Problem 1) Show that for any matrix $A$ with eigenvalues $\lambda_1, \ldots, \lambda_n$

$$\det(A) = \prod_{i=1}^{n} \lambda_i.$$  

Problem 2) Show that if $A, B$ are PSD then $\text{trace}(AB) \geq 0$.

Problem 3) Show that the maximum degree of vertices of a uniform spanning tree of a complete graph is $\Theta(\log(n) / \log \log(n))$.


Problem 4) We say that an unweighted graph $G = (V, E)$ is $k$-edge connected if for any cut $(S, \overline{S})$, $|E(S, \overline{S})| \geq k$. Recall that by Menger’s theorem a graph is $k$-edge-connected if for any pair of vertices $s, t$ there are at least $k$ edge disjoint paths from $s$ to $t$.

a) Suppose $G$ is connected and let $\epsilon = \max_{e \in E} \text{Reff}(e)$. Show that $G$ is $1/\epsilon$-edge connected.

b) Show that the converse of the above is not true. Bonus point: construct a $k$-edge-connected graph with $n$ vertices where $\max_{e \in E} \text{Reff}(e) \geq \Omega(n)/k$.

c) Show that for any simple unweighted $k$-edge-connected graph $G = (V, E)$ (with no parallel edges),

$$\sum_{e \in E} \text{Reff}(e)^2 \leq O(n/k).$$

Hint: Show that for any simple unweighted graph $G$, and any edge $e = \{u, v\}$, $\text{Reff}(e) \geq \frac{1}{d(u)+1} + \frac{1}{d(v)+1}$ where $d(u), d(v)$ are the degrees of $u, v$ are respectively.

Problem 5) In this exercise we want to show that the effective resistance is convex over the space of all positive definite matrices. For a PD matrix $A \in \mathbb{R}^{V \times V}$ let

$$\text{Reff}_A(s, t) = b_{s,t}^T A^{-1} b_{s,t}.$$  

Show that for any two matrix $A, B \succ 0$,

$$\text{Reff}_{A+B/2}(s, t) \leq \frac{1}{2}(\text{Reff}_A(s, t) + \text{Reff}_B(s, t)).$$

Hint: Use the Schur complement which says that for a positive definite matrix $A$,

$$\begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \succ 0 \text{ if and only if } C - B^T A^{-1} B \succeq 0.$$  

Problem 6) Prove that the expected size of the diameter of a uniform spanning tree of a complete graph on $n$ vertices is $\Omega(\sqrt{n})$.

Hint: Use the Aldous, Broder algorithm for sampling a uniform spanning tree.