You are free to collaborate to solve these assignments.

Problem 1) Show that for any matrix $A$ with eigenvalues $\lambda_1, \ldots, \lambda_n$

$$\det(A) = \prod_{i=1}^{n} \lambda_i.$$  

It is easy to see that

$$\det(tI + A) = \prod_{i=1}^{n} (t + \lambda_i).$$

Letting $t = 0$ proves the claim.

Problem 2) Show that if $A, B$ are PSD then $\text{trace}(AB) \geq 0$.

Since $B \succeq 0$ we can write $B = C^\top C$ for some matrix $C \in \mathbb{R}^{n \times n}$. Therefore it enough to show

$$\text{trace}(AB) = \text{trace}(C^\top AC) \geq 0.$$  

We show a stronger claim that is $CAC^\top$ is a PSD matrix so all of its eigenvalues are nonnegative and so is the trace. For any vector $x$ we can write

$$x^\top CAC^\top x = y^\top Ay \geq 0,$$

where $y = C^\top x$, and the last inequality uses that $A \succeq 0$.

Problem 3) Show that the maximum degree of vertices of a uniform spanning tree of a complete graph is $\Theta(\log(n) / \log \log(n))$.

Hint: Use the Prüfer code, http://en.wikipedia.org/wiki/Prüfer_sequence We can use the Chernoff bound to show that the degree of every very is at most $O(\log(n) / \log \log(n))$ using the negative correlation between the edges and Panconesi, Srinivasan theorem that we proved in Lecture 3 with probability $1 - 1/n^2$. By the union the degree of all vertices is at most $O(\log(n) / \log \log(n))$ with probability $1 - 1/n$.

We use the Prüfer code for the lower bound. First, recall that the Prüfer code gives a bijection between all spanning trees of a complete graph and all sequences of length $n - 2$ of numbers between $1, 2, \ldots, n$. Note that this gives a different proof that the number of spanning trees of a complete graph is exactly $n^{n-2}$. Say $a_1, \ldots, a_{n-2}$ is one these sequences. It is easy to see that the number of for every vertex $j$ the degree of $j$ in the spanning tree corresponding to the sequence is the number of occurrences of $j$ in this sequence plus 1. So, to find the maximum degree vertex it is enough to count the largest multiplicity of a number in a random sequence of length $n - 2$. This is essentially a balls and bins process where we throw $n - 2$ balls into $n$ bins. It is very well known that when we through $n$ balls into $n$ bins the largest number of balls in a bin is $\Omega(\log(n) / \log \log(n))$ with high probability. This proves the claim.

Problem 4) We say that an unweighted graph $G = (V, E)$ is $k$-edge connected if for any cut $(S, \overline{S})$, $|E(S, \overline{S})| \geq k$. Recall that by Menger’s theorem a graph is $k$-edge-connected if for any pair of vertices $s, t$ there are at least $k$ edge disjoint paths from $s$ to $t$. w
a) Suppose $G$ is connected and let $\epsilon = \max_{e \in E} \text{Reff}(e)$. Show that $G$ is $1/\epsilon$-edge connected.

This simply follows from the Nash-Williams theorem that we discussed in class. For any edge $(s, t)$, $\text{Reff}(s, t)$ is at least the inverse of the size of the minimum cut separating $s$ from $t$.

b) Show that the converse of the above is not true. Bonus point: construct a $k$-edge-connected graph with $n$ vertices where $\max_{e \in E} \text{Reff}(e) \geq \Omega(n)/k$.

Consider a graph $G$ with $n$ parallel paths, that is suppose there are $n$ parallel edges between $i, i+1$ for all $1 \leq i \leq n$. Also, add a single edge between 1, $n$. It is easy to see that this graph is $n$ connected, so $k = n$. But the effective resistance of the edge connecting 1, $n$ is about $1/2$.

c) Show that for any simple unweighted $k$-edge-connected graph $G = (V, E)$ (with no parallel edges),

$$\sum_{e \in E} \text{Reff}(e)^2 \leq O(n/k).$$

Hint: Show that for any simple unweighted graph $G$, and any edge $e = \{u, v\}$, $\text{Reff}(e) \geq \frac{1}{d(u) + 1} + \frac{1}{d(v) + 1}$ where $d(u), d(v)$ are the degrees of $u, v$ respectively.

First we prove the Hint. For any simple graph and any edge $(u, v)$ $\text{Reff}(u, v) \geq 1/(d(u) + 1) + 1/(d(v) + 1)$. To see that add a new node $w$ between $u, v$, that is remove $(u, v)$ and add $(u, w)$ and $(w, v)$ both with conductance 2. It is easy to see that this doesn’t change $\text{Reff}(u, v)$. Now, we use the Nash-Williams theorem to calculate $\text{Reff}(u, v)$ in the new graph,

$$\text{Reff}(u, v) \geq \frac{1}{\sum_{e \in \delta(v)} w(e)} + \frac{1}{\sum_{e \in \delta(u)} w(e)} = \frac{1}{2 + d(u) - 1} + \frac{1}{2 + d(v) - 1}.$$

Note that we are using the fact that $G$ is simple, so there is only one edge in $G$ connecting $u, v$.

