Higher eigenvalues of graphs

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Abstract— We present a general method for proving upper bounds on the eigenvalues of the graph Laplacian. In particular, we show that for any positive integer k, the k^{th} smallest eigenvalue of the Laplacian on a bounded-degree planar graph is $O(k/n)$. This bound is asymptotically tight for every k, as it is easily seen to be achieved for planar grids. We also extend this spectral result to graphs with bounded genus, graphs which forbid fixed minors, and other natural families. Previously, such spectral upper bounds were only known for $k = 2$, i.e. for the Fiedler value of these graphs. In addition, our result yields a new, combinatorial proof of the celebrated result of Korevaar in differential geometry.

1. Introduction

In combinatorial optimization, spectral methods are a class of techniques that use the eigenvectors of matrices associated with the underlying graphs. These matrices include the adjacency matrix, the Laplacian, and the random-walk matrix of a graph. One of the earliest applications of spectral methods is to graph partitioning, pioneered by Hall [23] and Donath and Hoffman [15], [16] in the early 1970s. Donath and Hoffman proposed using eigenvectors of the adjacency matrix. The use of the graph Laplacian for partitioning was introduced by Fiedler [18], [19], [20], who showed a connection between the second-smallest eigenvalue of the Laplacian of a graph and its connectivity. Since their inception, spectral methods have been used for solving a wide range of optimization problems, from graph coloring [7], [4] to image segmentation [35], [41] to web search [27], [10].

Analysis of the Fiedler value. In parallel with the practical development of spectral methods, progress on the mathematical front has has made connections between various graph properties and corresponding graph spectra. In 1970, independent from the work of Hall and of Donath and Hoffman, Cheeger [13] proved that the isoperimetric number of a continuous manifold can be bounded from above by the square root of the smallest eigenvalue of its Laplacian. Cheeger's inequality was then extended to graphs by Alon [2], Alon and Milman [3], and Sinclair and Jerrum [36]. They showed that if the Fiedler value of a graph — the second smallest eigenvalue of the Laplacian of the graph — is small, then partitioning the graph according to the values of the vertices in the associated eigenvector will produce a cut where the ratio of cut edges to the number of vertices in the cut is similarly small.

To explain the practical success of spectral partitioning algorithms in scientific computing and VLSI design, Spielman and Teng [38] proved a spectral theorem for planar graphs, which asserts that the Fiedler value of every bounded-degree planar graph with n vertices is $O(1/n)$. They also showed that the Fiedler value of a finite-element mesh in d dimensions with n vertices is $O(n^{-2/d})$. Kelner [25] then proved that the Fiedler value of a bounded-degree graph with n vertices and genus g is $O((g+1)/n)$. The proofs in [38], [25] critically use the inherent geometric structure of the planar graphs, meshes, and graphs with bounded genus. Recently, Biswal, Lee, and Rao [8] developed a new combinatorial approach for studying the Fiedler value; they resolved most of the open problems in [38]. In particular, they proved that the Fiedler value of a bounded-degree graph of n vertices without a K_h minor is $O((h^6 \log h)/n)$. These spectral theorems together with Cheeger's inequality on the Fiedler value immediately imply that one can use the spectral method to produce a partition as good as the best known partitioning methods for planar graphs [32], geometric graphs [33], graph with bounded genus [21], and graphs free of small complete minors [5].

Higher eigenvalues and our contribution. Although previous theoretical work most focuses on $k = 2$ (the Fiedler value of a graph), higher eigenvalues and eigenvectors are used in many practical algorithms [6], [11], [12], [41]. For example, Alpert and Yao [6] report that in the context of VLSI applications, the use of more eigenvectors produces better partitions; Spielman and Teng [37] showed that the embedding of a planar graph with the eigenvectors associated with the second and the third smallest eigenvalues of the graph Laplacian usually gives a nearly planar drawing; and Tolliver [40] shows experimentally that higher eigenvectors provide better multiway image segmentations.

In this paper, we prove the following theorem on higher

*Research partially supported by NSF grant CCF-0843915.
†Research partially supported by NSF grant CCF-0644037. Part of this work was completed during a visit of the author to the Massachusetts Institute of Technology.
‡Research partially supported by NSF grant CCF-0843915, an NSF Graduate Research Fellowship, and an Akamai Fellowship.
§Research partially supported by NSF grant CCF-0635102.

graph spectra, which concludes a long line of work on upper bounds for the eigenvalues of graphs.

Theorem 1.1 (Graph spectra)**.** *Let G be a constant-degree n-vertex graph either with a constant genus, or with a constant-sized forbidden minor. Then the k^{th} smallest eigenvalue of the Laplacian on G is $O(k/n)$.*

Our spectral theorem provides a theoretical justification of the experimental observation that when k is small, the k^{th} eigenvalues of the graphs arising in scientific computing, image processing, and VLSI design are usually small as well. We hope our result will lead to new progress in the analysis of practical spectral methods, and the design of new spectral algorithms.

For special graphs such as line graphs, grid graphs and complete binary trees, the higher spectra are known. The question of higher eigenvalues has also been resolved in the 2-dimensional Riemannian setting. Korevaar [28] answered the question of Yau by extending results of Hersch [24] and Yang and Yau [42] which provide a tight upper bound on the smallest eigenvalue of the Laplace operator of a Riemannian manifold: He proved that the k^{th} smallest eigenvalue of the Laplace operator of a genus g surface M is $O\left(\frac{k(g+1)}{\text{vol}(M)}\right)$.

However, the spectra of graphs may be more subtle than the spectra of surfaces For graphs with large diameter, the analysis of graph spectra resembles the analysis for surfaces. For example, Chung [14] gave an upper bound of $O(1/D^2)$ on the the Fiedler value, where D is the diameter of the graph. Grigor'yan and Yau [22] extended Korevaar's analysis to bounded genus graphs that have strong volume measure — these graphs have diameter $\Omega(\sqrt{n})$.

Bounded-degree planar graphs (and bounded genus graphs), however, may have diameter as small as $O(\log n)$, making it impossible to directly apply these diameter-based spectral analyses. For planar graphs and graphs of genus g, Spielman and Teng [38] and Kelner [25] showed that without any assumption on graph diameters, planarity and a genus bound are sufficient to guarantee that the Fiedler value is small. Their spectral theorems can be viewed as a discrete analogue of the results of Hersch and and Yang and Yau for compact 2-surfaces.

Our spectral theorem not only provides a discrete analog for Korevaar's theorem on higher eigenvalues, but also extends the higher-eigenvalue bounds to graphs with a bounded forbidden minor, a family that is more combinatorially defined. Because the Laplacian of a manifold can be approximated by that of a mesh graph, our result also provides a new, combinatorial proof of Korevaar's theorem. In fact—although we defer details to the full version—our analysis has a natural analog in the manifold setting as well.

