Invariant Generation for Complexity Analysis of Python Programs

Solutions

Structural Induction

(1) Recall the following definitions:

\[
\begin{align*}
\mathrm{len}(\varepsilon) &= 0 \\
\mathrm{len}(wa) &= \mathrm{len}(w) + 1, \text{for } w \in \Sigma^*, a \in \Sigma \\
x \cdot \varepsilon &= x, \text{for } x \in \Sigma^* \\
x \cdot wa &= (x \cdot w)a, \text{for } x \in \Sigma^*, a \in \Sigma
\end{align*}
\]

Consider the following recursive definition:

\[
\begin{align*}
\mathrm{stutter}(\varepsilon) &= \varepsilon \\
\mathrm{stutter}(wa) &= \mathrm{stutter}(w) \cdot aa, \text{for } w \in \Sigma^*, a \in \Sigma
\end{align*}
\]

Prove that \(\mathrm{len}(\mathrm{stutter}(w)) = 2\mathrm{len}(w)\) for all \(w \in \Sigma^*\).

Solution: Let \(P(w)\) be “\(\mathrm{len}(\mathrm{stutter}(w)) = 2\mathrm{len}(w)\)” for all \(w \in \Sigma^*\). We go by structural induction.

Base Case. Note that \(\mathrm{len}(\mathrm{stutter}(\varepsilon)) = \mathrm{len}(\varepsilon) = 0 = 2 \cdot 0 = 2 \cdot \mathrm{len}(\varepsilon)\). So, \(P(\varepsilon)\) is true.

Induction Hypothesis. Suppose that \(P(w)\) is true for some \(w \in \Sigma^*\).

Induction Step. Note that

\[
\begin{align*}
\mathrm{len}(\mathrm{stutter}(wa)) &= \mathrm{len}(\mathrm{stutter}(w) \cdot aa) \quad \text{[By Definition of \(\mathrm{stutter}\)]} \\
&= \mathrm{len}((\mathrm{stutter}(w) \cdot a)a) \quad \text{[By Definition of \(\bullet\)]} \\
&= \mathrm{len}(\mathrm{stutter}(w) \cdot a) + 1 \quad \text{[By Definition of \(\mathrm{len}\)]} \\
&= \mathrm{len}(\mathrm{stutter}(w)a) + 1 \\
&= \mathrm{len}(\mathrm{stutter}(w)) + 1 + 1 \quad \text{[By Definition of \(\mathrm{len}\)]} \\
&= 2\mathrm{len}(w) + 2 \quad \text{[By IH]} \\
&= 2(\mathrm{len}(w) + 1) \\
&= 2(\mathrm{len}(wa)) \quad \text{[Definition of \(\mathrm{len}\)]}
\end{align*}
\]

Thus, \(P(w)\) is true for all \(w \in \Sigma^*\) by structural induction.
Regular Expressions

(1) Write a regular expression that matches base 10 numbers (e.g., there should be no leading zeroes).

Solution:

\[ 0 \cup ((1 \cup 2 \cup 3 \cup 4 \cup 5 \cup 6 \cup 7 \cup 8 \cup 9)(0 \cup 1 \cup 2 \cup 3 \cup 4 \cup 5 \cup 6 \cup 7 \cup 8 \cup 9)^*) \]

(2) Write a regular expression that matches e-mail addresses.

Solution:

(3) Write a regular expression that matches all base-3 numbers that are divisible by 3.

Solution:

\[ (0 \cup 1 \cup 2)^*0 \]

(4) Write a regular expression that matches all binary strings that contain the substring "111".

Solution:

\[ (0 \cup 1)^*111(0 \cup 1)^* \]

(5) Write a regular expression that matches all binary strings that do not contain the substring "000".

Solution:

\[ (01 \cup 001 \cup 1)^*(0 \cup 00 \cup \varepsilon) \]

(6) Write a regular expression that matches all binary strings that contain the substring "111", but not the substring "000".

Solution:

\[ (01 \cup 001 \cup 1)^*(0 \cup 00 \cup \varepsilon)111(01 \cup 001 \cup 1)^*(0 \cup 00 \cup \varepsilon) \]

Induction

(1) Prove that every amount of postage that is at least 12 cents can be made from a combination of 4-cent and 5-cent stamps.

