    \[
    \frac{H ; C'[e + e] \Downarrow c'}{H ; C'[e + e] + e' \Downarrow c}
    \]

    Now assume $H ; C'[e + e] + e' \Downarrow c$ and show $H ; C'[e * 2] + e' \Downarrow c$: Any derivation of $H ; C'[e + e] + e' \Downarrow c$ must end with ADD, i.e., it looks like this (where $c = c' + c''$):

    \[
    \vdots \quad H ; C'[e + e] \Downarrow e' \quad H ; e' \Downarrow c' \quad H ; C'[e * 2] \Downarrow c''
    \]

    As argued above, the existence of a derivation of $H ; C'[e + e] \Downarrow c'$ ensures the existence of a derivation of $H ; C'[e * 2] \Downarrow c'$. So using ADD and the existence of a derivation of $H ; e' \Downarrow c''$, we can derive:

    \[
    \frac{H ; C'[e * 2] \Downarrow c'}{H ; C'[e * 2] + e' \Downarrow c}
    \]

- The other 3 cases are similar. (Try them out.)
Theorem: The two semantics below are equivalent, i.e., $H; e \Downarrow c$ if and only if $H; e \to^* c$.

\[
\begin{array}{c|c|c}
\text{CONST} & \text{VAR} & \text{ADD} \\

c & H; x \Downarrow H(x) & H; e_1 \Downarrow c_1 \\
\hline
H; c \Downarrow c & H; x \Downarrow H(x) & H; e_2 \Downarrow c_2 \\
\end{array}
\]

Proof: We prove the two directions separately.

First assume $H; e \Downarrow c$; show $\exists n. H; e \to^n c$. By induction on the height $h$ of derivation of $H; e \Downarrow c$:

- $h = 1$: Then the derivation must end with CONST or VAR. For CONST, $e$ is $c$ and trivially $H; e \to^0 c$. For VAR, $e$ is some $x$ where $H(x) = c$, so using SVAR, $H; e \to^1 c$.

- $h > 1$: Then the derivation must end with ADD, so $e$ is some $e_1 + e_2$ where $H; e_1 \Downarrow c_1$, $H; e_2 \Downarrow c_2$, and $e$ is $c_1 + c_2$. By induction $\exists n_1, n_2. H; e_1 \to^{n_1} c_1$ and $H; e_2 \to^{n_2} c_2$. Therefore, using the lemma below, $H; e_1 + e_2 \to^{n_1 + n_2 + 1} c_1 + c_2$, so ADD lets us derive $H; e_1 + e_2 \to^{n_1 + n_2 + 1} c$. We can then use the lemma below, $H; e_1 + e_2 \to e_1 + e_2$.

Lemma: If $H; e \to^n e'$, then $H; e_1 + e_2 \to^n e_1 + e_2$ and $H; e \to^n e' + e_2$.

Proof: By induction on $n$. If $n = 0$, the result is trivial because $e = e'$. If $n > 0$, then there exists some $e''$ such that $H; e \to^{n-1} e''$ and $H; e'' \to^1 e'$. So by induction $H; e_1 + e \to^{n-1} e_1 + e''$ and $H; e \to^{n-1} e'' + e_2$. Using SRIGHT and SLEFT respectively, $H; e'' \to^1 e'$ ensures $H; e_1 + e'' \to^1 e_1 + e'$ and $H; e'' \to^1 e' + e_2$. So with the inductive hypotheses, $H; e_1 + e \to^n e_1 + e'$ and $H; e + e_2 \to^n e' + e_2$.

Now assume $\exists n. H; e \to^n c$; show $H; e \Downarrow c$. By induction on $n$:

- $n = 0$: $c$ is $e$ and CONST lets us derive $H; c \Downarrow c$.

- $n > 0$: So $\exists e'. H; e \to e'$ and $H; e' \to^{n-1} c$. By induction $H; e' \Downarrow c$. So this lemma suffices: If $H; e \to e'$ and $H; e' \Downarrow c$, then $H; e \Downarrow c$. Prove the lemma by induction on height $h$ of derivation of $H; e \to e'$:

  - $h = 1$: Then the derivation ends with SVAR or SADD. For SVAR, $e$ is some $x$ and $e' = H(x) = c$. So with VAR we can derive $H; x \Downarrow H(x)$, i.e., $H; e \Downarrow c$. For SADD, $e$ is some $c_1 + c_2$ and $e' = c = c_1 + c_2$. So with ADD, we can derive $H; c_1 + c_2 \Downarrow c_1 + c_2$, i.e., $H; e \Downarrow c$. (Note the case $h = 1$ case may look a little weird because in fact this case $n = 1$, i.e., $e'$ must be a constant.)

  - $h > 1$: Then the derivation ends with SLEFT or SRIGHT. For SLEFT, the assumed derivations end like this:

    \[
    \begin{array}{c|c|c}
    \text{VAR} & \text{SLEFT} & \text{SRIGHT} \\
    H; e_1 \to e_1' & H; e_1' \Downarrow c_1 & H; e_2 \Downarrow c_2 \\
    H; e_1 + e_2 \to e_1' + e_2 & H; e_1' + e_2 \Downarrow c_1 + c_2 & H; e_2 \Downarrow c_2 \\
    \end{array}
    \]

Using $H; e_1 \to e_1'$, $H; e_1' \Downarrow c_1$, and the inductive hypothesis, $H; e_1 \Downarrow c_1$. Using this fact, $H; e_2 \Downarrow c_2$, and ADD, we can derive $H; e_1 + e_2 \Downarrow c_1 + c_2$.

For SRIGHT, the assumed derivations end like this:

\[
\begin{array}{c|c|c}
\text{VAR} & \text{SLEFT} & \text{SRIGHT} \\
H; e_2 \to e_2' & H; e_1 \Downarrow c_1 & H; e_2' \Downarrow c_2 \\
H; e_1 + e_2 \to e_1 + e_2' & H; e_1 + e_2' \Downarrow c_1 + c_2 & H; e_2 \Downarrow c_2 \\
\end{array}
\]

Using $H; e_2 \to e_2'$, $H; e_2' \Downarrow c_2$, and the inductive hypothesis, $H; e_2 \Downarrow c_2$. Using this fact, $H; e_1 \Downarrow c_1$, and ADD, we can derive $H; e_1 + e_2 \Downarrow c_1 + c_2$.

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