embeddings, flow, and cuts: an introduction

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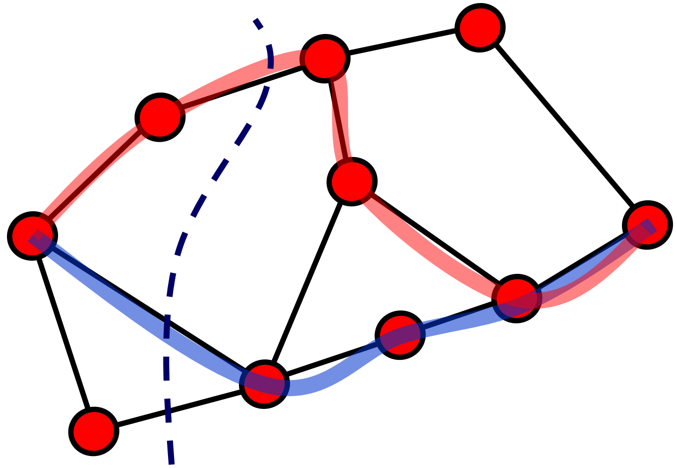
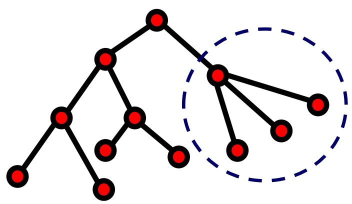
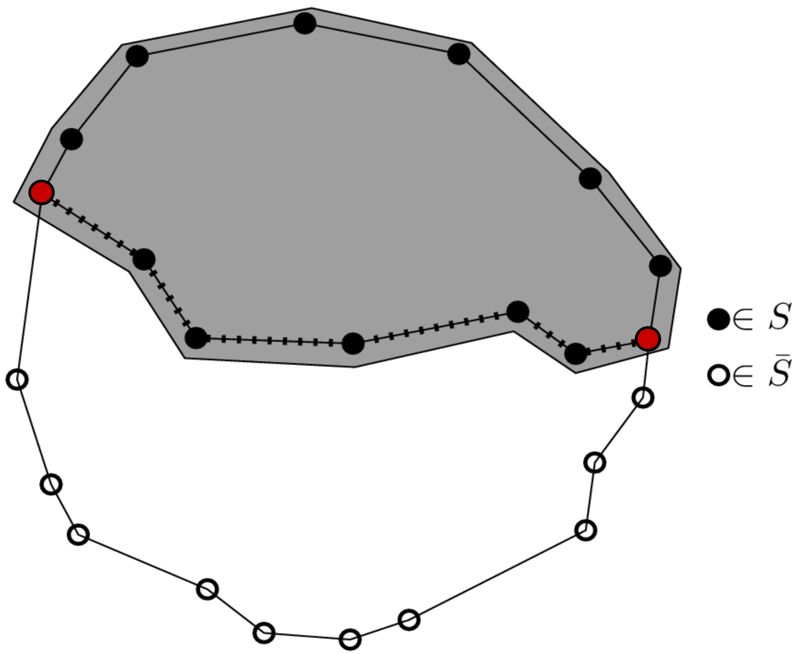
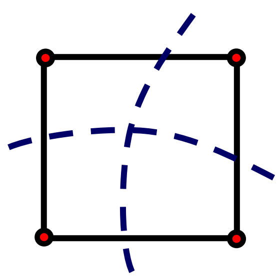
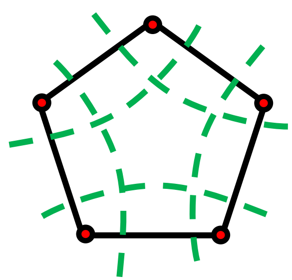
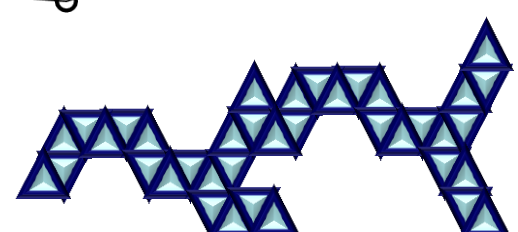




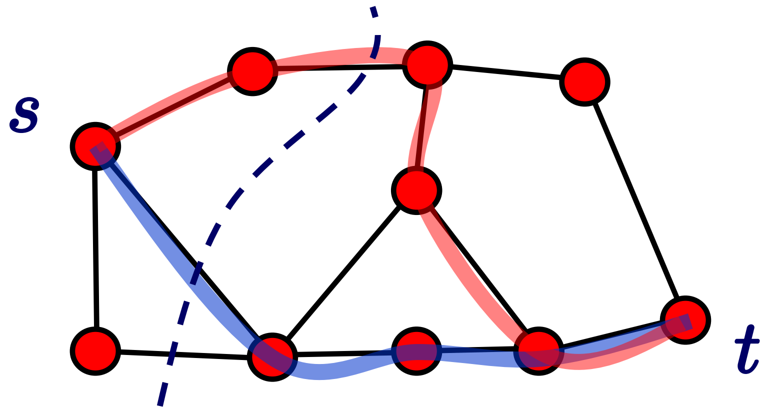








max-flow min-cut theorem

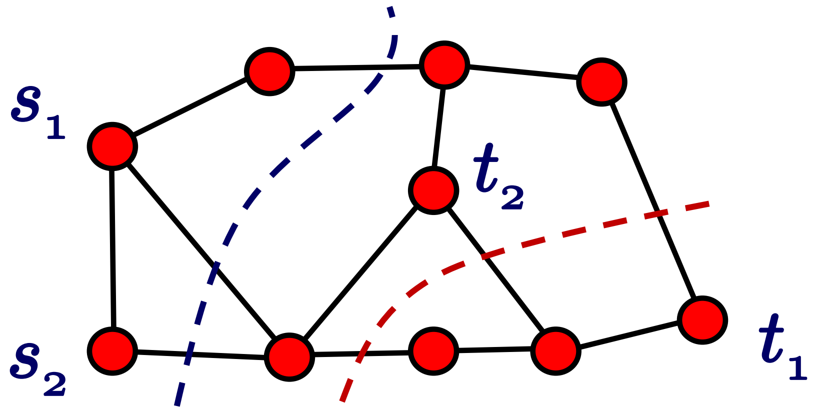


Flow network: Graph G and non-negative capacities on edges

Max-flow Min-Cut Theorem: Value of maximum flow $=$ value of minimum s-t cut

[Menger'27, Elias-Feinstein-Shannon'56, Ford-Fulkerson'56]:

