1) In this exercise we overview basic facts about eigenvalues of graphs.

a) Show that if $G$ is $d$-regular then the largest eigenvalue of the adjacency matrix of $G$ is equal to $d$.

It is easy to see that $1$ is an eigenvector of $A$ with eigenvalue $d$, as $A1 = d1$, so the max eigenvalue is at least $d$. On the other hand, $dI - A$ is PSD, so the max eigenvalue of $A$ is at most $d$.

b) Show that a $d$-regular graph is bipartite if and only if the smallest eigenvalue of the adjacency matrix is $-d$.

Say $G$ is bipartite with the two sides $X, Y$. It is easy to see that the vector that is 1 in the vertices of $X$ and $-1$ in the vertices of $Y$ is eigenvector of $A$ with eigenvalue $-d$. So, the minimum eigenvalue is at least $-d$. Now, let $M = dI + A$. Observe that for any vector $x$,

$$x^\top Mx = \sum_{u \sim v} (x(u) + x(v))^2.$$  

Therefore, $M$ is PSD, so the minimum eigenvalue of $A$ is at least $d$.

Conversely, if $G$ is not bipartite, we show that the minimum eigenvalue of $A$ is larger than $-d$. Equivalently, we can show that the minimum eigenvalue of $M$ is positive, i.e., $M$ is nonsingular. Observe that for a connected graph $G$,

$$\sum_{u \sim v} (x(u) + x(v))^2 = 0$$

if and only for any pair of adjacent vertices $x(u) = -x(v)$, i.e., $G$ is bipartite.

c) Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of the adjacency matrix of a $d$-regular graph $G$. Show that $d - \lambda_1, \ldots, d - \lambda_n$ are the eigenvalues of the Laplacian of $G$.

This simply follows from $L_G = dI - A$. Then eigenvalues of $-A$ are the negative of the eigenvalues of $A$, adding up with $dI$ just shifts every eigenvalue by $d$.

2) In this exercise we want to prove the Cayley’s theorem, that is the number of spanning trees of a complete graph on $n$ vertices is $n^{n-2}$. Show that all of the non-zero eigenvalues of the Laplacian of a complete graph is $n$. Use this to show that a complete graph has $n^{n-2}$ spanning trees.

By part (c) of the previous problem, it is sufficient to show that adjacency matrix of a complete graph has one eigenvalue that is $n - 1$ and the rest are $-1$. Let $A$ be the adjacency matrix of a complete graph. It is enough to show that $A + I$ has one eigenvalue that is $n$ and the rest are 0 (because $I$ shifts all eigenvalues by 1). Now, $A + I$ is a rank 1 matrix where every entry is 1, i.e., (recall that $1$ is the constant vector of norm 1 where every coordinate is $1/\sqrt{n}$)

$$A + I = n11^\top.$$ 

So, the claim follows.

3) Show that if $A \preceq B$ then the $i$-th eigenvalue of $A$ is at most the $i$-th eigenvalue of $B$.

This simply follows from variational characterization of eigenvalues. First observe that for any $k \geq 1$ and any $k$ orthogonal vectors $x_1, \ldots, x_k$ and any $x \in \text{span}\{x_1, \ldots, x_k\}$,

$$\frac{x^\top Ax}{x^\top x} \leq \frac{x^\top Bx}{x^\top x}.$$
Therefore, for any orthogonal vectors \( x_1, \ldots, x_k \),
\[
\max_{x \neq 0} \left\{ \frac{x^T A x}{x^T x} : x \in \text{span}\{x_1, \ldots, x_k\} \right\} \leq \max_{x \neq 0} \left\{ \frac{x^T B x}{x^T x} : x \in \text{span}\{x_1, \ldots, x_k\} \right\}
\]
So,
\[
\min_{x_1, \ldots, x_k \text{ orthogonal}} \max_{x \neq 0} \left\{ \frac{x^T A x}{x^T x} : x \in \text{span}\{x_1, \ldots, x_k\} \right\} \leq \min_{x_1, \ldots, x_k \text{ orthogonal}} \max_{x \neq 0} \left\{ \frac{x^T B x}{x^T x} : x \in \text{span}\{x_1, \ldots, x_k\} \right\}
\]
as required.

