Looking back, looking forward

This is the last lecture using IMP (hooray!). Done:
- Abstract syntax
- Operational semantics (large-step and small-step)
- Semantic properties of (sets of) programs
- “Pseudo-denotational” semantics

Now:
- Packet-filter languages and other examples
- Equivalence of programs in a semantics
- Equivalence of different semantics

Next lecture: Local variables, lambda-calculus
Packet Filters

A very simple view of packet filters:

▶ Some bits come in off the wire
▶ Some application(s) want the “packet” and some do not (e.g., port number)
▶ For safety, only the O/S can access the wire
▶ For extensibility, the applications accept/reject packets

Conventional solution goes to user-space for every packet and app that wants (any) packets

Faster solution: Run app-written filters in kernel-space
What we need

Now the O/S writer is defining the packet-filter language!

Properties we wish of (untrusted) filters:

1. Do not corrupt kernel data structures
2. Terminate (within a time bound)
3. Run fast (the whole point)

Should we download some C/assembly code? (Get 1 of 3)

Should we make up a language and “hope” it has these properties?
Language-based approaches

1. Interpret a language
   + clean operational semantics, + portable, - may be slow (+ filter-specific optimizations), - unusual interface

2. Translate a language into C/assembly
   + clean denotational semantics, + employ existing optimizers, - upfront cost, - unusual interface

3. Require a conservative subset of C/assembly
   + normal interface, - too conservative w/o help

IMP has taught us about (1) and (2) — we’ll get to (3)
A General Pattern

Packet filters move the code to the data rather than data to the code

General reasons: performance, security, other?

Other examples:
- Query languages
- Active networks
- Client-side web scripts (Javascript)
Equivalence motivation

- Program equivalence (we change the program):
  - code optimizer
  - code maintainer

- Semantics equivalence (we change the language):
  - interpreter optimizer
  - language designer
    - (prove properties for equivalent semantics with easier proof)

Note: Proofs may seem easy with the right semantics and lemmas
  - (almost never start off with right semantics and lemmas)

Note: Small-step operational semantics often has harder proofs, but models more interesting things
What is equivalence?

Equivalence depends on what is observable!
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Equivalence depends on what is observable!

- Partial I/O equivalence (if terminates, same ans)

- Total I/O equivalence (same termination behavior, same ans)

- Total heap equivalence (same termination behavior, same heaps)

- All (almost all?) variables have the same value

- Equivalence plus complexity bounds

- Is $O(2^n)$ really equivalent to $O(n)$?

- Is "runs within 10ms of each other" important?

- Syntactic equivalence (perhaps with renaming)

  Too strict to be interesting?

In PL, equivalence most often means total I/O equivalence
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Equivalence depends on what is *observable*!

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  - *while 1 skip* equivalent to everything
  - not transitive
- Total I/O equivalence (same termination behavior, same ans)
What is equivalence?

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- Total I/O equivalence (same termination behavior, same ans)
- Total heap equivalence (same termination behavior, same heaps)
  - All (almost all?) variables have the same value
- Equivalence plus complexity bounds
  - Is $O(2^n^n)$ really equivalent to $O(n)$?
  - Is “runs within 10ms of each other” important?
What is equivalence?

Equivalence depends on **what is observable**!

- Partial I/O equivalence (if terminates, same ans)
  - **while 1 skip** equivalent to everything
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- Total I/O equivalence (same termination behavior, same ans)
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In PL, equivalence most often means total I/O equivalence
Program Example: Strength Reduction

Motivation: Strength reduction
  ▶ A common compiler optimization due to architecture issues

Theorem: $H; e \ast 2 \Downarrow c$ if and only if $H; e + e \Downarrow c$

Proof sketch:
Program Example: Strength Reduction

Motivation: Strength reduction

▶ A common compiler optimization due to architecture issues

Theorem: $H; e \times 2 \downarrow c$ if and only if $H; e + e \downarrow c$

Proof sketch:

▶ Prove separately for each direction
Program Example: Strength Reduction

Motivation: Strength reduction
- A common compiler optimization due to architecture issues

Theorem: $H; e \times 2 \Downarrow c$ if and only if $H; e + e \Downarrow c$

Proof sketch:
- Prove separately for each direction
- Invert the assumed derivation, use hypotheses plus a little math to derive what we need
Program Example: Strength Reduction

