You are free to collaborate to solve these assignments.

Problem 1) Given a graph $G = (V,E)$, a metric is a mapping $d : V \times V \to \mathbb{R}_+$ such that $d(v, v) = 0$, $d(u,v) = d(v,u)$ for all $u,v$ and

$$d(u,v) \leq d(u,w) + d(w,v)$$

for all $u,v,w \in V$. We say $d(.,.)$ is an $L_1$ metric if we can assign vectors $\{x_v\}_{v \in V}$ such that for all $u,v$, $d(u,v) = \|x_u - x_v\|_1$. We say $d(.,.)$ is an $L_2$ metric if we can assign vectors $\{x_v\}_{v \in V}$ such that for all $u,v \in V$,

$$d(u,v) = \|x_u - x_v\|_2^2.$$

(a) Suppose $G$ is $d$-regular. The conductance of a set $S \subseteq V$ is defined as $\phi(S) = \frac{|E(S,\overline{S})|}{d \cdot |S|}$. The conductance of $G$ is defined as $\min_{S \subseteq V} \phi(S)$. Show that for any $L_1$ metric $d(.,.)$,

$$\phi(G) \leq O \left( \frac{\mathbb{E}_{u \sim v} d(u,v)}{\mathbb{E}_{u,v} d(u,v)} \right),$$

where the expectations are over the uniform distribution.

(b) Show that if $d(.,.)$ is the shortest path metric in a bounded degree expander then

$$\phi(G) \geq \Omega(\log n) \frac{\mathbb{E}_{u \sim v} d(u,v)}{\mathbb{E}_{u,v} d(u,v)}$$

This shows that the integrality gap of the Leighton-Rao relaxation for the sparsest cut problem is $\Omega(\log n)$.

(c) Show that any $L_1$ metric is an $L_2$ metric. It is optional to see that the converse of this is not necessarily true.

Problem 2) For $m < n$ we say a sequence of numbers $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_m$ interlaces $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$ if for any $1 \leq i \leq m$,

$$\alpha_i \geq \beta_i \geq \alpha_{n-m+i}.$$

Use properties of real stable polynomials to show that for any symmetric matrix $A$ and any set $S \subset [n]$, the eigenvalues of $A_S$ interlaces those of $A$.

Problem 3) Show that for any symmetric matrix $X$ and any integer $k \geq 1$ the sum of the $k$ largest eigenvalues of $X$ is a convex function of $X$. You are allowed to use tools in Chapter 3 of the Boyd’s book.

Problem 4) In this part we want to prove several properties of non-symmetric matrices.

a) Let $D$ be the diagonal matrix of degree of vertices of a graph $G$, and $A$ be the adjacency matrix of $G$. $P = D^{-1}A$ is called the transition probability matrix of of $G$. The normalized Laplacian matrix of a nonregular graph $G$ is defined as follows

$$L_G = D^{-1/2}L_G D^{-1/2} = I - D^{-1/2}AD^{-1/2}.$$
Show that if $0 = \lambda_1 \leq \cdots \leq \lambda_n$ are the eigenvalues of $L_G$, then
\[ 1 = 1 - \lambda_1 \geq \cdots \geq 1 - \lambda_n \]
are the eigenvalues of $P$.

b) Suppose $m \leq n$. Use the spectral theorem to show that any matrix $A \in \mathbb{R}^{m \times n}$ can be written as
\[ A = \sum_{i=1}^{m} \sigma_i u_i v_i^\top, \]
where $u_1, \ldots, u_m$ are orthonormal vectors which are called left singular vectors of $A$, $\sigma_1, \ldots, \sigma_m \geq 0$ are the singular values of $A$ and $v_1, \ldots, v_m$ are orthonormal vectors which are called right singular vectors of $A$.

c) The trace norm or nuclear norm of $A$ is defined as
\[ \|A\|_* = \text{trace}((A^\top A)^{1/2}). \]
Show that $\|A\|_* = \sum_{i=1}^{\min(m,n)} \sigma_i$.

d) The Frobenius norm of $A$ is defined as
\[ \|A\|_F = \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} A_{i,j}^2}. \]
Show that $\|A\|_F = \sqrt{\sum_{i=1}^{\min(m,n)} \sigma_i^2}$.

e) We say a matrix $U \in \mathbb{R}^{n \times n}$ is unitary if all of its singular values are equal to 1. Show that a unitary operator is norm preserving. That is for any vector $x \in \mathbb{R}^n$,
\[ \|Ux\| = \|x\|. \]

f) Show that for any matrix $A \in \mathbb{R}^{n \times m}$ where $n \geq m$ we have
\[ \|A\|_* = \max_{\text{unitary } U} \text{trace}(UA). \]
This is also known as the dual of the nuclear norm is the spectral norm, i.e., the optimum of the following convex program is $\|A\|*$,
\[ \min U \cdot A, \]
subject to $\|U\| \leq 1$. 