max-flow min-cut theorem



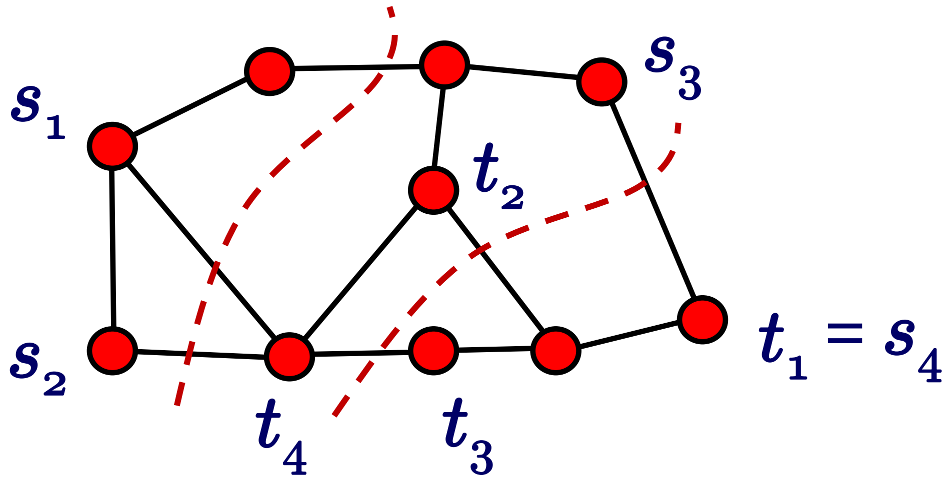
Flow network: Graph G and non-negative capacities on edges

Max-flow Min-Cut Theorem: Value of maximum flow = value of minimum s-t cut
[Menger'27, Elias-Feinstein-Shannon'56, Ford-Fulkerson'56]:

2-commodity MFMC Theorem: max **concurrent** flow = min-cut **[Hu'69]**

Implied by: **Every 4-point metric space embeds isometrically in the Euclidean plane equipped with the L_1 norm**

multi-commodity flows

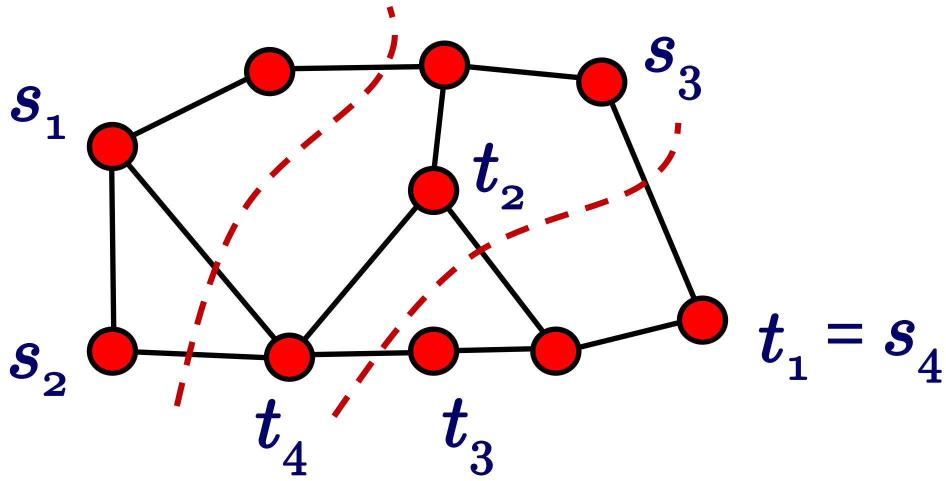


For a subset $S \subseteq V$ of vertices, we define its **sparsity** by

$$\Phi(S) = \frac{\text{capacity of edges across } S}{\text{number of demand pairs separated}}$$

Is there a Multi-flow/Sparse-cut theorem? **NO, even for 3 demand pairs**

multi-commodity flows



For a subset $S \subseteq V$ of vertices, we define its **sparsity** by

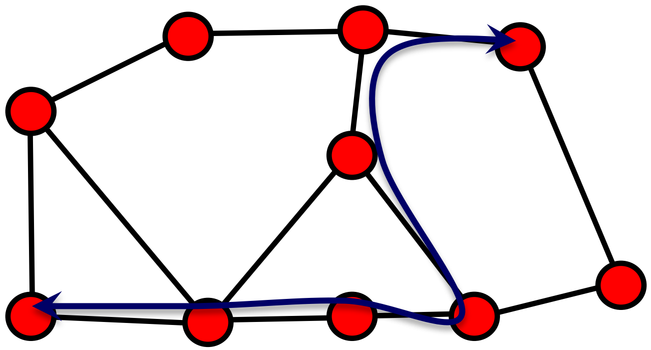
$$\Phi(S) = \frac{\text{capacity of edges across } S}{\text{number of demand pairs separated}}$$

Define **gap**(G) = worst possible ratio between min sparse cut and max multi-flow over all instances on G

[instance = capacities + demand pairs]

geometries on graphs

Consider an arbitrary undirected graph $G=(V, E)$ as a **topological template**.



Every length on edges **len** : $E \to [0,\infty]$ induces a **shortest-path geometry** on G.