Motivation: Strength reduction

- A common compiler optimization due to architecture issues

Theorem: $H ; e \cdot 2 \Downarrow c$ if and only if $H ; e + e \Downarrow c$

Proof sketch:

- Prove separately for each direction

- Invert the assumed derivation, use hypotheses plus a little math to derive what we need

- Hmm, doesn’t use induction. That’s because this theorem isn’t very useful...
Program Example: Nested Strength Reduction

Theorem: If $e'$ has a subexpression of the form $e \ast 2$, then $H ; e' \downarrow c'$ if and only if $H ; e'' \downarrow c'$ where $e''$ is $e'$ with $e \ast 2$ replaced with $e + e$
Program Example: Nested Strength Reduction

Theorem: If $e'$ has a subexpression of the form $e \ast 2$, then $H ; e' \Downarrow c'$ if and only if $H ; e'' \Downarrow c'$ where $e''$ is $e'$ with $e \ast 2$ replaced with $e + e$

First some useful metanotation:

$$C ::= [\cdot] \mid C + e \mid e + C \mid C \ast e \mid e \ast C$$

$C[e]$ is “$C$ with $e$ in the hole” (inductive definition of “stapling”)

Crisper statement of theorem:

$$H ; C[e \ast 2] \Downarrow c'$$ if and only if $$H ; C[e + e] \Downarrow c'$$
Program Example: Nested Strength Reduction

Theorem: If \( e' \) has a subexpression of the form \( e * 2 \), then \( H ; e' \Downarrow c' \) if and only if \( H ; e'' \Downarrow c' \) where \( e'' \) is \( e' \) with \( e * 2 \) replaced with \( e + e \)

First some useful metanotation:

\[
C ::= [\cdot] \mid C + e \mid e + C \mid C * e \mid e * C
\]

\( C[e] \) is “\( C \) with \( e \) in the hole” (inductive definition of “stapling”)

Crisper statement of theorem:

\( H ; C[e * 2] \Downarrow c' \) if and only if \( H ; C[e + e] \Downarrow c' \)

Proof sketch: By induction on structure ("syntax height") of \( C \)
  - The base case (\( C = [\cdot] \)) follows from our previous proof
  - The rest is a long, tedious, (and instructive!) induction
Proof reuse

As we cannot emphasize enough, proving is just like programming.

The proof of nested strength reduction had nothing to do with \( e \ast 2 \) and \( e + e \) except in the base case where we used our previous theorem.

A much more useful theorem would parameterize over the base case so that we could get the “nested \( X \)” theorem for any appropriate \( X \):

If \((H; e_1 \downarrow c) \text{ if and only if } (H; e_2 \downarrow c)\),
then \((H; C[e_1] \downarrow c') \text{ if and only if } (H; C[e_2] \downarrow c')\)

The proof is identical except the base case is “by assumption”
Small-step program equivalence

These sort of proofs also work with small-step semantics (e.g., our IMP statements), but tend to be more cumbersome, even to state.

Example: The statement-sequence operator is associative. That is,

(a) For all \( n \), if \( H ; s_1; (s_2; s_3) \xrightarrow{n} H' ; \text{skip} \) then there exist \( H'' \) and \( n' \) such that \( H ; (s_1; s_2); s_3 \xrightarrow{n'} H'' ; \text{skip} \) and \( H''(\text{ans}) = H'(\text{ans}) \).

(b) If for all \( n \) there exist \( H' \) and \( s' \) such that
\( H ; s_1; (s_2; s_3) \xrightarrow{n} H' ; s' \), then for all \( n \) there exist \( H'' \) and \( s'' \) such that \( H ; (s_1; s_2); s_3 \xrightarrow{n} H'' ; s'' \).

(Proof needs a much stronger induction hypothesis.)