Our work builds on the method of Biswal, Lee, and Rao [8], which uses multi-commodity flows to define a deformation of the graph geometry. This deformation is derived from certain kinds of optimal flows in the graph. To capture higher eigenvalues, we study a new flow problem, which we define in Section 1.2 and call *subset flows*; this notion may be independently interesting. As we discuss in the next section, these flows arise as dual objects of certain kinds of optimal spreading metrics on the graph. We use techniques from the theory of discrete metric spaces to build test vectors from spreading metrics, and we develop new combinatorial methods to understand the structure of optimal subset flows.

1.1. Outline of our approach

For the sake of clarity, we restrict ourselves for now to a discussion of bounding $\lambda_k(G)$ when $G = (V, E)$ is a bounded-degree planar graph with $n = |V|$. Towards this end, we first review the known methods for bounding $\lambda_2 = \lambda_2(G)$.

Bounding λ_2. By the variational characterization of eigenvalues, giving an upper bound on λ_2 requires finding a certain kind of embedding of G into the real line (see Section 1.2). Spielman and Teng [38] obtain an initial geometric representation using the Koebe-Andreev-Thurston circle packing theorem for planar graphs. Because of the need for finding a test vector which is orthogonal to the first eigenvector, one has to post-process this representation before it will yield a bound on λ_2. They use a topological argument to show the existence of an appropriate Möbius transformation which achieves this. (A similar step was used 40 years ago by Hersch [24] in the manifold setting.) Even in the arguably simpler setting of manifolds, no similar method is known for bounding λ_3, due to the lack of a rich enough family of circle-preserving transformations.

Our approach initiates with the arguments of Biswal, Lee, and Rao [8]. Instead of finding an external geometric representation, those authors begin by finding an appropriate *intrinsic* deformation of the graph, expressed via a non-negative vertex-weighting $\omega : V \to [0, \infty)$, which induces a corresponding shortest-path metric on G,

$$\mathsf{dist}_\omega(u, v) = \text{length of shortest } u\text{-}v \text{ path},$$

where the length of a path P is given by $\sum_{v \in P} \omega(v)$. The proper deformation ω is found via variational methods (it is the optimum of some convex program), and the heart of the analysis involves studying the geometry of the optimal solutions, via their dual formulation in terms of certain kinds of multi-commodity flows. Finally, techniques from the theory of metric embeddings are used to embed the resulting metric space $(V, \mathsf{dist}_\omega)$ into the real line, thus recovering an appropriate test vector to bound λ_2.

Controlling λ_k for $k \geq 3$. In order to bound higher eigenvalues, we need to produce a system of many disjoint test vectors. The first problem one encounters is that the optimal deformation ω might not contain enough information to produce more than a single vector if the geometry of the ω-deformed graph is degenerate, e.g. if $V = C \cup C'$ for two large clusters C, C' where C and C' are far apart, but each has small diameter.

Spreading metrics and padded partitions. To combat this, we would like to impose the constraint that no large set collapses in the metric dist_ω, i.e. that for any subset $C \subseteq V$ with $|C| \geq n/k$, the diameter of C is large. In order to produce such an ω by variational techniques, we have to specify this constraint (or one like it) in a convex way. We do this using the well-known *spreading metric* constraints (see, e.g. [17]) on all sets of size $\approx n/k$.

Given such a spreading weight ω, we show in Section 2 how to obtain a bound on λ_k by producing k smooth, disjoint bump functions on $(V, \mathsf{dist}_\omega)$, which then act as our k test vectors. The bump functions are produced using padded metric partitions (see, e.g. [30]), which are known to exist for all planar graphs from the seminal work of Klein, Plotkin, and Rao [26].

The spreading deformation, duality, and subset flows. At this point, to upper bound λ_k, it suffices to find a spreading weight ω with $\sum_{v \in V} \omega(v)^2$ small enough. In order to do this, in Section 2.3, we write a convex program to compute the best weight ω. The dual program involves a new kind of multi-commodity flow problem, which we now describe.

Consider a probability distribution μ on subsets $S \subseteq V$. For a flow F in G (see Section 1.2 for a review of multi-commodity flows), we write $F[u, v]$ for the total amount of flow sent from u to v, for any $u, v \in V$. In this case a *feasible μ-flow* is one which satisfies, for every $u, v \in V$,

$$F[u, v] \geq \Pr_\mu[u, v \in S],$$

where S is chosen according to the distribution μ. In other words, every set S places a demand of $\mu(S)$ between every pair $u, v \in S$. For instance, the classical all-pairs multi-commodity flow problem would be specified by choosing μ which concentrates all its weight on the entire vertex set V.

Given such a μ, the corresponding "subset flow" problem is to find a feasible μ-flow F so that the total ℓ_2-norm of the congestion of F at vertices is minimized (see Section 2.3 for a formal definition of the ℓ_2-congestion). Finally, bounding λ_k requires us to prove *lower bounds* on the congestion of *every possible* μ-flow with μ concentrated on sets of size $\approx n/k$.

An analysis of optimal subset flows: New crossing number inequalities. In the case of planar graphs G, we use a randomized rounding argument to relate the existence of a feasible μ-flow in G with small ℓ_2-congestion to the ability to draw certain kinds of graphs in the plane without too many edge crossings. This was done in [8], where the relevant combinatorial problem involved the number of edge crossings necessary to draw dense graphs in the plane, a question which was settled by Leighton [31], and Ajtai, Chvátal, Newborn, and Szemerédi [1].

In the present work, we have to develop new crossing *weight* inequalities for a "subset drawing" problem. Let $H = (U, F)$ be a graph with non-negative edge weights $W : F \to [0, \infty)$. Given a drawing of H in the plane, we define the crossing weight of the drawing as the total weight of all edge crossings, where two edges $e, e' \in F$ incur weight $W(e) \cdot W(e')$ when they cross. Write $\mathsf{cr}(H; W)$ for the minimal crossing weight needed to draw H in the plane. In Section 4, we prove the following theorem, which forms the technical core of our eigenvalue bound.

Theorem 1.2 (Subset crossing theorem)**.** *There exists a constant $C \geq 1$ such that if μ is any probability distribution on subsets of $[n]$ with $\mathbb{E}_{S\sim\mu}|S|^2 \geq C$, then the following holds. For $u, v \in [n]$, defining $W(u, v) = \Pr_{S\sim\mu}[u, v \in S]$, we have*

$$\mathsf{cr}([n]; W) \gtrsim \frac{1}{n} \left(\mathbb{E}_{S\sim\mu}|S|^2\right)^{5/2}.$$

Observe that the theorem is asymptotically tight for all values of $\mathbb{E}|S|^2$. Classical crossing bounds [1], [31] show that drawing an r-clique in the plane requires $\Omega(r^4)$ edge crossings. Thus if we take μ to be uniform on k disjoint subsets of size n/k, then the crossing weight is $\approx k \cdot (1/k)^2 \cdot (n/k)^4 = n^4/k^5$, which matches the lower bound $\frac{1}{n}(\mathbb{E}|S|^2)^{5/2} = \frac{1}{n}(n/k)^5$.