We consider the set of all (pseudo-)metrics induced on a given G:

Every metric on a **path** embeds isometrically into $\mathbb{R}$.

Every metric on a **tree** embeds isometrically into L_1.

The **complete graph** on n nodes supports every metric on n points.

embeddings and flow-cut gaps

Let $L_1 = L_1([0,1])$ (can also think of $\mathbb{R}^n$ equipped with the L_1 norm).

For a metric space (X, d), a mapping $f : X \to L_1$ is a D-**embedding** if

$$d(x, y) \leq \|f(x) - f(y)\|_1 \leq D \cdot d(x,y) \quad \text{for all } x,y \in X$$

For a metric space X:

$$c_1(X, d) = \inf \{ D : (X, d) \text{ admits a } D\text{-embedding into } L_1 \}$$

For a graph G:

$$c_1(G) = \sup \{ c_1(X, d) : (X, d) \text{ is supported on } G \}$$

Theorem [Linial-London-Rabinovich'92, Gupta-Newman-Rabinovich-Sinclair'99]:

For every graph G, $c_1(G) = \text{gap}(G)$

Bourgain 1985

Theorem [Linial-London-Rabinovich, Aumann-Rabani]:

If G has n vertices, then $\mathrm{gap}(G) = O(\log n)$
and this it tight (expander graphs)

For a graph family $\mathcal{F}$, define: $c_1(\mathcal{F}) = \sup \{ c_1(G) : G \in \mathcal{F} \}$

Which families admit a uniform multi-flow/cut gap,
i.e. when do we have $c_1(\mathcal{F}) < \infty$?

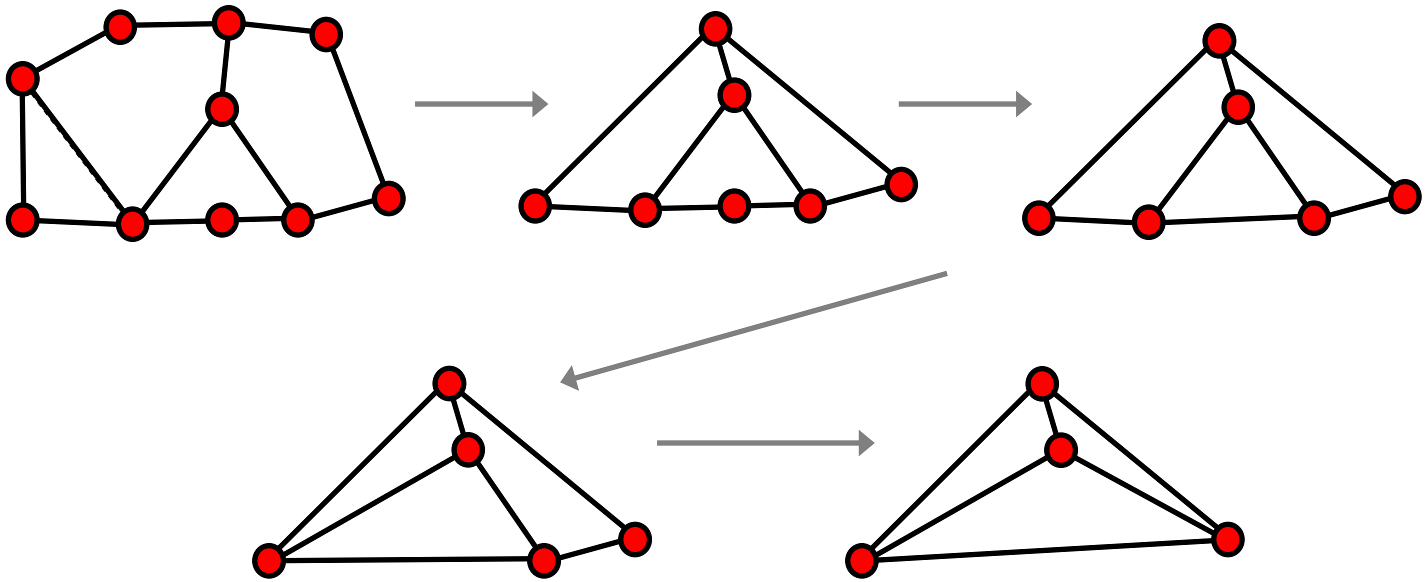
If $\mathcal{F}$ supports all finite metric space, then $c_1(\mathcal{F}) = \infty$.

Conjecture [GNRS 1999]:

$c_1(\mathcal{F}) < \infty$ if and only if $\mathcal{F}$ does not support all finite metric spaces.

the relation to graph minors

A graph **H** is a minor of **G** if it can be obtained from **G** by a sequence of edge deletions and contractions…



If **H** is a minor of **G**, then **G** supports every metric space that **H** supports.

So it suffices to consider graph families $\mathcal{F}$ which are closed under taking minors.

the GNRS conjecture

Conjecture [GNRS 1999]:

$c_1(\mathcal{F}) = \text{gap}(\mathcal{F})$ is finite if and only if $\mathcal{F}$ forbids some minor.

(infinite graphs) equivalent:

$c_1(G)$ is finite if and only if G forbids some **finite** minor.

