Oblivious Bounds on the Probability of Boolean Functions

WOLFGANG GATTERBAUER, Carnegie Mellon University
DAN SUCIU, University of Washington

This paper develops upper and lower bounds for the probability of Boolean functions by treating multiple occurrences of variables as independent and assigning them new individual probabilities. We call this approach *dissociation* and give an exact characterization of *optimal oblivious bounds*, i.e. when the new probabilities are chosen independent of the probabilities of all other variables. Our motivation comes from the *weighted model counting* problem (or, equivalently, the problem of computing the probability of a Boolean function), which is #P-hard in general. By performing several dissociations, one can transform a Boolean formula whose probability is difficult to compute, into one whose probability is easy to compute, and which is guaranteed to provide an upper or lower bound, respectively, on the probability of the original formula. Our new bounds shed light on the connection between previous relaxation-based and model-based approximations in the literature and unify them as concrete choices in a larger design space. We also show how our theory allows a standard relational database management systems (DBMS) to both upper and lower bound hard probabilistic queries.

Categories and Subject Descriptors: G.3 [**Probability and Statistics**]; H.2.m [**Database Management**]: Miscellaneous; I.1.1 [**Symbolic and algebraic manipulation**]: Expressions and Their Representation

General Terms: Algorithms, Theory

Additional Key Words and Phrases: Probabilistic databases, Weighted model counting, Boolean expressions, Oblivious approximations, Relaxation

1. INTRODUCTION

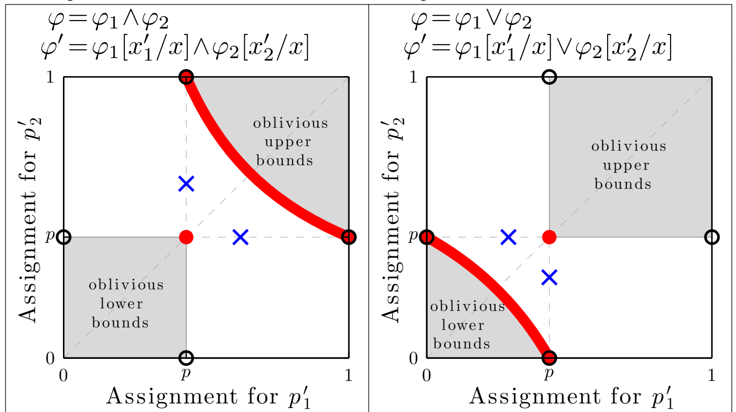
Query evaluation on probabilistic databases is based on weighted model counting for positive Boolean expressions. Since model counting is #P-hard in general, today's probabilistic database systems evaluate queries using one of the following three approaches: (1) incomplete approaches identify tractable cases (e.g., read-once formulas) either at the query-level [Dalvi and Suciu 2007; Dalvi et al. 2010] or the data-level [Olteanu and Huang 2008; Sen et al. 2010]; (2) exact approaches apply exact probabilistic inference, such as repeated application of Shannon expansion [Olteanu et al. 2009] or tree-width based decompositions [Jha et al. 2010]; and (3) approximate approaches either apply general purpose sampling methods [Jampani et al. 2008; Kennedy and Koch 2010; Re et al. 2007] or approximate the number of models of the Boolean lineage expression [Olteanu et al. 2010; Fink and Olteanu 2011].

This paper provides a new *algebraic framework* for approximating the probability of positive Boolean expressions. While our method was motivated by query evaluation on probabilistic databases, it is more general and applies to all problems that rely on weighted model counting, e.g., general probabilistic inference in graphical mod-

Authors' addresses: W. Gatterbauer (corresponding author), Tepper School of Business, Carnegie Mellon University; email: gatt@cmu.edu; D. Suciu, Computer Science and Engineering Department, University of Washington, Seattle.