The difficulty of proving Theorem 1.2 lies in controlling the extent to which μ is a mixture of three different types of "extremal" distributions: (1) μ is uniformly distributed on all sets of size r, (2) μ is concentrated on a single set of size r, and (3) μ is uniform over n/r disjoint sets, each of size r. In the actual proof, we deal with the corresponding cases: (1') μ is uniformly spread over edges, i.e. $\Pr_{S\sim\mu}[u, v \in S]$ is somewhat uniform over choices of $u, v \in V$. In this case, we have to take a global approach, showing that not only are there many intra-set crossings, but also a lot of crossing weight is induced by crossing edges coming from different sets. (2') $\Pr_{S\sim\mu}[u \in S]$ is unusually large for all $u \in V'$ with $|V'| \ll |V|$. In this case, there is a "density increment" on the induced subgraph $G[V']$, and we can apply induction. Finally, if we are in neither of the cases (1') or (2'), we are left to show that, in some sense, the distribution μ must be similar to case (3) above, in which case we can appeal to the classical dense crossing bounds applied to the complete graph on $S \cap S'$ where $S, S' \sim \mu$ are chosen i.i.d.

More general families. Clearly the preceding discussion was specialized to planar graphs. To handle general excluded-minor families, we can no longer deal with the notion of drawings, and we have to work directly with multi-commodity flows in graphs. To do this, we use the corresponding "flow crossing" theory developed in [8].

1.2. Preliminaries

1.2.1. Laplacian spectrum: Let $G = (V, E)$ be a finite, undirected graph. We consider the linear space $\mathbb{R}^V = \{f : V \to \mathbb{R}\}$ and define the Laplacian $\mathcal{L} : \mathbb{R}^V \to \mathbb{R}^V$ as the symmetric positive definite linear operator given by

$$(\mathcal{L}f)(v) = \sum_{u\sim v} f(v) - f(u).$$

which in matrix form could be written as $\mathcal{L} = D - A$ where A is the adjacency matrix of G and D the diagonal matrix whose entries are the vertex degrees. We wish to give upper bounds on the k^{th} eigenvalue of $\mathcal{L}$ for each k. To do this we consider the seminorm given by $\|f\|_{\mathcal{L}} = f \cdot \mathcal{L}f = \sum_{u\sim v}(f(u) - f(v))^2$ and restrict it to k-dimensional subspaces $U \subset \mathbb{R}^V$. By the spectral theorem, the maximum norm ratio $\|f\|_{\mathcal{L}}/\|f\|$ over U is minimized when U is spanned by the k eigenvectors of least eigenvalue, in which case its value is λ_k. Therefore if we exhibit a k-dimensional subspace U in which $\|f\|_{\mathcal{L}} \leq c$ for all unit vectors f, it follows that $\lambda_k \leq c$.

In particular, to describe such a subspace U it suffices to produce k test vectors $f_1, \ldots, f_k$ such that both $f_i \cdot f_j$ and $f_i \cdot \mathcal{L}f_j$ are zero for all $i \neq j$, and the norm ratios

$$\frac{\|f_i\|_{\mathcal{L}}}{\|f_i\|} = \frac{\sum_{u\sim v}(f(u) - f(v))^2}{\sum_u f(u)^2}$$

are all bounded by c. Both orthogonality conditions will hold if the f_i are disjoint both in their support and in the set of edges incident to their support.

1.2.2. Flows: Let $G = (V, E)$ be a finite, undirected graph, and for every pair $u, v \in V$, let $\mathcal{P}_{uv}$ be the set of all paths between u and v in G. Let $\mathcal{P} = \bigcup_{u,v\in V} \mathcal{P}_{uv}$.

Then a *flow in G* is a mapping $F : \mathcal{P} \to [0, \infty)$. For any $u, v \in V$, let $F[u, v] = \sum_{p\in\mathcal{P}_{uv}} F(p)$ be the amount of flow sent between u and v. We speak of the symmetric function $F[\cdot, \cdot] : V \times V \to [0, \infty)$ as the *demand function* of the flow F.

Our main technical theorem concerns a class of flows we call *subset flows*. Let μ be a probability distribution on subsets of V. Then F is a *μ-flow* if its demand has the form $F[u, v] = \Pr_{S\sim\mu}[u, v \in S]$. For $r \leq |V|$, we write $\mathcal{F}_r(G)$ for the set of all μ-flows in G with $\operatorname{supp}(\mu) \subseteq \binom{V}{r}$.

We say a flow F is an *integral flow* if it is supported on only one path p in each $\mathcal{P}_{uv}$, and a *unit flow* if every demand $F[u, v]$ is either 0 or 1. For an edge-weighted graph $G' = (V', E')$, we call F a *G'-flow* if its demand function is induced by the weights on G' under some vertex embedding $V' \to V$, and in this case we call G' a *demand graph* and G the *host graph*.

We define the *squared ℓ_2-congestion*, or simply *congestion*, of a flow F by $\mathsf{con}(F) = \sum_{v\in V} C_F(v)^2$, where $C_F(v) = \sum_{p\in\mathcal{P}:v\in p} F(p)$. This congestion can also be written as

$$\mathsf{con}(F) = \sum_{p,p'\in\mathcal{P}} \sum_{v\in p\cap p'} F(p)F(p')$$

and is therefore bounded below by a more restricted sum, the *intersection number*:

$$\mathsf{inter}(F) = \sum_{\substack{u,v,u',v' \\ |\{u,v,u',v'\}|=4}} \sum_{\substack{p\in\mathcal{P}_{uv} \\ p'\in\mathcal{P}_{u'v'}}} \sum_{x\in p\cap p'} F(p)F(p')$$

2. Eigenvalues and spreading weights

2.1. Padded partitions

Let (X, d) be a finite metric space. We will view a partition P of X as a collection of subsets, and also as a function $P : X \to 2^X$ mapping a point to the subset that contains it. We write $\beta(P, \Delta)$ for the infimal value of $\beta \geq 1$ such that

$$\left|\left\{x \in X : B(x, \Delta/\beta) \subseteq P(x)\right\}\right| \geq \frac{|X|}{2}.$$

Let $\mathcal{P}_\Delta$ be the set of all partitions P such that for every $S \in P$, $\operatorname{diam}(S) \leq \Delta$. Finally, we define

$$\beta_\Delta(X, d) = \inf\left\{\beta(P, \Delta) : P \in \mathcal{P}_\Delta\right\}.$$

The following theorem is a consequence [34] of the main theorem of Klein, Plotkin, and Rao [26].

