CS-XXX: Graduate Programming Languages

Lecture 9 — Simply Typed Lambda Calculus

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Types

Major new topic worthy of several lectures: Type systems
▶ Continue to use (CBV) Lambda Calculus as our core model
▶ But will soon enrich with other common primitives

This lecture:
▶ Motivation for type systems
▶ What a type system is designed to do and not do
  ▶ Definition of stuckness, soundness, completeness, etc.
▶ The Simply-Typed Lambda Calculus
  ▶ A basic and natural type system
  ▶ Starting point for more expressiveness later

Next lecture:
▶ Prove Simply-Typed Lambda Calculus is sound
Review: L-R CBV Lambda Calculus

\[ e ::= \lambda x. e \mid x \mid e \; e \]

\[ v ::= \lambda x. e \]

Implicit systematic renaming of bound variables

- \( \alpha \)-equivalence on expressions (“the same term”)

\[ e \to e' \]

\[
\begin{align*}
(\lambda x. e) \; v & \to e[v/x] \\
e_1 \; e_2 & \to e'_1 \; e_2 \\
v \; e_2 & \to v \; e'_2
\end{align*}
\]

\[ e_1[e_2/x] = e_3 \]

\[
\begin{align*}
x[e/x] & = e \\
y[e/x] & = y \\
\end{align*}
\]

\[
\begin{align*}
y \neq x \\
x[e/x] & = e \\
y[e/x] & = y \\
(\lambda y. e_1)[e/x] & = \lambda y. e'_1
\end{align*}
\]

\[
\begin{align*}
y \neq x \\
y \notin FV(e) \\
\end{align*}
\]
Introduction to Types

Naive thought: More powerful PLs are always better

- Be Turing Complete (e.g., Lambda Calculus or x86 Assembly)
- Have really flexible features (e.g., lambdas)
- Have conveniences to keep programs short

If this is the only metric, types are a step backward

- Whole point is to allow fewer programs
- A “filter” between abstract syntax and compiler/interpreter
  - Fewer programs in language means less for a correct implementation
- So if types are a great idea, they must help with other desirable properties for a PL...
Why types? (Part 1)

1. Catch “simple” mistakes early, even for untested code
   ▶ Example: “if” applied to “mkpair”
   ▶ Even if some too-clever programmer meant to do it
   ▶ Even though decidable type systems must be conservative
Why types? (Part 1)

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2. (Safety) Prevent getting stuck (e.g., \( x \, v \))
   ▶ Ensure execution never gets to a “meaningless” state
   ▶ But “meaningless” depends on the semantics
   ▶ Each PL typically makes some things type errors (again being conservative) and others run-time errors
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3. Enforce encapsulation (an abstract type)
   - Clients can’t break invariants
   - Clients can’t assume an implementation
   - Requires safety, meaning no “stuck” states that corrupt run-time (e.g., C/C++)
   - Can enforce encapsulation without static types, but types are a particularly nice way
Why types? (Part 2)

4. Assuming well-typedness allows faster implementations
   - Smaller interfaces enable optimizations
   - Don’t have to check for impossible states
   - Orthogonal to safety (e.g., C/C++)

5. Syntactic overloading
   - Have symbol lookup depend on operands’ types
   - Only modestly interesting semantically
   - Late binding (lookup via run-time types) more interesting

6. Detect other errors via extensions
   - Often via a “type-and-effect” system
   - Deep similarities in analyses suggest type systems a good way to think-about/define/prove what you’re checking
   - Uncaught exceptions, tainted data, non-termination, IO performed, data races, dangling pointers, ...
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We’ll focus on (1), (2), and (3) and maybe (6)
What is a type system?

Er, uh, you know it when you see it. Some clues:

▶ A decidable (?) judgment for classifying programs
  ▸ E.g., $e_1 + e_2$ has type int if $e_1, e_2$ have type int (else no type)

▶ A sound (?) abstraction of computation
  ▸ E.g., if $e_1 + e_2$ has type int, then evaluation produces an int (with caveats!))

▶ Fairly syntax directed
  ▸ Non-example (?): $e$ terminates within 100 steps

▶ Particularly fuzzy distinctions with abstract interpretation
  ▸ Possible topic for a later lecture
  ▸ Often a more natural framework for flow-sensitive properties
  ▸ Types often more natural for higher-order programs

This is a CS-centric, PL-centric view. Foundational type theory has more rigorous answers

▶ Later lecture: Typed PLs are like proof systems for logics
Plan for 3ish weeks

- Simply typed $\lambda$ calculus
- (Syntactic) Type Soundness (i.e., safety)
- Extensions (pairs, sums, lists, recursion)

*Break for the Curry-Howard isomorphism; continuations; midterm*

- Subtyping
- Polymorphic types (generics)
- Recursive types
- Abstract types
- Effect systems