DOI 10.1145/0000000.0000000 http://doi.acm.org/10.1145/0000000.0000000

	Conjunctive dissociation	Disjunctive dissociation



		Conjunctive dissociation	Disjunctive dissociation
Optimal oblivious bounds	Upper	$p_1' \cdot p_2' = p$	$p_1' = p \;\; p_2' = p$
	Lower	$p_1' = p \;\; p_2' = p$	$(1-p_1')\cdot(1-p_2') = 1-p$
Model-based bounds	Upper	$p_1' = p \;\; p_2' = 1$ (optimal)	$p_1' = p \;\; p_2' = 1$ (non-optimal)
	Lower	$p_1' = p \;\; p_2' = 0$ (non-optimal)	$p_1' = p \;\; p_2' = 0$ (optimal)
Relaxation & Comp.		$p_1' = p \;\; p_2' = \mathbb{P}[x \mid \varphi_1]$	$p_1' = p \;\; p_2' = \mathbb{P}[x \mid \neg\varphi_1]$

Fig. 1. Dissociation as framework that allows to determine optimal oblivious upper and lower bounds for the probabilities $\mathbf{p}' = \langle p_1', p_2' \rangle$ of dissociated variables. Oblivious here means that we assign new values after looking at only a limited scope of the expression. *Model-based* upper conjunctive and lower disjunctive bounds are obliviously optimal (they fall on the red line of optimal assignments), whereas lower conjunctive and upper disjunctive are not. *Relaxation & Compensation* is a form of dissociation which is not oblivious (p_2 is calculated with knowledge of φ_1) and does not, in general, guarantee to give an upper or lower bound.

els [Chavira and Darwiche 2008].[1] An important aspect of our method is that it is *not model-based* in the traditional sense. Instead, it enlarges the original variable space by treating multiple occurrences of variables as independent and assigning them new individual probabilities. We call this approach *dissociation*[2] and explain where existing relaxation-based and model-based approximations fit into this larger space of approximations. We characterize probability assignments that lead to guaranteed upper or lower bounds on the original expression and identify the best possible *oblivious* bounds, i.e. after looking at only a limited scope of the expression. We prove that for every model-based bound there is always a dissociation-based bound that is *as good or better*. And we illustrate how a standard relational DBMS can both upper and lower bound hard probabilistic conjunctive queries without self-joins with appropriate SQL queries that use dissociation in a query-centric way.

We briefly discuss our results: We want to compute the probability $\mathbb{P}[\varphi]$ of a Boolean expression φ when each of its Boolean variables x_i is set independently to true with some given probability $p_i = \mathbb{P}[x_i]$. Computing $\mathbb{P}[\varphi]$ is known to be #P-hard in general [Valiant 1979] and remains hard to even approximate [Roth 1996]. Our approach is to approximate $\mathbb{P}[\varphi]$ with $\mathbb{P}[\varphi']$ that is easier to compute. The new formula φ' is derived from φ through a sequence of *dissociation steps*, where each step replaces d distinct occurrences of some variable x in φ with d fresh variables $x_1', x_2', \ldots x_d'$. Thus,

[1] Note that *weighted model counting* is essentially the same problem as computing the probability $\mathbb{P}[\varphi]$ of a Boolean expression φ. Each truth assignment of the Boolean variables corresponds to one model whose weight is the probability of this truth assignment. Weighted model counting then asks for the sum of the weights of all satisfying assignments.

[2] *Dissociation* is the breaking of an existing association between related, but not necessarily identical items.

after applying dissociation repeatedly, we transform φ into another expression φ' and approximate $\mathbb{P}[\varphi]$ with $\mathbb{P}[\varphi']$. The question that we address in this paper is: how should we set the probabilities of the dissociated variables x'_i in order to ensure that $\mathbb{P}[\varphi']$ is a good approximation of $\mathbb{P}[\varphi]$? In particular, we seek conditions under which φ' is guaranteed to be either an upper bound $\mathbb{P}[\varphi'] \geq \mathbb{P}[\varphi]$ or a lower bound $\mathbb{P}[\varphi'] \leq \mathbb{P}[\varphi]$.

Our main result can be summarized as follows: Suppose that x occurs positively in φ. Dissociate it into two variables x'_1 and x'_2, such that the dissociated formula is $\varphi' = \varphi'_1 \wedge \varphi'_2$, and x'_1 occurs only in φ'_1, while x'_2 occurs only in φ'_2; in other words, $\varphi \equiv \varphi'_1[x'_1/x] \wedge \varphi'_2[x'_2/x]$. Let $p = \mathbb{P}[x]$, $p'_1 = \mathbb{P}[x'_1]$, $p'_2 = \mathbb{P}[x'_2]$ be their probabilities. Then (1) $\mathbb{P}[\varphi'] \geq \mathbb{P}[\varphi]$ iff $p'_1 \cdot p'_2 \geq p$, and (2) $\mathbb{P}[\varphi'] \leq \mathbb{P}[\varphi]$ iff $p'_1 \leq p$ and $p'_2 \leq p$. In particular, the best upper bounds are obtained by choosing any p'_1, p'_2 that satisfy $p'_1 \cdot p'_2 = p$, and the best lower bound is obtained by setting $p'_1 = p'_2 = p$. The "only if" direction holds assuming φ' satisfies certain mild conditions (e.g., it should not be redundant), and under the assumption that p'_1, p'_2 are chosen *obliviously*, i.e. they are functions only of $p = \mathbb{P}[x]$ and independent of the probabilities of all other variables. This restriction to oblivious probabilities guarantees the computation of the probabilities p'_1, p'_2 to be very simple.[3] Our result extends immediately to the case when the variable x is dissociated into several variables $x'_1, x'_2, \ldots, x'_d$, and also extends (with appropriate changes) to the case when the expressions containing the dissociated variables are separated by $\vee$ rather than $\wedge$ (see Fig. 1).