Theorem 2.1. *Let $G = (V, E)$ be a graph without a $K_{r,r}$ minor and (V, d) be any shortest-path semimetric on G, and let $\Delta > 0$. Then $\beta_\Delta(V, d) = O(r^2)$.*

In particular, if G is planar then $\beta_\Delta(V, d)$ is bounded by an absolute constant, and if G is of genus $g > 0$ then $\beta_\Delta(V, d) = O(g)$.

2.2. Spreading vertex weights

Consider a non-negative weight function $\omega : V \to \mathbb{R}_+$ on vertices, and extend ω to subsets $S \subseteq V$ via $\omega(S) = \sum_{v\in V} \omega(v)$. We associate a vertex-weighted shortest-path metric by defining

$$\mathsf{dist}_\omega(u, v) = \min_{p\in\mathcal{P}_{uv}} \omega(p).$$

Say that ω is (r, ε)-spreading if, for every $S \subseteq V$ with $|S| = r$, we have

$$\frac{1}{r^2} \sum_{u,v\in S} \mathsf{dist}_\omega(u, v) \geq \varepsilon \sqrt{\sum_{v\in V} \omega(v)^2}.$$

Write $\varepsilon_r(G, \omega)$ for the maximal value of ε for which ω is (r, ε)-spreading.

Theorem 2.2 (Higher eigenvalues)**.** *Let $G = (V, E)$ be any n-vertex graph with maximum degree $d_{\max}$, and let λ_k be* *the kth Laplacian eigenvalue of G. Then for any weight function $\omega : V \to \mathbb{R}_+$ with*

$$\sum_{v \in V} \omega(v)^2 = 1,$$

we have

$$\lambda_k \leq O\left(d_{\max} \frac{1}{\varepsilon^2 n} \left(\beta_{\varepsilon/2}(V, \mathsf{dist}_\omega)\right)^2\right),$$

where $\varepsilon = \varepsilon_{\lfloor n/4k \rfloor}(G, \omega)$.

Proof: Let ω be an $(\lfloor n/4k \rfloor, \varepsilon)$-spreading weight function. Let $V = C_1 \cup C_2 \cup \cdots \cup C_m$ be a partition of V into sets of diameter at most $\varepsilon/2$, and define for every $i \in [m]$,

$$\hat{C}_i = \left\{ x \in C_i : B(x, \varepsilon/(2\beta)) \subseteq C_i \right\},$$

where $\beta = \beta_{\varepsilon/2}(V, \mathsf{dist}_\omega)$. By the definition of β, there exists a choice of $\{C_i\}$ with

$$|\hat{C}_1 \cup \hat{C}_2 \cup \cdots \cup \hat{C}_m| \geq n/2.$$

Now, since $\mathrm{diam}(C_i) \leq \varepsilon/2$, we see that

$$\frac{1}{|C_i|^2} \sum_{u,v \in C_i} \mathsf{dist}_\omega(u, v) \leq \frac{\varepsilon}{2} = \frac{\varepsilon}{2} \sqrt{\sum_{v \in V} \omega(v)^2},$$

so by the $(\lfloor n/4k \rfloor, \varepsilon)$-spreading property of ω, we know that $|C_i| \leq n/4k$ for every $i \in [m]$.

Thus by taking disjoint unions of the sets $\{\hat{C}_i\}$ which are each of size at most $n/4k$, we can find sets $S_1, S_2, \ldots, S_{2k}$ with $|S_i| \geq n/4k$ and $\mathsf{dist}_\omega(S_i, S_j) \geq \varepsilon/(2\beta)$ for every $i \neq j$.

Letting $\tilde{S}_i$ be the $\varepsilon/(4\beta)$-neighborhood of S_i, we see that the sets $\{\tilde{S}_i\}$ are pairwise disjoint. Now define, for every $i \in [2k]$,

$$W(\tilde{S}_i) = \sum_{u \in \tilde{S}_i} \sum_{v : uv \in E} [\omega(u) + \omega(v)]^2$$

Clearly, we have

$$\sum_{i=1}^{2k} W(\tilde{S}_i) \leq 2 \sum_{uv \in E} [\omega(u) + \omega(v)]^2 \leq 4 d_{\max} \sum_{v \in V} \omega(v)^2 = 4 d_{\max}.$$

Hence by averaging, there exists a subcollection, say $\left\{\tilde{S}_1, \tilde{S}_2, \ldots, \tilde{S}_k\right\}$, with $W(\tilde{S}_i) \leq \frac{4d_{\max}}{k}$ for $i = 1, 2, \ldots, k$.

Finally, we define functions $f_1, f_2, \ldots, f_k : V \to \mathbb{R}$ by

$$f_i(x) = \max\left\{0, \frac{\varepsilon}{4\beta} - \mathsf{dist}_\omega(x, S_i)\right\}$$

so that f_i is supported on $\tilde{S}_i$.

Since each f_i is 1-Lipschitz and has $\mathrm{supp}(f_i) \subseteq \tilde{S}_i$, we have

$$\begin{aligned}
\sum_{uv \in E} |f_i(u) - f_i(v)|^2 &= \sum_{u \in \tilde{S}_i} \sum_{v : uv \in E} |f_i(u) - f_i(v)|^2 \\
&\leq \sum_{u \in \tilde{S}_i} \sum_{v : uv \in E} \mathsf{dist}_\omega(u, v)^2 \\
&= \sum_{u \in \tilde{S}_i} \sum_{v : uv \in E} [\omega(u) + \omega(v)]^2 \\
&= W(\tilde{S}_i) \leq \frac{4 d_{\max}}{k}.
\end{aligned}$$

Furthermore the functions have disjoint support and satisfy

$$\sum_{u \in V} f_i(u)^2 \geq \left(\frac{\varepsilon}{4\beta}\right)^2 |S_i| \gtrsim \left(\frac{\varepsilon^2}{\beta^2}\right) \frac{n}{k}.$$

Combining the preceding two estimates shows that for each f_i,

$$\frac{\sum_{uv \in E} |f_i(u) - f_i(v)|^2}{\sum_{u \in V} f_i(u)^2} \lesssim \frac{d_{\max}}{n} \left(\frac{\beta}{\varepsilon}\right)^2,$$

and the proof is complete by the discussion in Section 1.2.1. ■

The following results follow from Theorem 2.2, Theorem 2.1, Theorem 2.4, Lemma 3.2, Theorem 4.1, and Corollary 3.5.

