CS-XXX: Graduate Programming Languages

Lecture 17 — Recursive Types

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Where are we

- System F gave us type abstraction
  - code reuse
  - strong abstractions
  - different from real languages (like ML), but the right foundation

- This lecture: Recursive Types (different use of type variables)
  - For building unbounded data structures
  - Turing-completeness without a fix primitive

- Future lecture (?): Existential types (dual to universal types)
  - First-class abstract types
  - Closely related to closures and objects

- Future lecture (?): Type-and-effect systems
Recursive Types

We could add list types (list(τ)) and primitives ([], ::, match), but we want user-defined recursive types

Intuition:

\[
\text{type intlist} = \text{Empty} \mid \text{Cons int} \ast \text{intlist}
\]

Which is roughly:

\[
\text{type intlist} = \text{unit} + (\text{int} \ast \text{intlist})
\]

▶ Seems like a named type is unavoidable
  ▶ But that’s what we thought with let rec and we used fix

▶ Analogously to \text{fix } \lambda x. e, we’ll introduce \mu\alpha.τ
  ▶ Each \alpha “stands for” entire \mu\alpha.τ
Mighty $\mu$

In $\tau$, type variable $\alpha$ stands for $\mu\alpha.\tau$, bound by $\mu$

Examples (of many possible encodings):

- int list (finite or infinite): $\mu\alpha.\text{unit} + (\text{int} \times \alpha)$
- int list (infinite “stream”): $\mu\alpha.\text{int} \times \alpha$
  
  - Need laziness (thunking) or mutation to build such a thing
  
  - Under CBV, can build values of type $\mu\alpha.\text{unit} \rightarrow (\text{int} \times \alpha)$
- int list list: $\mu\alpha.\text{unit} + ((\mu\beta.\text{unit} + (\text{int} \times \beta)) \times \alpha)$

Examples where type variables appear multiple times:

- int tree (data at nodes): $\mu\alpha.\text{unit} + (\text{int} \times \alpha \times \alpha)$
- int tree (data at leaves): $\mu\alpha.\text{int} + (\alpha \times \alpha)$
Using $\mu$ types

How do we build and use int lists ($\mu\alpha.\text{unit} + (\text{int} \times \alpha)$)?

We would like:

- **empty list** = $A()$
  Has type: $\mu\alpha.\text{unit} + (\text{int} \times \alpha)$

- **cons** = $\lambda x : \text{int}. \lambda y : (\mu\alpha.\text{unit} + (\text{int} \times \alpha))$. $B((x, y))$
  Has type: $\text{int} \rightarrow (\mu\alpha.\text{unit} + (\text{int} \times \alpha)) \rightarrow (\mu\alpha.\text{unit} + (\text{int} \times \alpha))$

- **head** = $\lambda x : (\mu\alpha.\text{unit} + (\text{int} \times \alpha))$. $\text{match } x \text{ with } A() | B(y.1)$
  Has type: $(\mu\alpha.\text{unit} + (\text{int} \times \alpha)) \rightarrow (\text{unit} + \mu\alpha.\text{unit} + (\text{int} \times \alpha))$

- **tail** = $\lambda x : (\mu\alpha.\text{unit} + (\text{int} \times \alpha))$. $\text{match } x \text{ with } A() | B(y.2)$
  Has type: $(\mu\alpha.\text{unit} + (\text{int} \times \alpha)) \rightarrow (\text{unit} + \mu\alpha.\text{unit} + (\text{int} \times \alpha))$

But our typing rules allow none of this (yet)
Using \( \mu \) types

How do we build and use int lists \( (\mu\alpha.\text{unit} + (\text{int} \times \alpha)) \)?

We would like:

- empty list = \( \text{A}(() \)\)
  Has type: \( \mu\alpha.\text{unit} + (\text{int} \times \alpha) \)
Using \(\mu\) types

How do we build and use int lists \((\mu\alpha.\text{unit} + (\text{int} \times \alpha))\)?