4) Observe that for any PSD matrix \( A \), \( \det(A) \geq 0 \). This is because all eigenvalues of \( A \) are nonnegative and \( \det(A) \) is just the product of the eigenvalues of \( A \). A minor of a matrix \( A \in \mathbb{R}^{n \times n} \) is a square submatrix \( A_{S,T} \) where \( |S| = |T| \) but \( S \) is not necessarily equal to \( T \). We say \( A \) is totally positive if \( \det(A_{S,T}) \geq 0 \) for any square minor \( A_{S,T} \) of \( A \).

a) Recall that a principal minor of \( A \) is a square submatrix \( A_{S,S} \) for some \( S \subseteq [n] \), i.e., a minor \( A_{S,T} \) where \( S = T \). Show that if \( A \) is PSD then \( \det(A_S) \geq 0 \) for all principal minors of \( A \).

This holds because any principal minor of a PSD matrix is also PSD so its eigenvalues are nonnegative and the determinant is nonnegative. To see that a principal minor \( A_S \) is PSD note that for any vector \( x \) that is supported only on the set \( S \),
\[
x^T A x = x^T S A_S x_S
\]
where by \( x_S \) we mean the vector \( x \) restricted to the set \( S \). The LHS in the above is nonnegative because \( A \) is PSD, so the RHS is nonnegative, so \( A_S \) is PSD.

b) Show that a PSD matrix is not necessarily totally positive.

The Laplacian of a graph is PSD but not totally positive as it has negative entries.

c) Use the Cauchy-Binet formula to show that for any pair of totally positive matrices \( A, B \in \mathbb{R}^{n \times n} \), \( AB \) is also totally positive.

Fix a minor \((AB)_{S,T}\) where \(|S| = |T| = k\). Then, by Couchy-Binet,
\[
\det((AB)_{S,T}) = \det(A_{S,[n]}, B_{[n],T}) = \sum_{S' \subseteq [n]} \det(A_{S,S'}, B_{S',T}) = \sum_{S' \subseteq [n]} \det(A_{S,S'}) \det(B_{S',T}) \geq 0.
\]

Note that \( A_{S,[n]} \) is the rows of \( A \) corresponding to indices in \( S \). In the last inequality we use that \( A, B \) are totally positive matrices.

5) Say \( G = (V,E) \) and \( H = (V,E') \) are unweighted graphs on the same vertex set \( V \).

a) show that if \( L_G \preceq L_H \), then for any \( S \subseteq V \),
\[
|E(S, \bar{S})| \leq |E'(S, \bar{S})|,
\]
where \( E(S, \bar{S}), E'(S, \bar{S}) \) are the sets of edges in the cut \((S, \bar{S})\) in \( G, H \) respectively.

For a set \( S \subseteq V \) let \( 1_S \) be the indicator vector of the set \( S \). Since \( L_G \preceq L_H \),
\[
1_S^T L_G 1_S \leq 1_S^T L_H 1_S.
\]
But, for any graph $G$,
\[
\mathbf{1}_S^T L_G \mathbf{1}_S = \sum_{u \sim v} (\mathbf{1}_S(u) - \mathbf{1}_S(v))^2
= \sum_{u \sim v} I_{[u \in S, v \notin S \text{ or } v \in S, u \notin S]}
= |E(S, \overline{S})|.
\]

b) Show that the converse of the above statement is not necessarily true, i.e., construct $G, H$ such that $|E(S, \overline{S})| \leq |E'(S, \overline{S})|$ for all $S \subseteq V$ but $L_G \not\preceq L_H$.

Bonus point: Construct an example where $|E(S, \overline{S})| \leq |E'(S, \overline{S})|$ for all $S \subseteq V$ but $L_G \not\preceq \Omega(n)L_H$.

The idea is to use long paths. On the long paths we can define vectors where $x^T L_G x$ are very small by slowly changing the value of $x$ along the paths. This is also closely related to the performance of Cheeger inequalities on graphs similar to cycle. We will talk about Cheeger inequalities later in the course.

Let $G$ be a complete graph where the weight of each edge is $1/n$, and let $H$ be a path of length $n$ with edges $(i, i + 1)$ for all $1 \leq i \leq n - 1$. Then, for any set $S$ say of size $|S| = k$,
\[
|E(S, \overline{S})| = \frac{k(n - k)}{n^2} \leq 1 \leq |E'(S, \overline{S})|.
\]

Now, let $x(i) = i$. Then,
\[
x^T L_H x = \sum_{i=1}^{n-1} (x(i + 1) - x(i))^2 = n.
\]

While,
\[
x^T L_G x \geq \Omega(n^2).
\]

So,
\[
L_G \not\preceq \Omega(n)L_H.
\]