Corollary 2.3. *If G is planar, then*

$$\lambda_k \leq O\left(d_{\max} \frac{k}{n}\right).$$

If G is of genus $g > 0$, then

$$\lambda_k \leq O\left(d_{\max} g^3 \frac{k}{n}\right).$$

If G is K_h-minor-free, then

$$\lambda_k \leq O\left(d_{\max} h^6 \log h \frac{k}{n}\right).$$

2.3. *Spreading weights and subset flows*

We now show a duality between the optimization problem of finding a spreading weight ω and the problem of minimizing congestion in subset flows. The following theorem is proved by a standard Lagrange-multipliers argument.

Theorem 2.4 (Duality). *Let $G = (V, E)$ be a graph and let $r \leq |V|$. Then*

$$\max\left\{\varepsilon_r(G, \omega) \middle| \omega : V \to \mathbb{R}_+\right\} = \frac{1}{r^2} \min\left\{\sqrt{\mathsf{con}(F)} \middle| F \in \mathcal{F}_r(G)\right\}.$$

Proof: We shall write out the optimizations $\max_\omega \varepsilon_r(G, \omega)$ and $\frac{1}{r^2} \min_F \sqrt{\mathsf{con}(F)}$ as convex programs, and show that they are dual to each other. The equality then follows from Slater's condition [9, Ch. 5]:

Fact 2.5 (Slater's condition for strong duality)**.** *When the feasible region for a convex program* $(\mathbf{P})$ *has non-empty interior, the values of* $(\mathbf{P})$ *and its dual* $(\mathbf{P}^*)$ *are equal.*

We begin by expanding $\max_\omega \varepsilon_r(G,\omega)$ as a convex program $(\mathbf{P})$. Let $P \in \{0,1\}^{\mathcal{P}\times V}$ be the path incidence matrix; $Q \in \{0,1\}^{\mathcal{P}\times\binom{V}{2}}$ the path connection matrix; and $R \in \{0,1\}^{\binom{V}{r}\times\binom{V}{2}}$ a normalized set containment matrix, respectively defined as

$$P_{p,v} = \begin{cases} 1 & v \in p \\ 0 & \text{else} \end{cases}$$

$$Q_{p,uv} = \begin{cases} 1 & p \in \mathcal{P}_{uv} \\ 0 & \text{else} \end{cases}$$

$$R_{S,uv} = \begin{cases} 1/r^2 & \{u,v\} \subset S \\ 0 & \text{else.} \end{cases}$$

Then the convex program $(\mathbf{P}) = \max_\omega \varepsilon_r(G,\omega)$ is

$$\begin{array}{ll} \text{minimize} & -\varepsilon \\ \text{subject to} & \varepsilon\mathbf{1} \preceq Rd \quad Qd \preceq Ps \quad s^\top s \le 1 \\ & d \succeq 0 \qquad s \succeq 0 \end{array}.$$

Introducing Lagrange multipliers λ, μ, ν, the Lagrangian function is

$$L(d,s,\lambda,\mu,\nu) = -\varepsilon + \lambda^\top(\varepsilon\mathbf{1} - Rd) + \mu^\top(Qd - Ps) + \nu(s^\top s - 1)$$

so that $(\mathbf{P})$ and its dual $(\mathbf{P}^*)$ may be written as

$$(\mathbf{P}) = \inf_{\varepsilon,d,s}\sup_{\lambda,\mu,\nu} L(d,s,\lambda,\mu,\nu)$$

$$(\mathbf{P}^*) = \sup_{\lambda,\mu,\nu}\inf_{\varepsilon,d,s} L(d,s,\lambda,\mu,\nu).$$

Now we simplify $(\mathbf{P}^*)$. Rearranging terms in L, we have

$$\begin{aligned}(\mathbf{P}^*) &= \sup_{\lambda,\mu,\nu}\inf_{\varepsilon,d,s} (\lambda^\top\mathbf{1} - 1)\varepsilon + (\mu^\top Q - \lambda^\top R)d \\ &\quad + (\nu s^\top s - \mu^\top P s) - \nu \\ &= \sup_{\lambda,\mu,\nu}\inf_{\varepsilon}(\lambda^\top\mathbf{1} - 1)\varepsilon \\ &\quad + \inf_d(\mu^\top Q - \lambda^\top R)d + \inf_s(\nu s^\top s - \mu^\top P s) - \nu.\end{aligned}$$

Now the infima $\inf_\varepsilon(\lambda^\top\mathbf{1} - 1)\varepsilon$ and $\inf_d(\mu^\top Q - \lambda^\top R)d$ are either 0 or $-\infty$, so at optimum they must be zero and $\lambda^\top\mathbf{1} - 1 \ge 0$, $\mu^\top Q - \lambda^\top R \succeq 0$. With these two constraints, the optimization reduces to $\sup_{\mu,\nu}\inf_s(\nu s^\top s - \mu^\top P s) - \nu$. At optimum the gradient of the infimand is zero, so $s = \frac{P^\top\mu}{2\nu}$ and the infimum is $-\frac{\|P^\top\mu\|_2^2}{4\nu}$. Then at maximum $\nu = \frac{1}{2}\|P^\top\mu\|_2$, so that the supremand is $-\|P^\top\mu\|_2$. We have shown that $(\mathbf{P}^*)$ is the convex program

$$\begin{array}{ll} \text{maximize} & -\left\|P^\top\mu\right\|_2 \\ \text{subject to} & \mu^\top Q \succeq \lambda^\top R \quad \lambda^\top\mathbf{1} \ge 1 \\ & \mu \succeq 0 \qquad \lambda \succeq 0 \end{array}.$$

This program is precisely (the negative of) the program to minimize vertex 2-congestion of a subset flow in $\mathcal{F}_r(G)$, where the subset weights are normalized to unit sum. The proof is complete. ■

3. Congestion measures

In this section, we develop concepts that will enable us to give lower bounds on the congestion $\mathsf{con}(F)$ of all subset flows F in a given graph G.

Definition 3.1. *Let G be a graph, the* host graph*. For a weighted graph H, called the* demand graph*, define the G-*congestion of *H:*

$$\mathsf{con}_G H = \min_{F \text{ an } H\text{-flow in } G} \mathsf{con}(F)$$

*and the G-*intersection number of *H:*

$$\mathsf{inter}_G(H) = \min_{F \text{ an integral } H\text{-flow in } G} \mathsf{inter}(F).$$

The next lemma is proved via randomized rounding.