outline

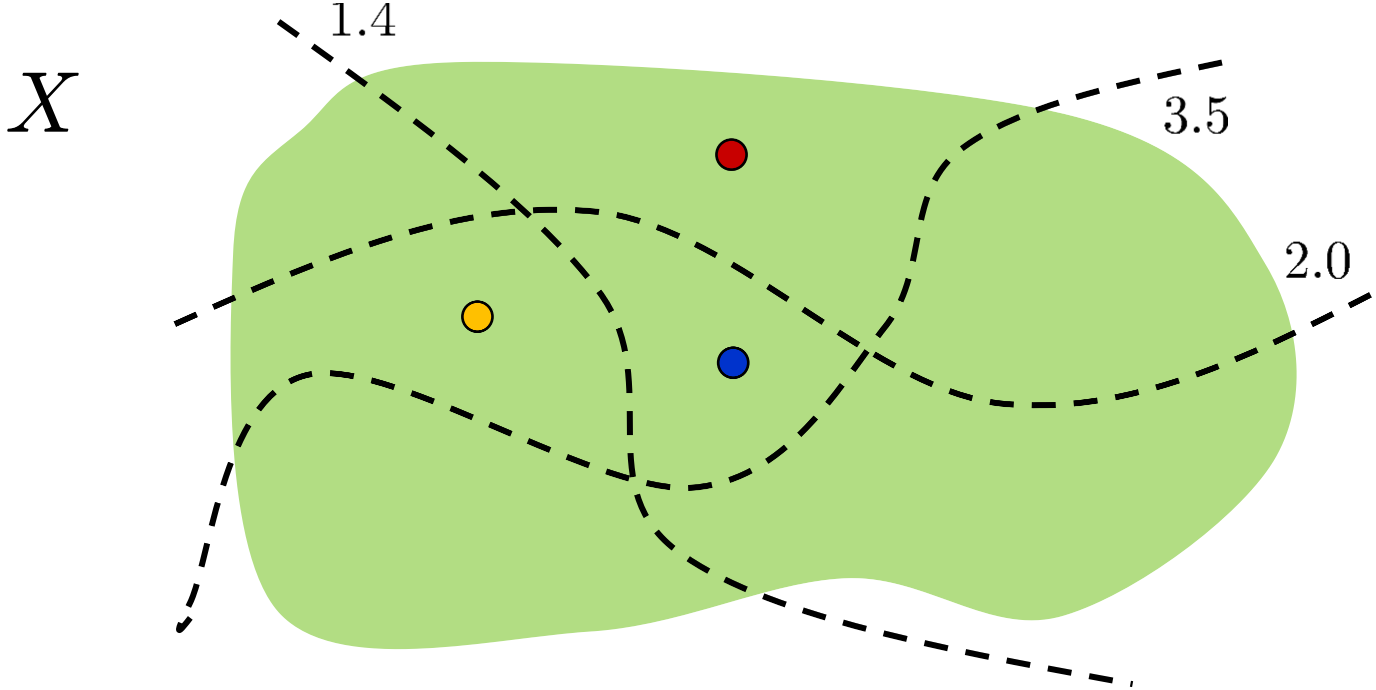
- Cuts and L_1 embeddings
- The (geometric) Okamura-Seymour theorem
- Planar graphs
- Sums of tetrahedra
- More general flow networks

L_1 pseudometrics and the cut cone

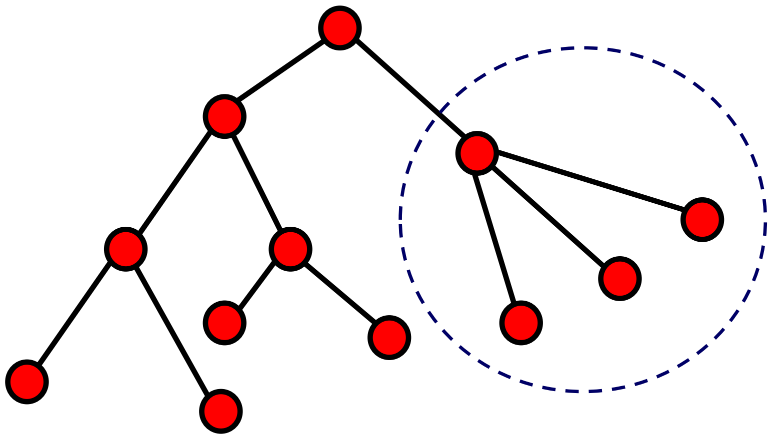
Consider a finite set X and a mapping $f : X \to L_1$.

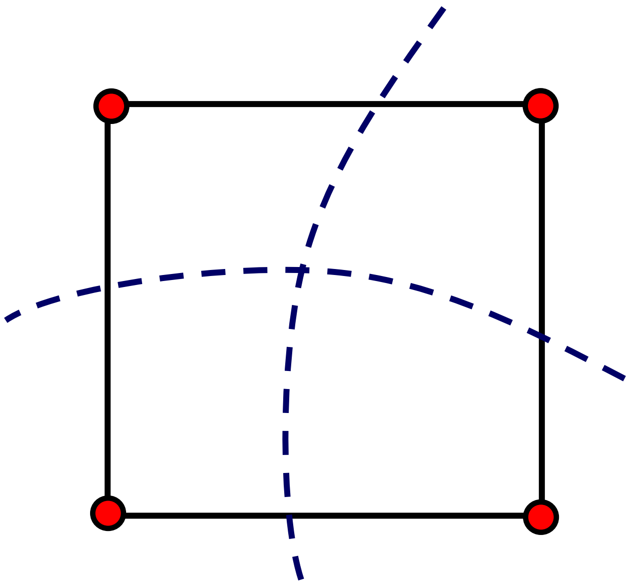
Fact: There exists a measure μ on subsets $S \subseteq X$ such that:

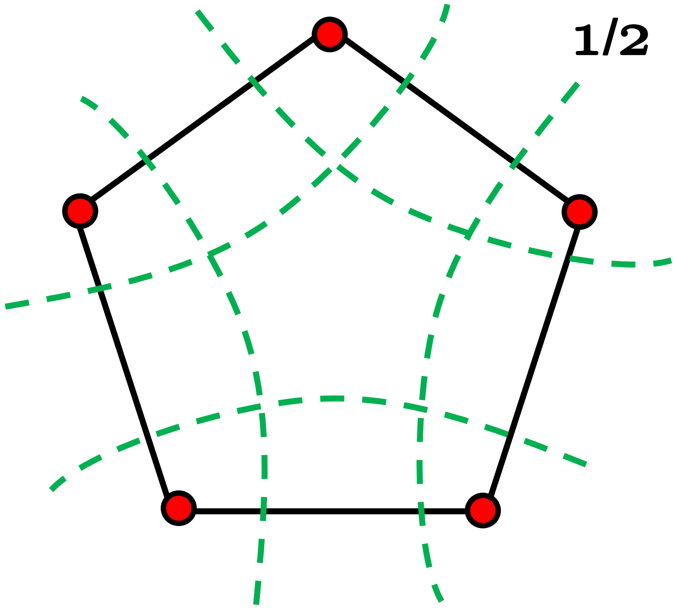
$$\|f(x) - f(y)\|_1 = \sum_{S \subseteq X} \mu(S) |\mathbf{1}_S(x) - \mathbf{1}_S(y)|$$