One way to avoid it: Prove large-step and small-step semantics equivalent, then prove program equivalences in whichever is easier.
Language Equivalence Example

IMP w/o multiply large-step:

\[
\begin{align*}
\text{CONST} & \quad \text{VAR} \\
H ; c & \Downarrow c & H ; x & \Downarrow H(x)
\end{align*}
\]

IMP w/o multiply small-step:

\[
\begin{align*}
\text{SVAR} & \\
H ; x & \rightarrow H(x)
\end{align*}
\]

\[
\begin{align*}
\text{SLEFT} & \\
H ; e_1 & \rightarrow e'_1 & H ; e_1 + e_2 & \rightarrow e'_1 + e_2
\end{align*}
\]

\[
\begin{align*}
\text{SADD} & \\
H ; c_1 + c_2 & \rightarrow c_1 + c_2
\end{align*}
\]

\[
\begin{align*}
\text{SRIGHT} & \\
H ; e_2 & \rightarrow e'_2 & H ; e_1 + e_2 & \rightarrow e_1 + e'_2
\end{align*}
\]

Theorem: Semantics are equivalent: \( H ; e \Downarrow c \) if and only if \( H ; e \rightarrow^* c \)

Proof: We prove the two directions separately...
Proof, part 1

First assume $H; e \downarrow c$ and show $\exists n. H; e \rightarrow^n c$
Proof, part 1

First assume $H; e \downarrow c$ and show $\exists n. H; e \rightarrow^n c$

Lemma (prove it!): If $H; e \rightarrow^n e'$, then $H; e_1 + e \rightarrow^n e_1 + e'$ and $H; e + e_2 \rightarrow^n e' + e_2$.

- Proof by induction on $n$
- Inductive case uses SLEFT and SRIGHT
Proof, part 1

First assume $H; e \downarrow c$ and show $\exists n. H; e \rightarrow^n c$

Lemma (prove it!): If $H; e \rightarrow^n e'$, then $H; e_1 + e \rightarrow^n e_1 + e'$
and $H; e + e_2 \rightarrow^n e' + e_2$.

- Proof by induction on $n$
- Inductive case uses $\text{SLEFT}$ and $\text{SRIGHT}$

Given the lemma, prove by induction on derivation of $H; e \downarrow c$
Proof, part 1

First assume $H; e \downarrow c$ and show $\exists n. H; e \rightarrow^n c$

Lemma (prove it!): If $H; e \rightarrow^n e'$, then $H; e_1 + e \rightarrow^n e_1 + e'$ and $H; e + e_2 \rightarrow^n e' + e_2$.

- Proof by induction on $n$
- Inductive case uses SLEFT and SRIGHT

Given the lemma, prove by induction on derivation of $H; e \downarrow c$

- **CONST**: Derivation with **CONST** implies $e = c$, and we can derive $H; c \rightarrow^0 c$
Proof, part 1

First assume $H \vdash e \Downarrow c$ and show $\exists n. \ H; e \rightarrow^n c$

Lemma (prove it!): If $H; e \rightarrow^n e'$, then $H; e_1 + e \rightarrow^n e_1 + e'$ and $H; e + e_2 \rightarrow^n e' + e_2$.

- Proof by induction on $n$
- Inductive case uses SLEFT and SRIGHT

Given the lemma, prove by induction on derivation of $H \vdash e \Downarrow c$

- **CONST**: Derivation with **CONST** implies $e = c$, and we can derive $H; c \rightarrow^0 c$
- **VAR**: Derivation with **VAR** implies $e = x$ for some $x$ where $H(x) = c$, so derive $H; e \rightarrow^1 c$ with **SVAR**
Proof, part 1

First assume $H; e \downarrow c$ and show $\exists n. H; e \rightarrow^n c$

Lemma (prove it!): If $H; e \rightarrow^n e'$, then $H; e_1 + e \rightarrow^n e_1 + e'$ and $H; e + e_2 \rightarrow^n e' + e_2$.

- Proof by induction on $n$
- Inductive case uses SLEFT and SRIGHT

Given the lemma, prove by induction on derivation of $H; e \downarrow c$

- **CONST**: Derivation with \texttt{CONST} implies $e = c$, and we can derive $H; c \rightarrow^0 c$

- **VAR**: Derivation with \texttt{VAR} implies $e = x$ for some $x$ where $H(x) = c$, so derive $H; e \rightarrow^1 c$ with \texttt{SVAR}

- **ADD**: ...
Part 1, continued

First assume $H; e \Downarrow c$ and show $\exists n. H; e \rightarrow^nc$

Lemma (prove it!): If $H; e \rightarrow^ne'$, then $H; e_1 + e \rightarrow^ne_1 + e'$ and $H; e_2 \rightarrow^ne' + e_2$.

Given the lemma, prove by induction on derivation of $H; e \Downarrow c$

- ... 