Lemma 3.2 (Rounding)**.** *For any graph G and unit flow F, there is an integral unit flow F^* with the same demand function such that*

$$\mathsf{inter}(F^*) \le \mathsf{inter}(F).$$

Consequently for every G and H

$$\mathsf{inter}_G(H) \le \min_{F \text{ an } H\text{-flow in } G} \mathsf{inter}(F) \le \mathsf{con}_G(H). \qquad (1)$$

Proof: We produce an integral flow F^* randomly by rounding F. For each pair of endpoints u,v, choose independently a path p_{uv} in $\mathcal{P}_{uv}$ with $\Pr[p_{uv} = p] = F(p)$ for each p. Then

$$\begin{aligned}\mathbb{E}[\mathsf{inter}(F^*)] &= \sum_{\substack{u,v,u',v' \\ |\{u,v,u',v'\}|=4}} \mathbb{E}\big[\,|p_{uv} \cap p_{u'v'}|\,\big] \\ &= \sum_{\substack{u,v,u',v' \\ |\{u,v,u',v'\}|=4}} \sum_{\substack{p\in\mathcal{P}_{uv} \\ p'\in\mathcal{P}_{u'v'}}} \sum_{x\in p\cap p'} F(p)F(p') = \mathsf{inter}(F)\end{aligned}$$

so that with positive probability we must have $\mathsf{inter}(F^*) \le \mathsf{inter}(F)$. Equation 1 follows because $\mathsf{inter}(F) \le \mathsf{con}(F)$ always. ■

Definition 3.3. *A* congestion measure *is a nonnegative function $\mathcal{C} = \mathsf{inter}_G$ on weighted graphs, for some host graph G, that satisfies the following equation for some constants $c(\mathcal{C}), a(\mathcal{C})$ and all graphs $H = (E,V)$:*

$$\mathcal{C}(H) \ge \frac{|E|^3}{c(\mathcal{C})|V|^2} - a(\mathcal{C})|V| \qquad (2)$$

In particular, $\mathcal{C}(K_n) \ge n^4/8c(\mathcal{C}) - a(\mathcal{C})n$.

The term $a(\mathcal{C})|V|$ in (2) is present for technical reasons. Throughout this and the following section, terms containing $a(\mathcal{C})$ should be thought of as negligible and ignored on a first reading.

Lemma 3.4. *Suppose that for some G and $k = k(G)$, every H obeys*

$$\mathsf{inter}_G(H) \ge |E(H)| - k|V(H)| - k^2. \qquad (3)$$

Then it follows that for every H,

$$\mathsf{inter}_G(H) \geq \frac{1}{18}\frac{|E(H)|^3}{k^2|V(H)|^2} - k|V(H)| \tag{4}$$

so that inter_G is a congestion measure with $c(\mathsf{inter}_G) = 18k^2$ and $a(\mathsf{inter}_G) = k$.

Proof: It suffices to consider $|E(H)| \geq 3k|V(H)|$ since otherwise the right-hand side of inequality (4) is negative.

Fix any H-flow F in G. Sample the nodes of H independently with probability p each to produce a new demand graph H' and flow $F' = F|_{H'}$. Then $\mathsf{inter}(F') \geq \mathsf{inter}_G(H') \geq |E(H')| - k|V(H')| - k^2$, and by taking expectations we have

$$p^4 \mathsf{inter}(F) \geq p^2|E(H)| - pk|V(H)| - k^2.$$

Choosing $p = 3k|V(H)|/|E(H)|$ and using the fact that $|E(H)|/|V(H)|^2 < 1$ we obtain (4). ■

The proof of the following follows [8].

Corollary 3.5. *If G is planar, then inter_G is a congestion measure with $c(\mathsf{inter}_G) = 162$ and $a(\mathsf{inter}_G) = 3$. If G is of genus $g > 0$, then inter_G is a congestion measure with $c(\mathsf{inter}_G) = O(g)$ and $a(\mathsf{inter}_G) = O(\sqrt{g})$. If G is K_h-minor-free, then inter_G is a congestion measure with $c(\mathsf{inter}_G) = O(h^2 \log h)$ and $a(\mathsf{inter}_G) = O(h\sqrt{\log h})$.*

Proof: If F is an integral H-flow with $\mathsf{inter}(F) > 0$, then some path in F and corresponding edge of H can be removed to yield an integral H'-flow F' with $\mathsf{inter}(F') \leq \mathsf{inter}(F) - 1$. Therefore to prove (3) it suffices to consider H with $\mathsf{inter}_G(H) = 0$ and show that $|E(H)| \leq k|V(H)| + k$. Then Lemma 3.4 will imply inter_G is a congestion measure with $c(\mathsf{inter}_G) = 18k^2$ and $a(\mathsf{inter}_G) = k$.

When G is planar, an H-flow F in G with $\mathsf{inter}(F) = 0$ gives a drawing of H in the plane without crossings, so that H itself is planar. Then an elementary application of the Euler characteristic gives

$$|E(H)| \leq 3|V(H)| - 6 < 3|V(H)|.$$

When G is of genus at most $g > 0$, the same argument gives

$$|E(H)| \leq 3|V(H)| + 6(g-1)$$

which suffices for $k = O(\sqrt{g})$.

For K_h-minor-free G and H with $\mathsf{inter}_G(H) = 0$, if H is bipartite with minimum degree 2, then Lemma 3.2 from [8] implies that H is K_h-minor-free, so that $|E(H)| \leq c_{KT}|V(H)|h\sqrt{\log h}$ by the theorem of Kostochka [29] and Thomason [39]. For general H, by taking a random partition we can obtain a bipartite subgraph H' with $E(H') \geq E(H)/2$, so that $\mathsf{inter}_G(H) = 0$ implies

$$|E(H)| \leq 2c_{KT}h\sqrt{\log h}|V(H)|.$$

■

In the next section, we will also require the following lemma.

Lemma 3.6. *Let $\mathcal{C} = \mathsf{inter}_G$ be a congestion measure and μ a subset distribution on a vertex set V. Write H_μ for the graph on V with edge weights $H_\mu(u,v) = \Pr_{S\sim\mu}[u,v \in S]$. Then*

$$\mathcal{C}(H_\mu) \geq \frac{1}{8c(\mathcal{C})}\mathbb{E}_{S\sim\mu,S'\sim\mu}\left[|S\cap S'|^4\right] - a(\mathcal{C})\mathbb{E}_{S\sim\mu,S'\sim\mu}\left[|S\cap S'|\right].$$