Homework: Adding back mutation
Omitted: Type inference
Adding constants

Enrich the Lambda Calculus with integer constants:

- Not strictly necessary, but makes types seem more natural

\[
e \ ::= \ \lambda x. \ e \mid x \mid e \ e \mid c
\]

\[
v \ ::= \ \lambda x. \ e \mid c
\]

*No new operational-semantics rules since constants are values*

We could add + and other *primitives*

- Then we would need new rules (e.g., 3 small-step for +)
- Alternately, parameterize “programs” by primitives: 
  \[
  \lambda plus. \ \lambda times. \ ... \ e
  \]
  - Like Pervasives in OCaml
  - A great way to keep language definitions small
Key issue: can a program “get stuck” (reach a “bad” state)?

- **Definition:** \( e \) is stuck if \( e \) is not a value and there is no \( e' \) such that \( e \rightarrow e' \)

- **Definition:** \( e \) can get stuck if there exists an \( e' \) such that \( e \rightarrow^* e' \) and \( e' \) is stuck
  - In a deterministic language, \( e \) “gets stuck”

Most people don’t appreciate that stuckness depends on the operational semantics

- Inherent given the definitions above
What’s stuck?

Given our language, what are the set of stuck expressions?

- Note: Explicitly defining the stuck states is unusual

\[
\begin{align*}
  e & ::= \lambda x. e \mid x \mid ee \mid c \\
  v & ::= \lambda x. e \mid c
\end{align*}
\]

\[
\begin{array}{l}
  (\lambda x. e) v \rightarrow e[v/x] \\
  e_1 \rightarrow e'_1 \\
  e_1 e_2 \rightarrow e'_1 e_2 \\
  e_2 \rightarrow e'_2 \\
  v e_2 \rightarrow v e'_2
\end{array}
\]

(Hint: The full set is recursively defined.)

\[
S ::= 
\]
What’s stuck?

Given our language, what are the set of stuck expressions?

- Note: Explicitly defining the stuck states is unusual

\[
\begin{align*}
  e & ::= \lambda x. e \mid x \mid e \ e \mid c \\
  v & ::= \lambda x. e \mid c \\

  \quad \frac{(\lambda x. e) \ v \rightarrow e[v/x]}{(\lambda x. e) v \rightarrow e[v/x]} & \quad \frac{e_1 \rightarrow e_1'}{e_1 \ e_2 \rightarrow e_1' \ e_2} & \quad \frac{e_2 \rightarrow e_2'}{v \ e_2 \rightarrow v \ e_2'}
\end{align*}
\]

(Hint: The full set is recursively defined.)

\[
S ::= x \mid c \ v \mid S \ e \mid v \ S
\]
What’s stuck?

Given our language, what are the set of stuck expressions?

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\[
\begin{align*}
  e &::= \lambda x. e \mid x \mid e \ e \mid c \\
  v &::= \lambda x. e \mid c
\end{align*}
\]

\[
\begin{align*}
  (\lambda x. e) \ v &\rightarrow e[v/x] \\
  e_1 \rightarrow e'_1 \\
  e_1 \ e_2 &\rightarrow e'_1 \ e_2 \\
  e_2 &\rightarrow e'_2 \\
  v \ e_2 &\rightarrow v \ e'_2
\end{align*}
\]

(Hint: The full set is recursively defined.)

\[
S ::= x \mid c \ v \mid S \ e \mid v \ S
\]

Note: Can have fewer stuck states if we add more rules

- Example: Javascript

- Example: 

\[
c \ v \rightarrow v
\]

- In unsafe languages, stuck states can set the computer on fire
Soundness and Completeness

A *type system* is a judgment for classifying programs
- “accepts” a program if some complete derivation gives it a type, else “rejects”

A *sound* type system never accepts a program that can get stuck
- No false negatives

A *complete* type system never rejects a program that can’t get stuck
- No false positives

It is typically *undecidable* whether a stuck state can be reachable
- Corollary: If we want an *algorithm* for deciding if a type system accepts a program, then the type system cannot be sound and complete
- We’ll choose soundness, try to reduce false positives in practice
Wrong Attempt

\[ \tau ::= \text{int} | \text{fn} \]

\[ \vdash e : \tau \]

\[ \vdash \lambda x. e : \text{fn} \quad \vdash c : \text{int} \quad \vdash e_1 : \text{fn} \quad \vdash e_2 : \text{int} \]

\[ \vdash e_1 e_2 : \text{int} \]
Wrong Attempt

\[ \tau ::= \ \text{int} \ | \ \text{fn} \]

\[ \vdash e : \tau \]

\[ \vdash \lambda x. e : \text{fn} \]

\[ \vdash c : \text{int} \]

\[ \vdash e_1 : \text{fn} \]

\[ \vdash e_2 : \text{int} \]

\[ \vdash e_1 \ e_2 : \text{int} \]