- ADD: Derivation with ADD implies $e = e_1 + e_2$, $c = c_1 + c_2$, $H; e_1 \Downarrow c_1$, and $H; e_2 \Downarrow c_2$ for some $e_1, e_2, c_1, c_2$. 
First assume \( H ; e \downarrow c \) and show \( \exists n. H ; e \rightarrow^n c \)

Lemma (prove it!): If \( H ; e \rightarrow^n e' \), then \( H ; e_1 + e \rightarrow^n e_1 + e' \) and \( H ; e + e_2 \rightarrow^n e' + e_2 \).

Given the lemma, prove by induction on derivation of \( H ; e \downarrow c \)

- ... 
- **ADD**: Derivation with **ADD** implies \( e = e_1 + e_2, \ c = c_1 + c_2, \ H ; e_1 \downarrow c_1, \) and \( H ; e_2 \downarrow c_2 \) for some \( e_1, e_2, c_1, c_2 \). By induction (twice), \( \exists n_1, n_2. \ H ; e_1 \rightarrow^{n_1} c_1 \) and \( H ; e_2 \rightarrow^{n_2} c_2 \).
First assume $H; e \downarrow c$ and show $\exists n. H; e \rightarrow^n c$

Lemma (prove it!): If $H; e \rightarrow^n e'$, then $H; e_1 + e \rightarrow^n e_1 + e'$ and $H; e + e_2 \rightarrow^n e' + e_2$.

Given the lemma, prove by induction on derivation of $H; e \downarrow c$

- ...  
- **ADD**: Derivation with ADD implies $e = e_1 + e_2$, $c = c_1 + c_2$, $H; e_1 \downarrow c_1$, and $H; e_2 \downarrow c_2$ for some $e_1, e_2, c_1, c_2$.

By induction (twice), $\exists n_1, n_2. H; e_1 \rightarrow^{n_1} c_1$ and $H; e_2 \rightarrow^{n_2} c_2$.

So by our lemma $H; e_1 + e_2 \rightarrow^{n_1} c_1 + e_2$ and $H; c_1 + e_2 \rightarrow^{n_2} c_1 + c_2$. 
Part 1, continued

First assume $H; e \Downarrow c$ and show $\exists n. H; e \rightarrow^n c$

Lemma (prove it!): If $H; e \rightarrow^n e'$, then $H; e_1 + e \rightarrow^n e_1 + e'$ and $H; e + e_2 \rightarrow^n e' + e_2$.

Given the lemma, prove by induction on derivation of $H; e \Downarrow c$

- ... 
- ADD: Derivation with ADD implies $e = e_1 + e_2$, $c = c_1 + c_2$, $H; e_1 \Downarrow c_1$, and $H; e_2 \Downarrow c_2$ for some $e_1, e_2, c_1, c_2$.
  
  By induction (twice), $\exists n_1, n_2. H; e_1 \rightarrow^{n_1} c_1$ and $H; e_2 \rightarrow^{n_2} c_2$.

  So by our lemma $H; e_1 + e_2 \rightarrow^{n_1} c_1 + e_2$ and $H; c_1 + e_2 \rightarrow^{n_2} c_1 + c_2$.

  By SADD $H; c_1 + c_2 \rightarrow c_1 + c_2$. 
Part 1, continued

First assume \( H; e \downarrow c \) and show \( \exists n. H; e \rightarrow^n c \)

Lemma (prove it!): If \( H; e \rightarrow^n e' \), then \( H; e_1 + e \rightarrow^n e_1 + e' \) and \( H; e + e_2 \rightarrow^n e' + e_2 \).

Given the lemma, prove by induction on derivation of \( H; e \downarrow c \)

- ... 

- \textbf{ADD}: Derivation with ADD implies \( e = e_1 + e_2, c = c_1 + c_2, \)
  \( H; e_1 \downarrow c_1, \) and \( H; e_2 \downarrow c_2 \) for some \( e_1, e_2, c_1, c_2 \).
  By induction (twice), \( \exists n_1, n_2. H; e_1 \rightarrow^{n_1} c_1 \) and
  \( H; e_2 \rightarrow^{n_2} c_2 \).
  So by our lemma \( H; e_1 + e_2 \rightarrow^{n_1} c_1 + e_2 \) and
  \( H; c_1 + e_2 \rightarrow^{n_2} c_1 + c_2 \).
  By \textbf{SADD} \( H; c_1 + c_2 \rightarrow c_1 + c_2 \).
  So \( H; e_1 + e_2 \rightarrow^{n_1+n_2+1} c \).
Proof, part 2

Now assume $\exists n. H; e \rightarrow^n c$ and show $H ; e \downarrow c$. 
Proof, part 2

Now assume $\exists n. \ H; e \rightarrow^n c$ and show $H ; e \downarrow c$.