simple examples



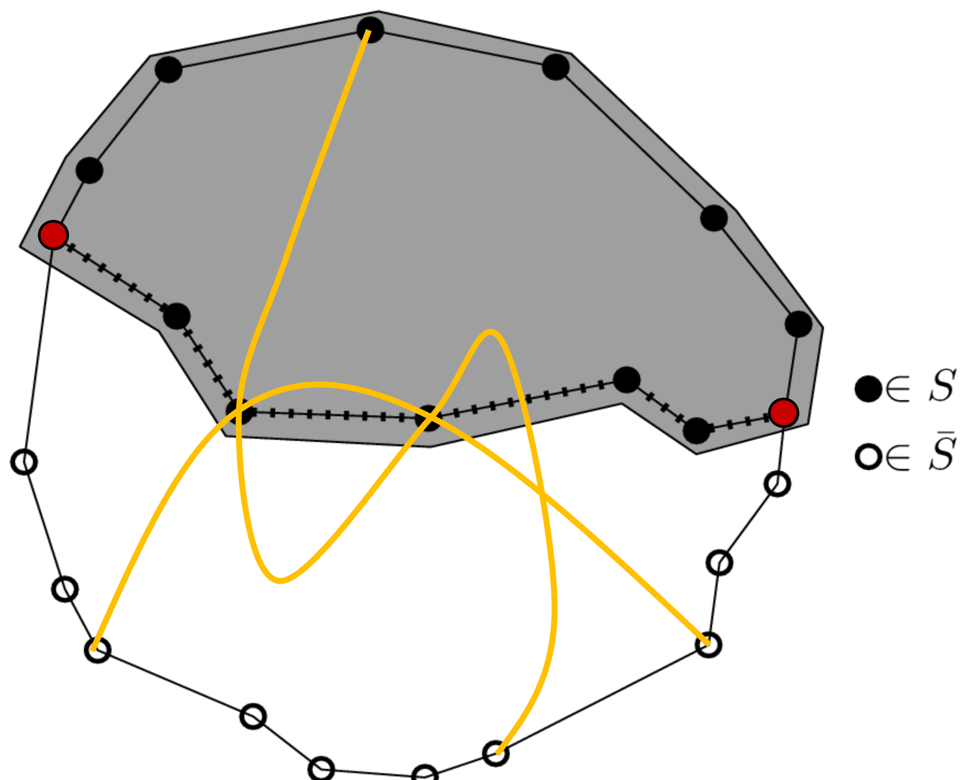




If G is a (weighted) planar graph and F is the outer face, there is a 1-Lipschitz mapping $f : G \rightarrow L_1$ which is an isometry on F.

Proof: Assume G is unit-weighted and bipartite.

G



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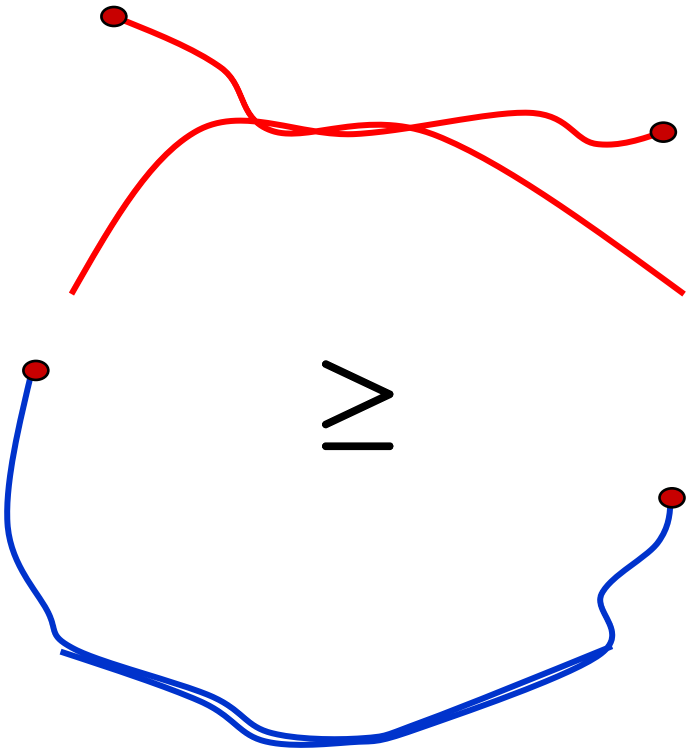
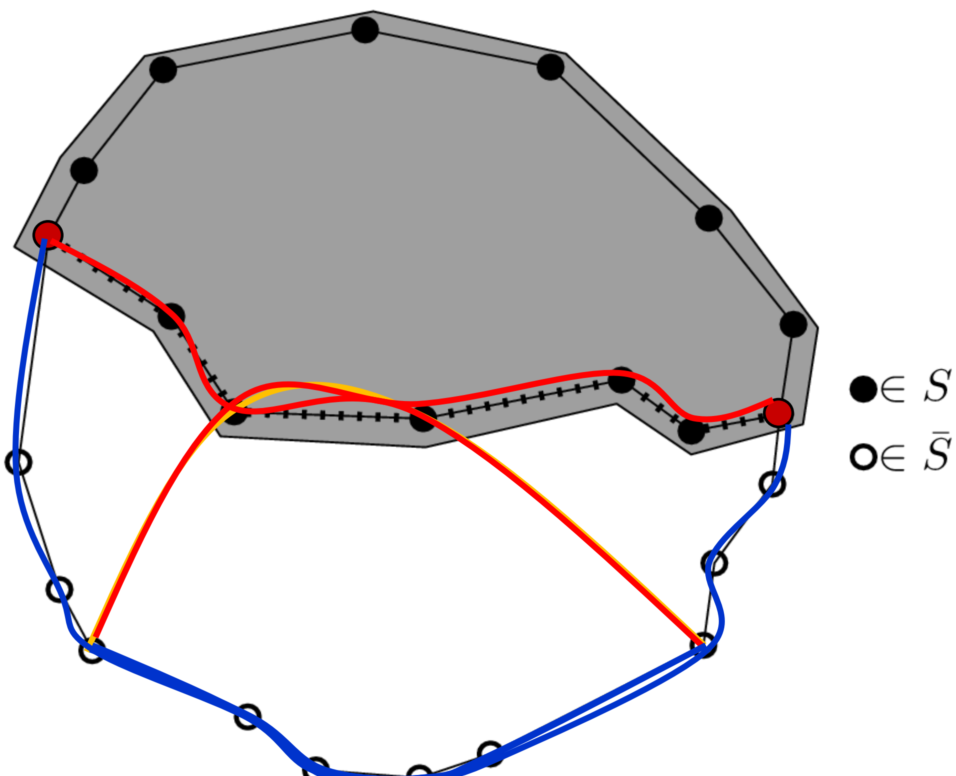
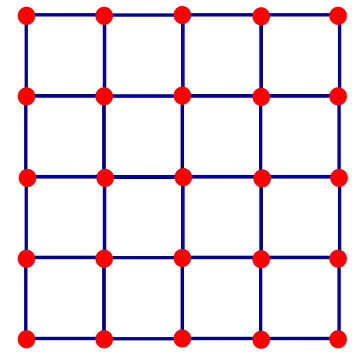