We would like:

- empty list = \(A()\)
  Has type: \(\mu\alpha.\text{unit} + (\text{int} \times \alpha)\)

- \(\text{cons} = \lambda x: \text{int}. \lambda y: (\mu\alpha.\text{unit} + (\text{int} \times \alpha)). B((x, y))\)
  Has type:
  \[\text{int} \rightarrow (\mu\alpha.\text{unit} + (\text{int} \times \alpha)) \rightarrow (\mu\alpha.\text{unit} + (\text{int} \times \alpha))\]
Using $\mu$ types

How do we build and use int lists ($\mu\alpha.\text{unit} + (\text{int} \times \alpha)$)?

We would like:

- empty list = $A(())$
  Has type: $\mu\alpha.\text{unit} + (\text{int} \times \alpha)$

- cons = $\lambda x:\text{int}. \lambda y:(\mu\alpha.\text{unit} + (\text{int} \times \alpha)). B((x, y))$
  Has type:
  \[
  \text{int} \rightarrow (\mu\alpha.\text{unit} + (\text{int} \times \alpha)) \rightarrow (\mu\alpha.\text{unit} + (\text{int} \times \alpha))
  \]

- head =
  \[
  \lambda x:(\mu\alpha.\text{unit} + (\text{int} \times \alpha)). \text{match } x \text{ with } A_. A(()). B. B(y.1)
  \]
  Has type: $(\mu\alpha.\text{unit} + (\text{int} \times \alpha)) \rightarrow (\text{unit} + \text{int})$
Using \( \mu \) types

How do we build and use int lists \( (\mu \alpha. \text{unit} + (\text{int} \times \alpha)) \)?

We would like:

- empty list = \( A(() \) \)
  Has type: \( \mu \alpha. \text{unit} + (\text{int} \times \alpha) \)
- cons = \( \lambda x: \text{int}. \lambda y: (\mu \alpha. \text{unit} + (\text{int} \times \alpha)). B((x, y)) \)
  Has type:
  \[
  \text{int} \rightarrow (\mu \alpha. \text{unit} + (\text{int} \times \alpha)) \rightarrow (\mu \alpha. \text{unit} + (\text{int} \times \alpha))
  \]
- head =
  \( \lambda x: (\mu \alpha. \text{unit} + (\text{int} \times \alpha)). \text{match } x \text{ with } A_. A(() \mid B y. B(y.1)) \)
  Has type:
  \( (\mu \alpha. \text{unit} + (\text{int} \times \alpha)) \rightarrow (\text{unit} + \text{int}) \)
- tail =
  \( \lambda x: (\mu \alpha. \text{unit} + (\text{int} \times \alpha)). \text{match } x \text{ with } A_. A(() \mid B y. B(y.2)) \)
  Has type:
  \( (\mu \alpha. \text{unit} + (\text{int} \times \alpha)) \rightarrow (\text{unit} + \mu \alpha. \text{unit} + (\text{int} \times \alpha)) \)
Using \( \mu \) types

How do we build and use int lists \((\mu\alpha.\text{unit} + (\text{int} \times \alpha))\)?

We would like:

- empty list = \( A(()) \)
  
  Has type: \( \mu\alpha.\text{unit} + (\text{int} \times \alpha) \)

- cons = \( \lambda x: \text{int}. \lambda y:(\mu\alpha.\text{unit} + (\text{int} \times \alpha)). B((x, y)) \)
  
  Has type:
  \[
  \text{int} \to (\mu\alpha.\text{unit} + (\text{int} \times \alpha)) \to (\mu\alpha.\text{unit} + (\text{int} \times \alpha))
  \]

- head =
  \[
  \lambda x:(\mu\alpha.\text{unit} + (\text{int} \times \alpha)). \text{match } x \text{ with } A_. \ A(()) \mid B y. B(y.1)
  \]
  