Proof by induction on $n$:
Proof, part 2

Now assume $\exists n. \ H; e \rightarrow^n c$ and show $H ; e \Downarrow c$.

Proof by induction on $n$:

- $n = 0$: $e$ is $c$ and $\text{CONST}$ lets us derive $H ; c \Downarrow c$
Proof, part 2

Now assume $\exists n. \ H; e \rightarrow^n c$ and show $H ; e \Downarrow c$.

Proof by induction on $n$:

- $n = 0$: $e$ is $c$ and $\text{CONST}$ lets us derive $H ; c \Downarrow c$
- $n > 0$: (Clever: break into first step and remaining ones)
  $\exists e'. \ H; e \rightarrow e'$ and $H; e' \rightarrow^{n-1} c$. 


Proof, part 2

Now assume \( \exists n. \ H; e \rightarrow^n c \) and show \( H ; e \downarrow c \).

Proof by induction on \( n \):

- \( n = 0 \): \( e \) is \( c \) and \textsc{const} lets us derive \( H ; c \downarrow c \)
- \( n > 0 \) (Clever: break into \textit{first} step and remaining ones)
  - \( \exists e'. \ H; e \rightarrow e' \) and \( H; e' \rightarrow^{n-1} c \).
  - By induction \( H ; e' \downarrow c \).
Proof, part 2

Now assume $\exists n. \ H ; e \rightarrow^n c$ and show $H ; e \Downarrow c$.

Proof by induction on $n$:

- $n = 0$: $e$ is $c$ and $\text{CONST}$ lets us derive $H ; c \Downarrow c$
- $n > 0$: (Clever: break into first step and remaining ones)
  - $\exists e'$. $H ; e \rightarrow e'$ and $H ; e' \rightarrow^{n-1} c$.
  - By induction $H ; e' \Downarrow c$.
  - So this lemma suffices: If $H ; e \rightarrow e'$ and $H ; e' \Downarrow c$, then $H ; e \Downarrow c$. 
Proof, part 2

Now assume $\exists n. \ H ; e \rightarrow^n c$ and show $H ; e \downarrow c$.

Proof by induction on $n$:

- $n = 0$: $e$ is $c$ and $\text{CONST}$ lets us derive $H ; c \downarrow c$
- $n > 0$: (Clever: break into first step and remaining ones)
  $\exists e'. \ H ; e \rightarrow e'$ and $H ; e' \rightarrow^{n-1} c$.
  By induction $H ; e' \downarrow c$.
  So this lemma suffices: If $H ; e \rightarrow e'$ and $H ; e' \downarrow c$, then $H ; e \downarrow c$.

Prove the lemma by induction on derivation of $H ; e \rightarrow e'$:

- SVAR: ...
- SADD: ...
- SLEFT: ...
- SRIGHT: ...
Part 2, key lemma

Lemma: If $H; e \rightarrow e'$ and $H; e' \downarrow c$, then $H; e \downarrow c$.

Prove the lemma by induction on derivation of $H; e \rightarrow e'$:
Part 2, key lemma

Lemma: If $H; e \rightarrow e'$ and $H; e' \downarrow c$, then $H; e \downarrow c$.

Prove the lemma by induction on derivation of $H; e \rightarrow e'$:

- **SVAR**: Derivation with `SVAR` implies $e$ is some $x$ and $e' = H(x) = c$, so derive, by `VAR`, $H; x \downarrow H(x)$. 

- **SADD**: Derivation with `SADD` implies $e$ is some $c_1 + c_2$ and $e' = c_1 + c_2 = c$, so derive, by `ADD` and two `CONST`, $H; c_1 + c_2 \downarrow c_1 + c_2$.

- **SLEFT**: Derivation with `SLEFT` implies $e = e_1 + e_2$ and $e' = e'_1 + e_2$ and $H; e_1 \rightarrow e'_1$ for some $e_1, e_2, e'_1$. Since $e' = e'_1 + e_2$ inverting assumption $H; e'_1 \downarrow c$ gives $H; e'_1 \downarrow c_1$, $H; e_2 \downarrow c_2$ and $c = c_1 + c_2$. Applying the induction hypothesis to $H; e_1 \rightarrow e'_1$ and $H; e'_1 \downarrow c_1$ gives $H; e_1 \downarrow c_1$.