Planar embedding conjecture (circa 1992):

$c_1(\mathcal{F})$ is finite when $\mathcal{F}$ = family of planar graphs (equivalent to $c_1(\mathbb{Z}^2)$ finite)

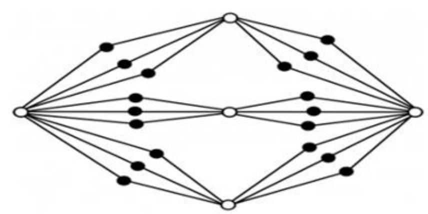
Theorem [Gupta-Newman-Rabinovich-Sinclair'99]:

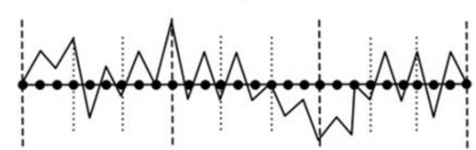
$c_1(\mathcal{F})$ is finite when $\mathcal{F}$ = family of series-parallel graphs

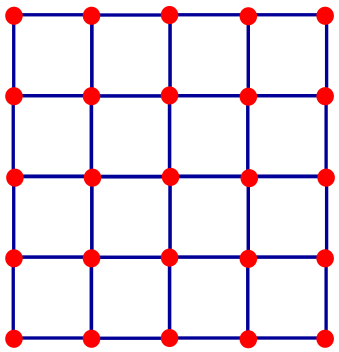
Theorem [L-Ragahvendra'07, Chakrabarti-L-Jaffe-Vincent'08]:

$c_1(\{\text{series-parallel graphs}\}) = 2$

Lower bound based on quantitative differentiation







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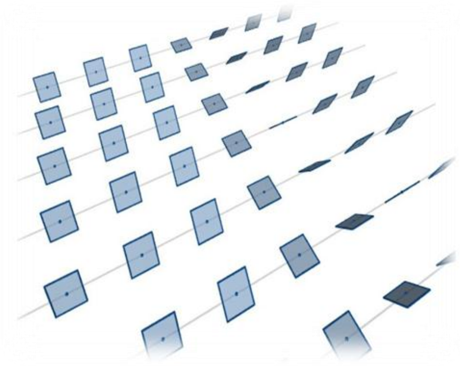
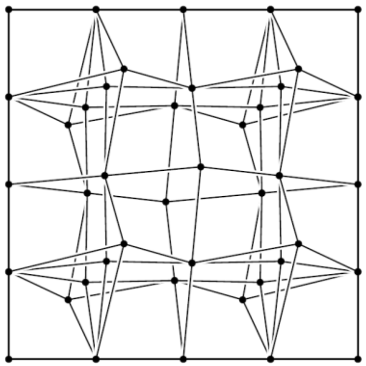
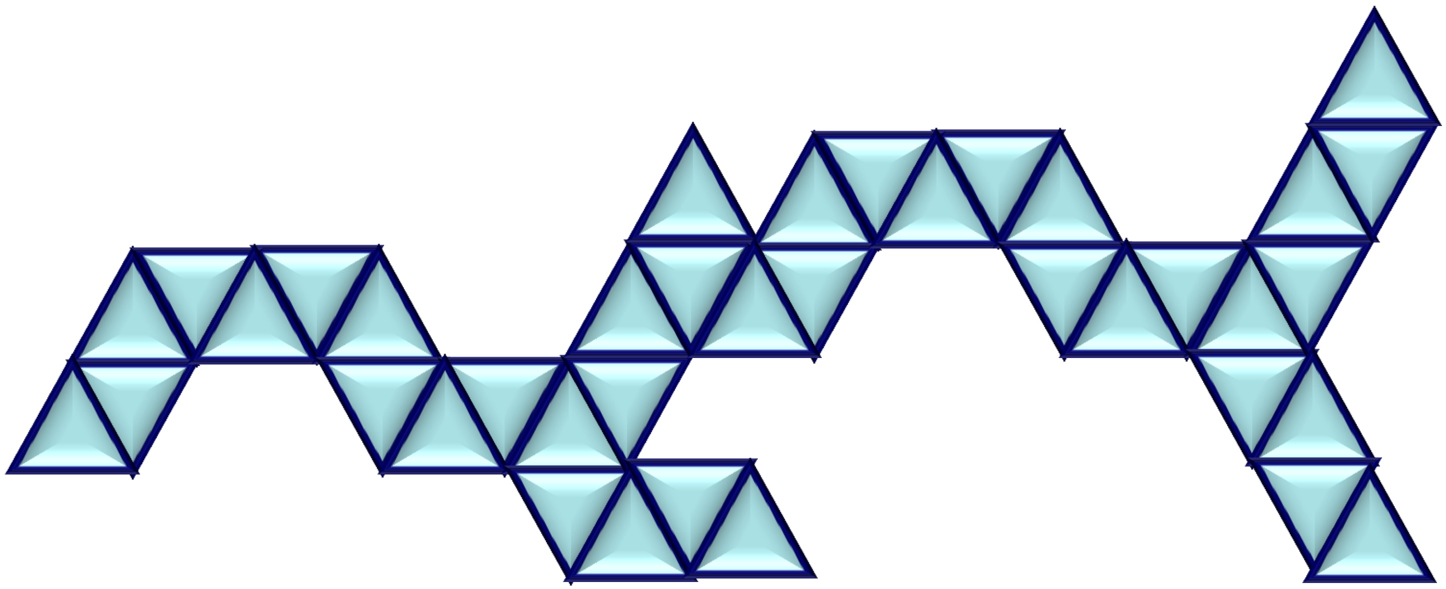