Proof of Lemma 3.6: For any flow F let the vertices of H_μ be identified with the corresponding vertices of G. For each u,v with $H_\mu(u,v) > 0$, let F_{uv} denote the unique path $p \in \mathcal{P}_{uv}$ with $F > 0$. Then, taking F in subscripts over integral H_μ-flows in G,

$$\begin{aligned}
\mathcal{C}(H_\mu) &= \min_F \mathsf{inter}(F) \\
&= \min_F \sum_{\substack{u,v,u',v' \\ |\{u,v,u',v'\}|=4}} |\{F_{uv} \cap F_{u'v'}\}| \Pr_{S\sim\mu}[u,v\in S] \Pr_{S'\sim\mu}[u',v'\in S'] \\
&= \min_F \mathbb{E}_{S\sim\mu,S'\sim\mu}\left[\sum_{\substack{u,v\in S,u',v'\in S' \\ |\{u,v,u',v'\}|=4}} |\{F_{uv}\cap F_{u'v'}\}|\right] \\
&\geq \min_F \mathbb{E}_{S\sim\mu,S'\sim\mu}\left[\mathsf{inter}(F|_{S\cap S'})\right]
\end{aligned}$$

where the inequality comes by restricting the sum to terms with $u,v,u',v' \in S\cap S'$. Then since the minimum expectation is bounded below by the expected minimum, it follows that

$$\begin{aligned}
\mathcal{C}(H_\mu) &\geq \mathbb{E}_{S\sim\mu,S'\sim\mu}\left[\min_F \mathsf{inter}(F|_{S\cap S'})\right] \\
&= \mathbb{E}_{S\sim\mu,S'\sim\mu}\left[\mathcal{C}(K_{|S\cap S'|})\right]
\end{aligned}$$

and the conclusion follows. ■

4. Congestion for subset flows

Theorem 4.1. *There is a universal constant $c_0 > 0$ such that the following holds. Let μ be any probability distribution on subsets of $[n]$. For $u,v \in [n]$, define*

$$F(u,v) = \Pr_{S\sim\mu}[u,v\in S]$$

and let G_μ be the graph on $[n]$ weighted by F. Let $\mathcal{C}$ be a congestion measure. Then

$$\mathcal{C}(G_\mu) \gtrsim \frac{1}{c(\mathcal{C})n}\left(\mathbb{E}|S|^2\right)^{5/2} - c_0\frac{a(\mathcal{C})}{n}\mathbb{E}|S|^2$$

As in the previous section, the terms involving $a(\mathcal{C})$ should be thought of as negligible.

Corollary 4.2. *If μ is supported on $\binom{[n]}{r}$ for some r, then $\mathcal{C}(G_\mu) \gtrsim \frac{r^5}{c(\mathcal{C})n} - \frac{a(\mathcal{C})r^2}{n}$. In particular, if $r = \Omega(a(\mathcal{C}) \cdot c(\mathcal{C}))^{1/3}$, then*

$$\mathcal{C}(G_\mu) \gtrsim \frac{r^5}{c(\mathcal{C})n}.$$

Proof of Theorem 4.1: We will freely use the fact that

$$\mathbb{E}|S|^2 = \sum_{u,v} F(u,v).$$

Also, put $F(u) = \Pr_{S\sim\mu}[u \in S]$ for $u \in [n]$.

The proof will proceed by induction on n, and will be broken into three cases.

Case I: Light edges.

Let $E(\alpha,\beta) = \{(u,v) : \alpha \le F(u,v) \le \beta\}$.

Claim 4.3. *For every $\beta \in [0,1]$, we have*

$$\mathcal{C}(G_\mu) \gtrsim \frac{\left(\sum_{(u,v)\in E(0,\beta)} F(u,v)\right)^3}{\beta c(\mathcal{C}) n^2} - \beta^2 a(\mathcal{C}) n. \tag{5}$$

Proof: First, observe that by (2), the subgraph consisting of the edges in $E(\alpha,\alpha')$ contributes at least

$$\alpha^2 \frac{|E(\alpha,\alpha')|^3}{c(\mathcal{C})n^2} - \alpha'^2 a(\mathcal{C}) n$$

to $\mathcal{C}(G_\mu)$ for every $\alpha,\alpha' \in [0,1]$. Therefore letting $E_i = E\left(2^{-i-1}\beta, 2^{-i}\beta\right)$, we have

$$\mathcal{C}(G_\mu) \gtrsim \frac{1}{c(\mathcal{C})n^2} \sum_{i=0}^{\infty} 2^{-2i}\beta^2 |E_i|^3 - a(\mathcal{C}) n \sum_{i=0}^{\infty} 2^{-2i}\beta^2.$$

Let $F_i = \sum_{(u,v)\in E_i} F(u,v)$ so that $|E_i| \ge (2^i/\beta) F_i$, and then

$$\mathcal{C}(G_\mu) \gtrsim \frac{1}{\beta c(\mathcal{C}) n^2} \sum_{i=0}^{\infty} 2^i F_i^3 - \beta^2 a(\mathcal{C}) n,$$

but also $\sum_{i=0}^{\infty} F_i = \sum_{u,v\in E(0,\beta)} F(u,v)$. Thus (5) is proved by noting that

$$\begin{aligned}\sum_{i=0}^{\infty} F_i &= \sum_{i=0}^{\infty} \left(2^{-i/3} \cdot 2^{i/3} F_i\right) \\ &\le \left(\sum_{i=0}^{\infty} 2^{-i/2}\right)^{2/3} \left(\sum_{i=0}^{\infty} 2^i F_i^3\right)^{1/3} \\ &< 2.27 \left(\sum_{i=0}^{\infty} 2^i F_i^3\right)^{1/3},\end{aligned}$$

using Hölder's inequality. ∎

Now fix $\beta = \sqrt{\frac{1}{n^2}\sum_{u,v} F(u,v)}$. If we have

$$\sum_{(u,v)\in E(0,\beta)} F(u,v) \ge \frac{1}{4} \sum_{u,v} F(u,v),$$

then the conclusion follows from Claim 4.3.

Case II: Heavy endpoints.

Define the set of "heavy" vertices,

$$H_K = \{u : F(u) \ge K\beta\}.$$

Observe that

$$\sum_{u\in[n]} F(u) = \mathop{\mathbb{E}}_{S\sim\mu} |S| \le \sqrt{\mathop{\mathbb{E}}_{S\sim\mu} |S|^2} = \sqrt{\sum_{u,v} F(u,v)} = \beta n,$$

hence $|H_K| \le n/K$ by Markov's inequality.

Let $E_H = \{(u,v) : u,v \in H_K\}$ be the set of edges both of whose endpoints are heavy, and apply the statement of the Theorem inductively to the induced graph on H_K to conclude that

$$\begin{aligned}\mathcal{C}(G_\mu) \gtrsim & \frac{K}{c(\mathcal{C})n} \left(\sum_{(u,v)\in E_H} F(u,v)\right)^{5/2} \\ & - \frac{a(\mathcal{C})}{n} \sum_{(u,v)\in E_H} F(u,v).\end{aligned} \tag{6}$$

Consequently, by taking $K = 32$, if $\sum_{(u,v)\in E_H} F(u,v) \ge \frac{1}{4}\sum_{u,v} F(u,v)$ then

$$K \left(\frac{\sum_{(u,v)\in E_H} F(u,v)}{\sum_{u,v} F(u,v)}\right)^{5/2} \ge 1$$

and the conclusion again follows.