- **SRIGHT**: Analogous to `SLEFT`.
Part 2, key lemma

Lemma: If $H; e \rightarrow e'$ and $H; e' \Downarrow c$, then $H; e \Downarrow c$.

Prove the lemma by induction on derivation of $H; e \rightarrow e'$:
- **svar**: Derivation with **svar** implies $e$ is some $x$ and $e' = H(x) = c$, so derive, by **var**, $H; x \Downarrow H(x)$.
- **sadd**: Derivation with **sadd** implies $e$ is some $c_1 + c_2$ and $e' = c_1 + c_2 = c$, so derive, by **add** and two **const**, $H; c_1 + c_2 \Downarrow c_1 + c_2$. 
Part 2, key lemma

Lemma: If $H; e \rightarrow e'$ and $H; e' \Downarrow c$, then $H; e \Downarrow c$.

Prove the lemma by induction on derivation of $H; e \rightarrow e'$:

- **svar**: Derivation with **svar** implies $e$ is some $x$ and $e' = H(x) = c$, so derive, by **var**, $H; x \Downarrow H(x)$.

- **sadd**: Derivation with **sadd** implies $e$ is some $c_1 + c_2$ and $e' = c_1 + c_2 = c$, so derive, by **add** and two **const**, $H; c_1 + c_2 \Downarrow c_1 + c_2$.

- **sleft**: Derivation with **sleft** implies $e = e_1 + e_2$ and $e' = e_1' + e_2$ and $H; e_1 \rightarrow e_1'$ for some $e_1, e_2, e_1'$.
Part 2, key lemma

Lemma: If $H; e \rightarrow e'$ and $H; e' \Downarrow c$, then $H; e \Downarrow c$.

Prove the lemma by induction on derivation of $H; e \rightarrow e'$:

- **SVAR:** Derivation with **SVAR** implies $e$ is some $x$ and $e' = H(x) = c$, so derive, by **VAR**, $H; x \Downarrow H(x)$.

- **SADD:** Derivation with **SADD** implies $e$ is some $c_1 + c_2$ and $e' = c_1 + c_2 = c$, so derive, by **ADD** and two **CONST**, $H; c_1 + c_2 \Downarrow c_1 + c_2$.

- **SLEFT:** Derivation with **SLEFT** implies $e = e_1 + e_2$ and $e' = e'_1 + e_2$ and $H; e_1 \rightarrow e'_1$ for some $e_1, e_2, e'_1$.
  
  Since $e' = e'_1 + e_2$ inverting assumption $H; e' \Downarrow c$ gives $H; e'_1 \Downarrow c_1$, $H; e_2 \Downarrow c_2$ and $c = c_1 + c_2$.
Part 2, key lemma

Lemma: If $H; e \rightarrow e'$ and $H; e' \downarrow c$, then $H; e \downarrow c$.

Prove the lemma by induction on derivation of $H; e \rightarrow e'$:

- **SVAR**: Derivation with SVAR implies $e$ is some $x$ and $e' = H(x) = c$, so derive, by VAR, $H; x \downarrow H(x)$.

- **SADD**: Derivation with SADD implies $e$ is some $c_1 + c_2$ and $e' = c_1 + c_2 = c$, so derive, by ADD and two CONST, $H; c_1 + c_2 \downarrow c_1 + c_2$.

- **SLEFT**: Derivation with SLEFT implies $e = e_1 + e_2$ and $e' = e'_1 + e_2$ and $H; e_1 \rightarrow e'_1$ for some $e_1, e_2, e'_1$. Since $e' = e'_1 + e_2$ inverting assumption $H; e' \downarrow c$ gives $H; e'_1 \downarrow c_1$, $H; e_2 \downarrow c_2$ and $c = c_1 + c_2$. Applying the induction hypothesis to $H; e_1 \rightarrow e'_1$ and $H; e'_1 \downarrow c_1$ gives $H; e_1 \downarrow c_1$. 

- **SRIGHT**: Analogous to SLEFT.
Part 2, key lemma

Lemma: If $H ; e \rightarrow e'$ and $H ; e' \downarrow c$, then $H ; e \downarrow c$.