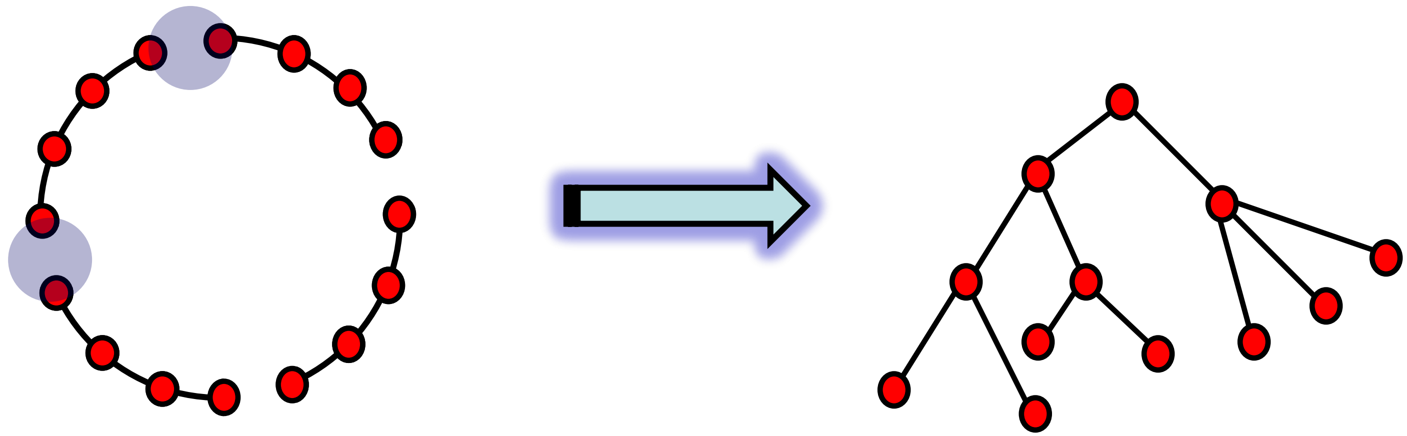
summing tetrahedra



Open problem: What about summing tetrahedra along triangles?
(What about summing $(k+1)$-cliques along k cliques?)

[L-Sidiropoulos'09]: Answer is positive (bounded distortion) if we only take sums along paths.

random simplifying the topology

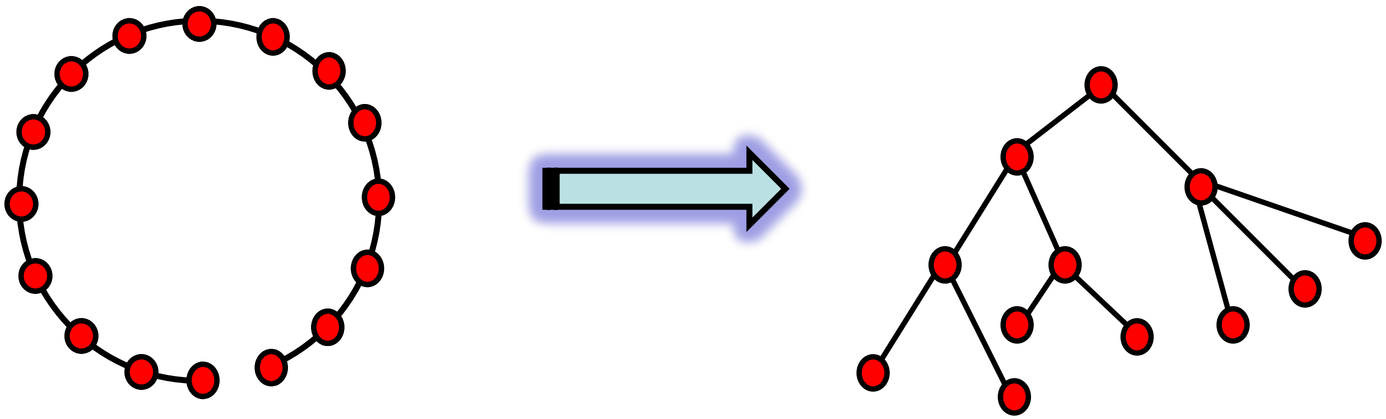


Consider a metric space (X, d_X) and a random mapping $F : X \to Y$, where (Y, d_Y) is a random metric space. This is a **probabilistic** D**-embedding** if