Case III: Heavy edges, light endpoints.

Define $E_{HL} = \{(u,v) : F(u,v) > \beta, \{u,v\} \not\subseteq H_K\} = \overline{E(0,\beta) \cup E_H}$. Let $\kappa = (2a(\mathcal{C}) \cdot c(\mathcal{C}))^{1/3}$, so that $\frac{\kappa^4}{c(\mathcal{C})} \ge 2a(\mathcal{C})\kappa$. By Lemma 3.6, we have

$$\begin{aligned}&\mathcal{C}(G_\mu) \gtrsim \frac{1}{c(\mathcal{C})} \mathop{\mathbb{E}}_{\substack{S\sim\mu,\\ S'\sim\mu}} \left[|S\cap S'|^4 \mathbf{1}_{|S\cap S'|\ge\kappa}\right] \\ &- a(\mathcal{C}) \mathop{\mathbb{E}}_{\substack{S\sim\mu,\\ S'\sim\mu}} \left[|S\cap S'| \mathbf{1}_{|S\cap S'|\ge\kappa}\right] \\ &\ge \frac{1}{2c(\mathcal{C})} \mathop{\mathbb{E}}_{\substack{S\sim\mu,\\ S'\sim\mu}} \left[|S\cap S'|^4 \mathbf{1}_{|S\cap S'|\ge\kappa}\right] \\ &= \frac{1}{2c(\mathcal{C})} \sum_{u\in[n]} \Pr[u\in S]^2 \mathop{\mathbb{E}}_{\substack{S\sim\mu,\\ S'\sim\mu}} \left[|S\cap S'|^3 \mathbf{1}_{|S\cap S'|\ge\kappa} \middle| u\in S\cap S'\right] \\ &\ge \frac{1}{2c(\mathcal{C})} \sum_{u:\beta\le F(u)\le K\beta} F(u)^2 \mathop{\mathbb{E}}_{\substack{S\sim\mu,\\ S'\sim\mu}} \left[|S\cap S'|^3 \mathbf{1}_{|S\cap S'|\ge\kappa} \middle| u\in S\cap S'\right] \\ &\ge \frac{\beta^2}{2c(\mathcal{C})} \sum_{u:\beta\le F(u)\le K\beta} \mathop{\mathbb{E}}_{\substack{S\sim\mu,\\ S'\sim\mu}} \left[|S\cap S'|^3 \middle| u\in S\cap S'\right] - \frac{K^2\beta^2}{2c(\mathcal{C})} n\kappa^3.\end{aligned}$$

Now, since $\frac{K^2\beta^2}{2c(\mathcal{C})} n\kappa^3 = \frac{K^2 a(C)}{n} \mathbb{E}|S|^2$, to finish the proof we need only show that

$$\sum_{u:\beta\le F(u)\le K\beta} \mathop{\mathbb{E}}_{\substack{S\sim\mu,\\ S'\sim\mu}} \left[|S\cap S'|^3 \middle| u\in S\cap S'\right] \gtrsim n\left(\mathbb{E}|S|^2\right)^{3/2}. \tag{7}$$

Now for each $u \in [n]$ let μ_u denote the distribution μ conditioned on u being a member of the set. Let I_{vS} denote the indicator variable $[v \in S]$, so that $\Pr[v \in S \mid u \in S]$ can be rewritten as $\mathbb{E}_{S\sim\mu_u}[I_{vS}]$. In this case,

$$\begin{aligned}
&\mathop{\mathbb{E}}_{\substack{S\sim\mu,\\ S'\sim\mu}}\left[|S\cap S'|^3 \,\middle|\, u\in S\cap S'\right]\\
&\quad= \sum_{v,v',v''}\mathop{\mathbb{E}}_{\substack{S\sim\mu_u,\\ S'\sim\mu_u}}\left[I_{vS}I_{v'S}I_{v''S}I_{vS'}I_{v'S'}I_{v''S'}\right]\\
&\quad= \mathop{\mathbb{E}}_{\substack{S\sim\mu_u,\\ S'\sim\mu_u}}\left[\Big(\sum_v I_{vS}I_{vS'}\Big)^3\right]\\
&\quad\geq \Big(\mathop{\mathbb{E}}_{\substack{S\sim\mu_u,\\ S'\sim\mu_u}}\Big[\sum_v I_{vS}I_{vS'}\Big]\Big)^3\\
&\quad= \Big(\sum_v \mathop{\mathbb{E}}_{S\sim\mu_u}[I_{vS}]^2\Big)^3
\end{aligned}$$

Therefore the left hand side of (7) is at least

$$\begin{aligned}
&\sum\nolimits_{u:\beta\le F(u)\le K\beta}\left(\sum_v \Pr[v\in S\mid u\in S]^2\right)^3\\
&\quad\geq \frac{1}{K^6}\sum_{u:\beta\le F(u)\le K\beta}|\{v: F(u,v)/F(u)\geq 1/K\}|^3\\
&\quad\geq \frac{1}{K^6}\sum_{u:\beta\le F(u)\le K\beta}|\{v: F(u,v)\geq \beta\}|^3 . \qquad (8)
\end{aligned}$$

Now, every edge $(u,v)\in E_{HL}$ has either $F(u)\le K\beta$ or $F(v)\le K\beta$. In particular, $F(u,v)\le K\beta$, which means that

$$|E_{HL}| \geq \frac{\sum_{(u,v)\in E_{HL}} F(u,v)}{K\beta}. \qquad (9)$$

Each of the edges in E_{HL} appears at least once in the sum (8), all among vertices of weight at least β (since $F(u)\geq F(u,v)$.) Therefore by the power-mean inequality, the left hand side of (7) is at least

$$\begin{aligned}
&\frac{1}{K^6}\sum_{u:\beta\le F(u)\le K\beta}|\{v:(u,v)\in E_{HL}\}|^3\\
&\quad\geq \frac{1}{K^6 n^2}\left(\sum_{u:\beta\le F(u)\le K\beta}|\{v:(u,v)\in E_{HL}\}|\right)^3\\
&\quad\geq \frac{1}{K^6 n^2}|E_{HL}|^3
\end{aligned}$$

and when $\sum_{(u,v)\in E_{HL}} F(u,v)\geq \frac{1}{2}\sum_{u,v}F(u,v)$ it follows from (9) that this is at least

$$\frac{1}{K^9 n^2\beta^3}\left(\sum_{u,v}F(u,v)\right)^3 \gtrsim n\left(\mathbb{E}|S|^2\right)^{3/2},$$

completing the proof. ■

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