Prove the lemma by induction on derivation of $H ; e \rightarrow e'$:

- **svar**: Derivation with svar implies $e$ is some $x$ and $e' = H(x) = c$, so derive, by var, $H ; x \downarrow H(x)$.

- **sadd**: Derivation with sadd implies $e$ is some $c_1 + c_2$ and $e' = c_1 + c_2 = c$, so derive, by add and two const, $H ; c_1 + c_2 \downarrow c_1 + c_2$.

- **sleft**: Derivation with sleft implies $e = e_1 + e_2$ and $e' = e'_1 + e_2$ and $H ; e_1 \rightarrow e'_1$ for some $e_1, e_2, e'_1$. Since $e' = e'_1 + e_2$ inverting assumption $H ; e' \downarrow c$ gives $H ; e'_1 \downarrow c_1, H ; e_2 \downarrow c_2$ and $c = c_1 + c_2$. Applying the induction hypothesis to $H ; e_1 \rightarrow e'_1$ and $H ; e'_1 \downarrow c_1$ gives $H ; e_1 \downarrow c_1$. So use add, $H ; e_1 \downarrow c_1$, and $H ; e_2 \downarrow c_2$ to derive $H ; e_1 + e_2 \downarrow c_1 + c_2$.
Part 2, key lemma

Lemma: If $H; e \rightarrow e'$ and $H; e' \Downarrow c$, then $H; e \Downarrow c$.

Prove the lemma by induction on derivation of $H; e \rightarrow e'$:

- **SVAR:** Derivation with **SVAR** implies $e$ is some $x$ and $e' = H(x) = c$, so derive, by **VAR**, $H; x \Downarrow H(x)$.

- **SADD:** Derivation with **SADD** implies $e$ is some $c_1 + c_2$ and $e' = c_1 + c_2 = c$, so derive, by **ADD** and two **CONST**, $H; c_1 + c_2 \Downarrow c_1 + c_2$.

- **SLEFT:** Derivation with **SLEFT** implies $e = e_1 + e_2$ and $e' = e'_1 + e_2$ and $H; e_1 \rightarrow e'_1$ for some $e_1, e_2, e'_1$. Since $e' = e'_1 + e_2$ inverting assumption $H; e' \Downarrow c$ gives $H; e'_1 \Downarrow c_1, H; e_2 \Downarrow c_2$ and $c = c_1 + c_2$. Applying the induction hypothesis to $H; e_1 \rightarrow e'_1$ and $H; e'_1 \Downarrow c_1$ gives $H; e_1 \Downarrow c_1$. So use **ADD**, $H; e_1 \Downarrow c_1$, and $H; e_2 \Downarrow c_2$ to derive $H; e_1 + e_2 \Downarrow c_1 + c_2$.

- **SRIGHT:** Analogous to **SLEFT**
The cool part, redux

Step through the $\text{SLEFT}$ case more visually:

By assumption, we must have derivations that look like this:

\[
\begin{align*}
H; e_1 & \rightarrow e'_1 \\
H; e_1 + e_2 & \rightarrow e'_1 + e_2
\end{align*}
\]

\[
\begin{align*}
H; e'_1 & \downarrow c_1 \\
H; e'_1 + e_2 & \downarrow c_1 + c_2
\end{align*}
\]

Grab the hypothesis from the left and the left hypothesis from the right and use induction to get $H; e_1 \downarrow c_1$.

Now go grab the one hypothesis we haven’t used yet and combine it with our inductive result to derive our answer:

\[
\begin{align*}
H; e_1 & \downarrow c_1 \\
H; e_2 & \downarrow c_2
\end{align*}
\]

\[
H; e_1 + e_2 \downarrow c_1 + c_2
\]
A nice payoff

Theorem: The small-step semantics is deterministic:
if $H; e \rightarrow^* c_1$ and $H; e \rightarrow^* c_2$, then $c_1 = c_2$

Not obvious (see $sleft$ and $sright$), nor do I know a direct proof.

Given $(((1 + 2) + (3 + 4)) + (5 + 6)) + (7 + 8)$ there are many execution sequences, which all produce 36 but with different intermediate expressions.