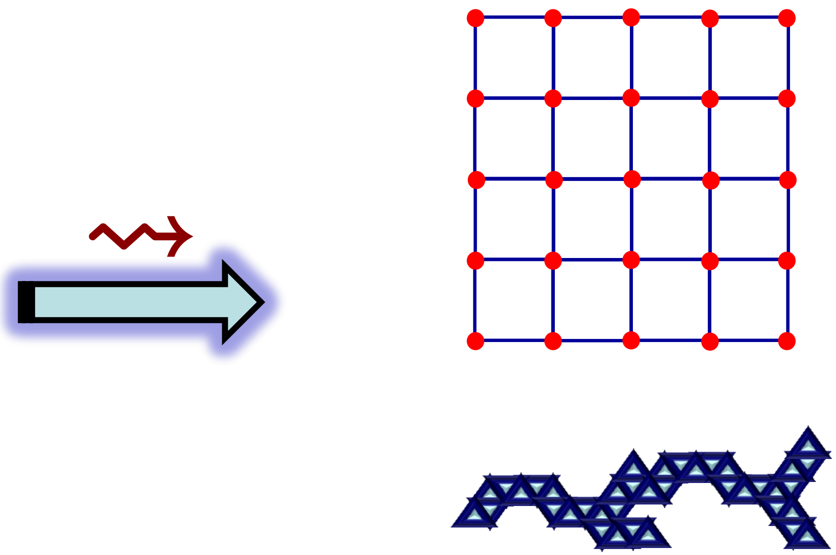
1. $d_Y(F(x), F(y)) \geq d_X(x, y)$ with probability one
2. $\mathbb{E}\left[d_Y(F(x), F(y)) \right] \leq D \cdot d_X(x, y)$

For two graph families $\mathcal{F}$ and $\mathcal{G}$, we write $\mathcal{F} \rightsquigarrow \mathcal{G}$ if there exists a **D** such that every metric supported on $\mathcal{F}$ probabilistically **D**-embeds into a metric supported on $\mathcal{G}$.

random simplifying the topology



GNRS conjecture



the k-sum conjecture

The k-sum conjecture:

For every $k \in \mathbb{N}$, $c_1(\oplus_k\mathcal{F})$ is finite whenever $c_1(\mathcal{F})$ is finite.

Take two graphs **G** and **G'** and fix a **k**-clique in each. Identify the cliques.

$\oplus_k\mathcal{F}$ is the closure of $\mathcal{F}$ under the operation of taking **k**-sums.

Example: $\oplus_1\{$ ●—● $\}$ is the family of all (connected) trees.

Conjecture is true for $k=1$: $c_1(\oplus_1\mathcal{F}) = c_1(\mathcal{F})$ for any family $\mathcal{F}$.

The k-sum conjecture:

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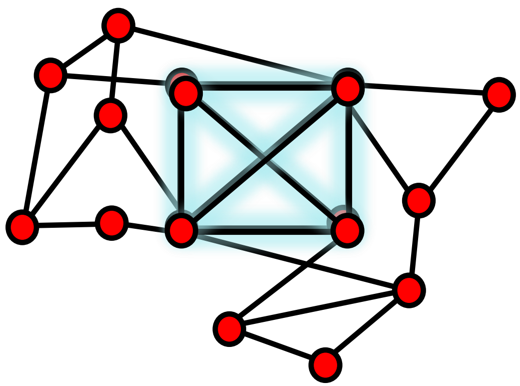
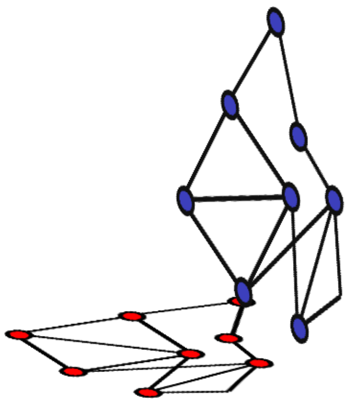
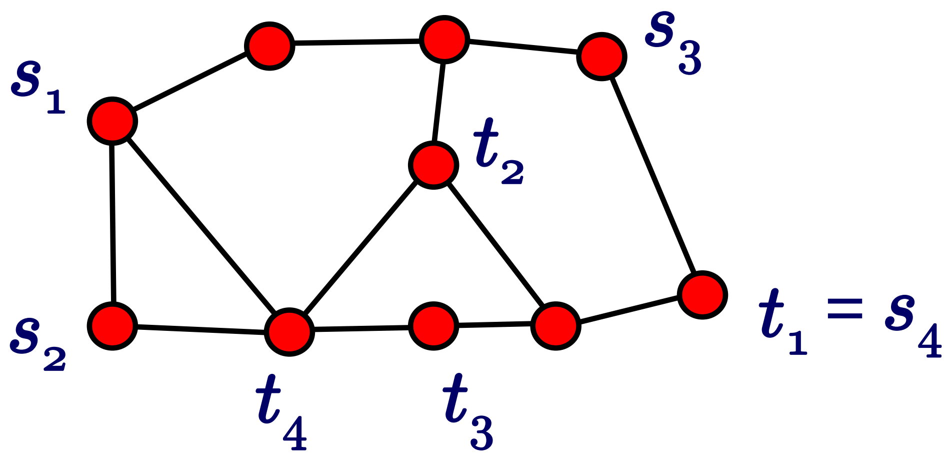




GNRS conjecture ⟺ **Planar embedding conjecture + k-sum conjecture**

node capacities and beyond



What if we have capacities on **nodes** instead of edges?

- Okamura-Seymour analogue is now (trivially) false.
- But an $O(1)$-approximate version is true! [L-Mendel-Moharrami'13]
- Very different techniques from the edge case
  (Random Lipschitz maps into trees, random Whitney decompositions)