Example 1.1 (*2CNF Dissociation*). For a simple illustration of our main result, consider a Positive-Partite-2CNF expression with $|E|$ clauses

$$\varphi = \bigwedge_{(i,j)\in E} (x_i \vee y_j) \tag{1}$$

for which calculating its probability is already #P-hard [Provan and Ball 1983]. If we dissociate all occurrences of all m variables x_i, then the expression becomes:

$$\varphi' = \bigwedge_{(i,j)\in E} (x'_{i,j} \vee y_j) \tag{2}$$

which is equivalent to $\bigwedge_j \big(y_j \vee \bigwedge_{i,j} x'_{i,j}\big)$. This is a read-once expression whose probability can always be computed in PTIME [Gurvich 1977]. Our main result implies the following: Let $p_i = \mathbb{P}[x_i]$, $i \in [m]$ be the probabilities of the original variables and denote $p'_{i,j} = \mathbb{P}[x'_{i,j}]$ the probabilities of the fresh variables. Then (1) if $\forall i \in [m] : p'_{i,j_1} \cdot p'_{i,j_2} \cdots p'_{i,j_{d_i}} = p_i$, then φ' is an upper bound ($\mathbb{P}[\varphi'] \geq \mathbb{P}[\varphi]$); (2) if $\forall i \in [m] : p'_{i,j_1} = p'_{i,j_2} = \ldots = p'_{i,j_{d_i}} = p_i$, then φ' is a lower bound ($\mathbb{P}[\varphi'] \leq \mathbb{P}[\varphi]$). Furthermore, these are the best possible *oblivious bounds*, i.e. where $p'_{i,j}$ depends only on $p_i = \mathbb{P}[x_i]$ and is chosen independently of other variables in φ. ■

We now explain how dissociation generalizes two other approximation methods in the literature (Fig. 1 gives a high-level summary and Sect. 5 the formal details).

Relaxation & Compensation. This is a framework by Choi and Darwiche [2009; 2010] for approximate probabilistic inference in graphical models. The approach performs exact inference in an approximate model that is obtained by relaxing equiv-

[3] Our usage of the term *oblivious* is inspired by the notion of oblivious routing algorithms [Valiant 1982] which use only local information and which can therefore be implemented very efficiently. Similarly, our *oblivious* framework forces p'_1, p'_2 to be computed only as a function of p, without access to the rest of φ. One can always find values p'_1, p'_2 for which $\mathbb{P}[\varphi] = \mathbb{P}[\varphi']$. However, to find those value in general, one has to first compute $q = \mathbb{P}[\varphi]$, then find appropriate values p'_1, p'_2 for which the equality $\mathbb{P}[\varphi'] = q$ holds. This is not practical, since our goal is to compute q in the first place.

alence constraints in the original model, i.e. by removing edges. The framework allows one to improve the resulting approximations by compensating for the relaxed constraints. In the particular case of a conjunctive Boolean formula $\varphi = \varphi_1 \wedge \varphi_2$, *relaxation* refers to substituting any variable x that occurs in both φ_1 and φ_2 with two fresh variables x'_1 in φ_1 and x'_2 in φ_2. *Compensation* refers to then setting their probabilities $p'_1 = \mathbb{P}[x'_1]$ and $p'_2 = \mathbb{P}[x'_2]$ to $p'_1 = p$ and $p'_2 = \mathbb{P}[x|\varphi_1]$. This new probability assignment is justified by the fact that, if x is the only variable shared by φ_1 and φ_2, then compensation ensures that $\mathbb{P}[\varphi'] = \mathbb{P}[\varphi]$ (we will show this claim in Prop. 5.1). In general, however, φ_1, φ_2 have more than one variable in common, and in this case we have $\mathbb{P}[\varphi'] \neq \mathbb{P}[\varphi]$ for the same compensation. Thus in general, compensation is applied as a heuristics. Furthermore, it is then not known whether compensation provides an upper or lower bound.