  Has type: \( (\mu\alpha.\text{unit} + (\text{int} \times \alpha)) \to (\text{unit} + \text{int}) \)

- tail =
  \[
  \lambda x:(\mu\alpha.\text{unit} + (\text{int} \times \alpha)). \text{match } x \text{ with } A_. \ A(()) \mid B y. B(y.2)
  \]
  
  Has type:
  \[
  (\mu\alpha.\text{unit} + (\text{int} \times \alpha)) \to (\text{unit} + \mu\alpha.\text{unit} + (\text{int} \times \alpha))
  \]

But our typing rules allow none of this (yet)
Using $\mu$ types (continued)

For empty list $= A(())$, one typing rule applies:

$$
\Delta; \Gamma \vdash e : \tau_1 \quad \Delta \vdash \tau_2 \\
\Delta; \Gamma \vdash A(e) : \tau_1 + \tau_2
$$

So we could show

$$
\Delta; \Gamma \vdash A(()) : \text{unit} + (\text{int} \times (\mu\alpha.\text{unit} + (\text{int} \times \alpha)))
$$

(since $FTV(\text{int} \times \mu\alpha.\text{unit} + (\text{int} \times \alpha)) = \emptyset \subseteq \Delta$)
Using $\mu$ types (continued)

For empty list $= A(())$, one typing rule applies:

$$
\frac{
\Delta; \Gamma \vdash e : \tau_1 \\
\Delta \vdash \tau_2
}{
\Delta; \Gamma \vdash A(e) : \tau_1 + \tau_2
}
$$

So we could show

$$
\Delta; \Gamma \vdash A() : \text{unit} + (\text{int} \ast (\mu \alpha. \text{unit} + (\text{int} \ast \alpha)))
$$

(since $FTV(\text{int} \ast \mu \alpha. \text{unit} + (\text{int} \ast \alpha)) = \emptyset \subseteq \Delta$)

But we want $\mu \alpha. \text{unit} + (\text{int} \ast \alpha)$
Using $\mu$ types (continued)

For empty list $= A(())$, one typing rule applies:

\[
\Delta; \Gamma \vdash e : \tau_1 \quad \Delta \vdash \tau_2
\]

\[
\Delta; \Gamma \vdash A(e) : \tau_1 + \tau_2
\]

So we could show

\[
\Delta; \Gamma \vdash A(()) : \text{unit} + (\text{int} \times (\mu\alpha.\text{unit} + (\text{int} \times \alpha)))
\]

(since $FTV(\text{int} \times \mu\alpha.\text{unit} + (\text{int} \times \alpha)) = \emptyset \subseteq \Delta$)

But we want $\mu\alpha.\text{unit} + (\text{int} \times \alpha)$

Notice: $\text{unit} + (\text{int} \times (\mu\alpha.\text{unit} + (\text{int} \times \alpha)))$ is

$(\text{unit} + (\text{int} \times \alpha))[\mu\alpha.\text{unit} + (\text{int} \times \alpha)]/\alpha$
Using $\mu$ types (continued)

For empty list $= A(())$, one typing rule applies:

$$
\Delta; \Gamma \vdash e : \tau_1 \\
\Delta \vdash \tau_2
\frac{}{\Delta; \Gamma \vdash A(e) : \tau_1 + \tau_2}
$$

So we could show

$$\Delta; \Gamma \vdash A(()) : \text{unit} + (\text{int} \times (\mu\alpha.\text{unit} + (\text{int} \times \alpha)))$$

(since $FTV(\text{int} \times \mu\alpha.\text{unit} + (\text{int} \times \alpha)) = \emptyset \subseteq \Delta$)

But we want $\mu\alpha.\text{unit} + (\text{int} \times \alpha)$

Notice: $\text{unit} + (\text{int} \times (\mu\alpha.\text{unit} + (\text{int} \times \alpha)))$ is