1. NO: can get stuck, e.g., \((\lambda x. \ y)\ 3\)

2. NO: too restrictive, e.g., \((\lambda x. \ x \ 3) \ (\lambda y. \ y)\)

3. NO: types not preserved, e.g., \((\lambda x. \ \lambda y. \ y)\ 3\)
Getting it right

1. Need to type-check function bodies, which have free variables
2. Need to classify functions using argument and result types

For (1): $\Gamma ::= \cdot \mid \Gamma, x : \tau$ and $\Gamma \vdash e : \tau$

- Require whole program to type-check under empty context $\cdot$

For (2): $\tau ::= \text{int} \mid \tau \rightarrow \tau$

- An infinite number of types:
  \begin{align*}
  \text{int} \rightarrow \text{int}, (\text{int} \rightarrow \text{int}) \rightarrow \text{int}, \text{int} \rightarrow (\text{int} \rightarrow \text{int}), \ldots
  \end{align*}

Concrete syntax note: $\rightarrow$ is right-associative, so $\tau_1 \rightarrow \tau_2 \rightarrow \tau_3$ is $\tau_1 \rightarrow (\tau_2 \rightarrow \tau_3)$
STLC Type System

\[ \tau ::= \text{int} \mid \tau \to \tau \]
\[ \Gamma ::= \cdot \mid \Gamma, x:\tau \]

\[ \Gamma \vdash e : \tau \]

\[ \begin{align*}
\Gamma & \vdash c : \text{int} \\
\Gamma, x : \tau_1 & \vdash e : \tau_2 \\
\Gamma & \vdash \lambda x. \; e : \tau_1 \to \tau_2 \\
\Gamma & \vdash e_1 : \tau_2 \to \tau_1 \\
\Gamma & \vdash e_2 : \tau_2 \\
\Gamma & \vdash e_1 \; e_2 : \tau_1
\end{align*} \]

The *function-introduction* rule is the interesting one...
A closer look

\[\Gamma, x : \tau_1 \vdash e : \tau_2\]
\[\frac{}{\Gamma \vdash \lambda x. e : \tau_1 \to \tau_2}\]

Where did \(\tau_1\) come from?

- Our rule “inferred” or “guessed” it
- To be syntax directed, change \(\lambda x. e\) to \(\lambda x : \tau. e\) and use that \(\tau\)

Can think of “adding \(x\)” as shadowing or requiring \(x \not\in \text{Dom}(\Gamma)\)

- Systematic renaming (\(\alpha\)-conversion) ensures \(x \not\in \text{Dom}(\Gamma)\) is not a problem
A closer look

\[ \frac{\Gamma, x : \tau_1 \vdash e : \tau_2}{\Gamma \vdash \lambda x. \ e : \tau_1 \rightarrow \tau_2} \]

Is our type system too restrictive?

- That’s a matter of opinion
- But it does reject programs that don’t get stuck

Example: \((\lambda x. \ (x \ (\lambda y. \ y)) \ (x \ 3)) \ \lambda z. \ z\)

- Does not get stuck: Evaluates to 3
- Does not type-check:
  - There is no \(\tau_1, \tau_2\) such that \(x : \tau_1 \vdash (x \ (\lambda y. \ y)) \ (x \ 3) : \tau_2\) because you have to pick one type for \(x\)
Always restrictive

Whether or not a program “gets stuck” is undecidable:
- If \( e \) has no constants or free variables, then \( e(3\ 4) \) or \( e\ x \) gets stuck if and only if \( e \) terminates (cf. the halting problem)

Old conclusion: “Strong types for weak minds”
- Need a back door (unchecked casts)

Modern conclusion: Unsafe constructs almost never worth the risk
- Make “false positives” (rejecting safe program) rare enough
  - Have compile-time resources for “fancy” type systems
- Make workarounds for false positives convenient enough
How does STLC measure up?

So far, STLC is sound:
- As language dictators, we decided \( e \) \( v \) and undefined variables were “bad” meaning neither values nor reducible.
- Our type system is a conservative checker that an expression will never get stuck.

But STLC is far too restrictive:
- In practice, just too often that it prevents safe and natural code reuse.
- More fundamentally, it’s not even Turing-complete:
  - Turns out all (well-typed) programs terminate.
  - A good-to-know and useful property, but inappropriate for a general-purpose PL.
  - That’s okay: We will add more constructs and typing rules.
Type Soundness

We will take a *syntactic* (operational) approach to soundness/safety

- The popular way since the early 1990s

Theorem (Type Safety): If $\vdash e : \tau$ then $e$ diverges or $e \rightarrow^n v$

for an $n$ and $v$ such that $\vdash v : \tau$

- That is, if $\vdash e : \tau$, then $e$ cannot get stuck

Proof: Next lecture