Proof:
Large-step evaluation is deterministic (easy induction proof)
Small-step and large-step are equivalent (just proved that)
So small-step is deterministic
Convince yourself a deterministic and a nondeterministic semantics cannot be equivalent
A nice payoff

Theorem: The small-step semantics is deterministic:
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Theorem: The small-step semantics is deterministic:
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- Given $((((1 + 2) + (3 + 4)) + (5 + 6)) + (7 + 8))$ there are many execution sequences, which all produce 36 but with different intermediate expressions

Proof:

- Large-step evaluation is deterministic (easy induction proof)
- Small-step and and large-step are equivalent (just proved that)
- So small-step is deterministic
- Convince yourself a deterministic and a nondeterministic semantics cannot be equivalent
Conclusions

- Equivalence is a subtle concept
- Proofs “seem obvious” only when the definitions are right

Some other language-equivalence claims:

- Replace `while rule` with `H`, `e` $\Downarrow$ `c` $\leq$ `0` `H`; `while e s` $\rightarrow$ `H`; `skip` `H`; `e` $\Downarrow$ `c` $>$ `0` `H`; `while e s` $\rightarrow$ `H`; `s`;

Equivalent to our original language

- Change syntax of heap and replace `assign` and `var` rules with `H`, `x` $\Rightarrow$ `e` $\rightarrow$ `H`, `x` $\Rightarrow$ `e` $\rightarrow$ `H`, `skip` `H`, `H(x)` $\Rightarrow$ `c` `H`;

NOT equivalent to our original language
Conclusions

- Equivalence is a subtle concept
- Proofs “seem obvious” only when the definitions are right
- Some other language-equivalence claims:

Replace **WHILE** rule with

\[
\begin{align*}
H ; e \downarrow c & \quad c \leq 0 \\
\hline
H ; \textbf{while} e \ s & \rightarrow H ; \textbf{skip}
\end{align*}
\]

\[
\begin{align*}
H ; e \downarrow c & \quad c > 0 \\
\hline
H ; \textbf{while} e \ s & \rightarrow H ; s; \textbf{while} e \ s
\end{align*}
\]
Conclusions

- Equivalence is a subtle concept
- Proofs “seem obvious” only when the definitions are right
- Some other language-equivalence claims:

Replace **while** rule with

\[
\frac{H ; e \downarrow c \quad c \leq 0}{H ; \text{while } e \ s \rightarrow H ; \text{skip}} \quad \frac{H ; e \downarrow c \quad c > 0}{H ; \text{while } e \ s \rightarrow H ; s ; \text{while } e \ s}
\]

Equivalent to our original language
Conclusions

- Equivalence is a subtle concept
- Proofs “seem obvious” only when the definitions are right
- Some other language-equivalence claims:

Replace WHILE rule with

\[
\begin{align*}
H ; e \downarrow c & \quad c \leq 0 \\
\hline
H ; \text{while } e s & \rightarrow H ; \text{skip}
\end{align*}
\]

\[
\begin{align*}
H ; e \downarrow c & \quad c > 0 \\
\hline
H ; \text{while } e s & \rightarrow H ; s \text{; while } e s
\end{align*}
\]

Equivalent to our original language

Change syntax of heap and replace ASSIGN and VAR rules with

\[
\begin{align*}
H ; x := e & \rightarrow H, x \mapsto e ; \text{skip}
\end{align*}
\]

\[
\begin{align*}
H ; H(x) \downarrow c
\end{align*}
\]

\[
\begin{align*}
H ; x \downarrow c
\end{align*}
\]
Conclusions

- Equivalence is a subtle concept
- Proofs “seem obvious” only when the definitions are right
- Some other language-equivalence claims:

Replace \texttt{while} rule with

\[
\begin{align*}
&H ; e \downarrow c \quad c \leq 0 \\
&\quad H ; \text{while } e \ s \rightarrow H ; \text{skip}
\end{align*}
\]

\[
\begin{align*}
&H ; e \downarrow c \quad c > 0 \\
&\quad H ; \text{while } e \ s \rightarrow H ; \ s ; \text{while } e \ s
\end{align*}
\]

Equivalent to our original language

Change syntax of heap and replace \texttt{ASSIGN} and \texttt{VAR} rules with

\[
\begin{align*}
&H ; x := e \rightarrow H, x \mapsto e ; \text{skip} \\
&H ; H(x) \downarrow c
\end{align*}
\]

\[
\begin{align*}
&H ; x \downarrow c
\end{align*}
\]

\textit{NOT} equivalent to our original language