$(\text{unit} + (\text{int} \times \alpha))[(\mu\alpha.\text{unit} + (\text{int} \times \alpha))/\alpha]$

The key: Subsumption — recursive types are equal to their “unrolling”
Return of subtyping

Can use *subsumption* and these subtyping rules:

\[
\begin{align*}
\text{ROLL} & \quad \tau[(\mu\alpha.\tau)/\alpha] \leq \mu\alpha.\tau \\
\text{UNROLL} & \quad \mu\alpha.\tau \leq \tau[(\mu\alpha.\tau)/\alpha]
\end{align*}
\]

Subtyping can “roll” or “unroll” a recursive type

Can now give empty-list, cons, and head the types we want:
Constructors use roll, destructors use unroll

Notice how little we did: One new form of type \((\mu\alpha.\tau)\) and two new subtyping rules

(Skipping: Depth subtyping on recursive types is very interesting)
Despite additions being minimal, must reconsider how recursive types change STLC and System F:

- **Erasure (no run-time effect): unchanged**

- **Termination: changed!**
  - \((\lambda x: \mu \alpha. \alpha \rightarrow \alpha. \; x \; x)(\lambda x: \mu \alpha. \alpha \rightarrow \alpha. \; x \; x)\)
  - In fact, we’re now Turing-complete without fix (actually, can type-check every closed \(\lambda\) term)

- **Safety: still safe, but Canonical Forms harder**

- **Inference: Shockingly efficient for “STLC plus \(\mu\)”**
  (A great contribution of PL theory with applications in OO and XML-processing languages)
Syntax-directed $\mu$ types

Recursive types via subsumption “seems magical”

Instead, we can make programmers tell the type-checker where/how to roll and unroll

“Iso-recursive” types: remove subtyping and add expressions:

\[
\begin{align*}
\tau & ::= \ldots | \mu \alpha.\tau \\
 e & ::= \ldots | \text{roll}_{\mu \alpha.\tau} e \mid \text{unroll} e \\
 v & ::= \ldots | \text{roll}_{\mu \alpha.\tau} v \\
\end{align*}
\]

\[
\begin{align*}
 e & \rightarrow e' \\
 \text{roll}_{\mu \alpha.\tau} e & \rightarrow \text{roll}_{\mu \alpha.\tau} e' \\
 \text{unroll} e & \rightarrow \text{unroll} e' \\
 \text{unroll} (\text{roll}_{\mu \alpha.\tau} v) & \rightarrow v
\end{align*}
\]

\[
\begin{align*}
\Delta; \Gamma & \vdash e : \tau[(\mu \alpha.\tau)/\alpha] \\
\Delta; \Gamma & \vdash \text{roll}_{\mu \alpha.\tau} e : \mu \alpha.\tau \\
\Delta; \Gamma & \vdash e : \mu \alpha.\tau \\
\Delta; \Gamma & \vdash \text{unroll} e : \tau[(\mu \alpha.\tau)/\alpha]
\end{align*}
\]
Syntax-directed, continued

Type-checking is syntax-directed / No subtyping necessary

Canonical Forms, Preservation, and Progress are simpler

This is an example of a key trade-off in language design:
  ► Implicit typing can be impossible, difficult, or confusing
  ► Explicit coercions can be annoying and clutter language with no-ops
  ► Most languages do some of each

Anything is decidable if you make the code producer give the implementation enough “hints” about the “proof”
ML datatypes revealed

How is $\mu \alpha . \tau$ related to
type $t = \text{Foo of int} \mid \text{Bar of int} \times t$

Constructor use is a “sum-injection” followed by an *implicit roll*

- So Foo $e$ is really $\text{roll}_t \text{Foo}(e)$
- That is, Foo $e$ has type $t$ (the rolled type)

A pattern-match has an *implicit unroll*

- So match $e$ with... is really match $\text{unroll} \ e$ with...

This “trick” works because different recursive types use different
tags – so the type-checker knows *which* type to roll to