1) In this exercise we overview basic facts about eigenvalues of graphs.
   a) Show that if $G$ is $d$-regular then the largest eigenvalue of the adjacency matrix of $G$ is equal to $d$.
   b) Show that a $d$-regular graph is bipartite if and only if the smallest eigenvalue of the adjacency matrix is $-d$.
   c) Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of the adjacency matrix of a $d$-regular graph $G$. Show that $d - \lambda_1, \ldots, d - \lambda_n$ are the eigenvalues of the Laplacian of $G$.

2) In this exercise we want to prove the Cayley’s theorem, that is the number of spanning trees of a complete graph on $n$ vertices is $n^{n-2}$. Show that all of the non-zero eigenvalues of the Laplacian of a complete graph is $n$. Use this to show that a complete graph has $n^{n-2}$ spanning trees.

3) Show that if $A \preceq B$ then the $i$-th eigenvalue of $A$ is at most the $i$-th eigenvalue of $B$.

4) Observe that for any PSD matrix $A$, $\det(A) \geq 0$. This is because all eigenvalues of $A$ are nonnegative and $\det(A)$ is just the product of the eigenvalues of $A$. A minor of a matrix $A \in \mathbb{R}^{n \times n}$ is a square submatrix $A_{S,T}$ where $|S| = |T|$ but $S$ is not necessarily equal to $T$. We say $A$ is totally positive if $\det(A_{S,T}) \geq 0$ for any square minor $A_{S,T}$ of $A$.
   a) Recall that a principal minor of $A$ is a square submatrix $A_{S,S}$ for some $S \subseteq [n]$, i.e., a minor $A_{S,T}$ where $S = T$. Show that if $A$ is PSD then $\det(A_{S}) \geq 0$ for all principal minors of $A$.
   b) Show that a PSD matrix is not necessarily totally positive.
   c) Use the Cauchy-Binet formula to show that for any pair of totally positive matrices $A, B \in \mathbb{R}^{n \times n}$, $AB$ is also totally positive.

5) Say $G = (V,E)$ and $H = (V,E')$ are unweighted graphs on the same vertex set $V$.
   a) show that if $L_G \preceq L_H$, then for any $S \subseteq V$,
      $$|E(S, \overline{S})| \leq |E'(S, \overline{S})|,$$
      where $E(S, \overline{S}), E'(S, \overline{S})$ are the sets of edges in the cut $(S, \overline{S})$ in $G, H$ respectively.
   b) Show that the converse of the above statement is not necessarily true, i.e., construct $G, H$ such that $|E(S, \overline{S})| \leq |E'(S, \overline{S})|$ for all $S \subseteq V$ but $L_G \not\preceq L_H$.
      Bonus point: Construct an example where $|E(S, \overline{S})| \leq |E'(S, \overline{S})|$ for all $S \subseteq V$ but $L_G \not\preceq \Omega(n)L_H$. 