(2) Show that a set of \( n \) elements has \( 2^n \) subsets.

Strong Induction

(1) Let the “Tribonacci” sequence be defined by \( T_1 = T_2 = T_3 = 1 \) and \( T_n = T_{n-3} + T_{n-2} + T_{n-3} \) for \( n \geq 4 \). Prove that \( T^n < 2^n \) for all positive integers \( n \).
Number of subsets: Show that a set of $n$ elements has $2^n$ subsets.

Proof: We will prove by induction that, for all $n \in \mathbb{Z}_+$, the following holds:

\[ P(n) \quad \text{Any set of } n \text{ elements has } 2^n \text{ subsets.} \]

Base case: Since any 1-element set has 2 subsets, namely the empty set and the set itself, and $2^1 = 2$, the statement $P(n)$ is true for $n = 1$.

Induction step:

- Let $k \in \mathbb{Z}_+$ be given and suppose $P(k)$ is true, i.e., that any $k$-element set has $2^k$ subsets.
- We seek to show that $P(k + 1)$ is true as well, i.e., that any $(k + 1)$-element set has $2^{k+1}$ subsets.
- Let $A$ be a set with $(k + 1)$ elements.
- Let $a$ be an element of $A$, and let $A' = A - \{a\}$ (so that $A'$ is a set with $k$ elements).
- We classify the subsets of $A$ into two types: (I) subsets that do not contain $a$, and (II) subsets that do contain $a$.
- The subsets of type (I) are exactly the subsets of the set $A'$. Since $A'$ has $k$ elements, the induction hypothesis can be applied to this set and we get that there are $2^k$ subsets of type (I).
- The subsets of type (II) are exactly the sets of the form $B = B' \cup \{a\}$, where $B'$ is a subset of $A'$. By the induction hypothesis there are $2^k$ such sets $B'$, and hence $2^k$ subsets of type (II).
- Since there are $2^k$ subsets of each of the two types, the total number of subsets of $A$ is $2^k + 2^k = 2^{k+1}$.
- Since $A$ was an arbitrary $(k + 1)$-element set, we have proved that any $(k + 1)$-element set has $2^{k+1}$ subsets. Thus $P(k + 1)$ is true, completing the induction step.

Conclusion: By the principle of induction, $P(n)$ is true for all $n \in \mathbb{Z}_+$. 
Let the “Tribonacci sequence” be defined by $T_1 = T_2 = T_3 = 1$ and $T_n = T_{n-1} + T_{n-2} + T_{n-3}$ for $n \geq 4$. Prove that $T_n < 2^n$ for all $n \in \mathbb{Z}_+$.

**Proof:** We will prove by strong induction that, for all $n \in \mathbb{Z}_+$,

$$T_n < 2^n$$

Base case: We will need to check (*) directly for $n = 1, 2, 3$ since the induction step (below) is only valid when $k \geq 3$. For $n = 1, 2, 3$, $T_n$ is equal to 1, whereas the right-hand side of (*) is equal to $2^1 = 2$, $2^2 = 4$, and $2^3 = 8$, respectively. Thus, (*) holds for $n = 1, 2, 3$.

Induction step: Let $k \geq 3$ be given and suppose (*) is true for all $n = 1, 2, \ldots, k$. Then

$$T_{k+1} = T_k + T_{k-1} + T_{k-2} \quad \text{(by recurrence for } T_n)$$

$$< 2^k + 2^{k-1} + 2^{k-2} \quad \text{(by strong ind. hyp. (*) with } n = k, k-1, \text{ and } k-2)$$

$$= 2^k + 2^{k-1} \left( \frac{1}{2} + \frac{1}{4} + \frac{1}{8} \right)$$

$$= 2^k + \frac{7}{8}$$

$$< 2^{k+1}.$$ 

Thus, (*) holds for $n = k + 1$, and the proof of the induction step is complete.

**Conclusion:** By the principle of strong induction, it follows that (*) is true for all $n \in \mathbb{Z}_+$. 