For any edge $(u, v)$ let $\epsilon_{u,v}$ be the slack of this edge,

$$\epsilon_{u,v} = \text{Reff}(u, v) - \frac{1}{d(u) + 1} - \frac{1}{d(v) + 1}.$$

Note that by the Hint, $\epsilon_{u,v} \geq 0$, in addition $\epsilon_{u,v} \leq \text{Reff}(u, v) \leq 1$. First we upper bound $\sum_{u \sim v} \epsilon_{u,v}$, and we prove the claim. We start with a simpler claim that is we upper bound $\sum_{u \sim v} \epsilon_{u,v}^2$.

$$\sum_{u \sim v} \epsilon_{u,v} = \sum_{u \sim v} \left( \text{Reff}(u, v) - \frac{1}{d(u) + 1} - \frac{1}{d(v) + 1} \right)$$

$$= n - 1 - \sum_v \frac{d(v)}{d(v) + 1}$$

$$= -1 + \sum_v \frac{1}{d(v) + 1} \leq \frac{n}{k}.$$

The last inequality uses that $d(v) \geq k$ for all $v$. Since $0 \leq \epsilon_{u,v} \leq 1$ for all $u \sim v$, $\sum_{u \sim v} \epsilon_{u,v}^2 \leq n/k$.

This is just by noting that in the worst case $\epsilon_{u,v} = 1$ for $n/k$ of the edges and it is zero everywhere else.
Now, we are ready to prove the claim.

\[
\sum_{u \sim v} \text{Reff}(u, v)^2 = \sum_{u \sim v} \left( \frac{1}{d(u) + 1} + \frac{1}{d(v) + 1} + \epsilon_{u,v} \right)^2 \\
\leq 4 \sum_{u \sim v} \left( \frac{1}{d(u) + 1} \right)^2 + \left( \frac{1}{d(v) + 1} \right)^2 + \epsilon_{u,v}^2 \\
= 4 \sum_v d(v) \left( \frac{1}{d(v) + 1} \right)^2 + 4 \sum_{u \sim v} \epsilon_{u,v}^2 \\
\leq 4 \sum_v \frac{1}{d(v) + 1} + \frac{4n}{k} = O(n/k).
\]

Problem 5) In this exercise we want to show that the effective resistance is convex over the space of all positive definite matrices. This generalizes the convexity proof that we covered in the notes. For a PD matrix \( A \in \mathbb{R}^{V \times V} \) let

\[
\text{Reff}_A(s, t) = b^\top_A A^{-1} b_{s,t}.
\]

Show that for any two matrix \( A, B > 0 \),

\[
\text{Reff}_{(A+B)/2}(s, t) \leq \frac{1}{2} (\text{Reff}_A(s, t) + \text{Reff}_B(s, t)).
\]

Hint: Use the Schur complement which says that for a positive definite matrix \( A \),

\[
\begin{bmatrix} A & B \\ B^\top & C \end{bmatrix} \succeq 0 \text{ if and only if } C - B^\top A^{-1} B \succeq 0.
\]

For the proof we use the convexity of the space of PD (or PSD) matrices. Recall that if \( A, B \) are PD then so is \( A + B \). For any vector \( x \) we have,

\[
\frac{1}{2} \begin{bmatrix} A & x^\top \\ x & x^\top A^{-1} x \end{bmatrix} + \frac{1}{2} \begin{bmatrix} B & x^\top \\ x & x^\top B^{-1} x \end{bmatrix} = \begin{bmatrix} \frac{1}{2}A + \frac{1}{2}B & x^\top \\ x & \frac{1}{2}x^\top A^{-1} x + \frac{1}{2}x^\top B^{-1} x \end{bmatrix}.
\]

Since \( A, B > 0 \), by the Schur complement both of the matrices in the LHS are PSD. So, their average, i.e., the RHS matrix is also PSD. Since \( A, B > 0 \), \( \frac{1}{2}A + \frac{1}{2}B \geq 0 \). So, applying Schur complement to the RHS we get the claim.

Problem 6) Prove that the expected size of the diameter of a uniform spanning tree of a complete graph on \( n \) vertices is \( \Omega(\sqrt{n}) \).

Hint: Use the Aldous,Broder algorithm for sampling a uniform spanning tree.

Say \( X_1, \ldots, X_t \) are the first \( t \) steps of a random walk in a complete graph. By the algorithm of Aldous, Broder it is enough to show that for \( t = \Theta(\sqrt{n}) \) the with probability 0.99 all of the vertices of the sequence \( X_1, \ldots, X_t \) are distinct (in this case all of the edges \( (X_i, X_{i+1}) \) belong to the sampled tree and the diameter will be \( \Omega(\sqrt{n}) \). Now a random walk of length \( t \) in a complete graph is just a sequence of length \( t \) of numbers in the range \( 1, 2, \ldots, n \). It is easy to see that for a small enough constant \( c \), if \( t = c\sqrt{n} \) then all of the vertices of the sequence are distinct with probability 0.99.