Indeed, let $p'_1 = p$, $p'_2 = \mathbb{P}[x|\varphi_1]$ be the probabilities set by the compensation method. Recall that our condition for $\mathbb{P}[\varphi']$ to be an upper bound is $p'_1 \cdot p'_2 \geq p$, but we have $p'_1 \cdot p'_2 = p \cdot \mathbb{P}[x|\varphi_1] \leq p$. Thus, the compensation method does not satisfy our oblivious upper bound condition. Similarly, because of $p'_1 = p$ and $p'_2 \geq p$, these values fail to satisfy our oblivious lower bound condition. Thus, relaxation is neither a guaranteed upper bound, nor a guaranteed lower bound. In fact, relaxation is not oblivious at all (since p'_2 is computed from the probabilities of all variables, not just $\mathbb{P}[x]$). This enables it to be an exact approximation in the special case of a single shared variable, but fails to guarantee any bounds in general.

Model-based approximations. Another technique for approximation described by Fink and Olteanu [2011] is to replace φ with another expression whose set of models is either a subset or superset of those of φ. Equivalently, the upper bound is a formula φ_U such that $\varphi \Rightarrow \varphi_U$, and the lower bound is φ_L such that $\varphi_L \Rightarrow \varphi$. We show in this paper, that if φ is a positive Boolean formula, then all upper and lower model-based bounds can be obtained by repeated dissociation: the model-based upper bound is obtained by repeatedly setting probabilities of dissociated variables to 1, and the model-based lower bound by setting the probabilities to 0. While the thus generated model-based upper bounds for conjunctive expressions correspond to optimal oblivious dissociation bounds, the model-based lower bounds for conjunctive expressions are *not optimal* and can always be improved by dissociation.

Indeed, consider first the upper bound for conjunctions: the implication $\varphi \Rightarrow \varphi_U$ holds iff there exists a formula φ_1 such that $\varphi \equiv \varphi_1 \wedge \varphi_U$.[4] Pick a variable x, denote $p = \mathbb{P}[x]$ its probability, dissociate it into x'_1 in φ_1 and x'_2 in φ_U, and set their probabilities as $p'_1 = 1$ and $p'_2 = p$. Thus, φ_U remains unchanged (except for the renaming of x to x'_2), while in φ_1 we have set $x_1 = 1$. By repeating this process, we eventually transform φ_1 into true (Recall that our formula is monotone). Thus, model-based upper bounds are obtained by repeated dissociation and setting $p'_1 = 1$ and $p'_2 = p$ at each step. Our results show that this is only one of many oblivious upper bounds as any choices with $p'_1 p'_2 \geq p$ lead to an oblivious upper bound for conjunctive dissociations.

Consider now the lower bound: the implication $\varphi_L \Rightarrow \varphi$ is equivalent to $\varphi_L \equiv \varphi \wedge \varphi_2$. Then there is a sequence of finitely many conjunctive dissociation steps, which transforms φ into $\varphi \wedge \varphi_2$ and thus into φ_L. At each step, a variable x is dissociated into x'_1 and x'_2, and their probabilities are set to $p'_1 = p$ and $p'_2 = 0$, respectively.[5] According to our result, this choice is not optimal: instead one obtains a tighter bound by also setting $p'_2 = p$, which no longer corresponds to a model-based lower bound.

[4] Fink and Olteanu [2011] describe their approach for approximating DNF expressions only. However, the idea of model-based bounds applies equally well to arbitrary Boolean expressions, including those in CNF.
[5] The details here are more involved and are given in detail in Sect. 5.2.

Thus, *model-based lower bounds for conjunctive expressions are not optimal* and can always be improved by using dissociation.

Our dual result states the following for the case when the two formulas are connected with disjunction $\vee$ instead of conjunction $\wedge$: (1) the dissociation is an upper bound iff $p'_1 \geq p$ and $p'_2 \geq p$, and (2) it is a lower bound iff $(1-p'_1)(1-p'_2) \geq 1-p$. We can see that model-based approximation gives an optimal lower bound for disjunctions, because $(1-p'_1)(1-p'_2) = 1 \cdot (1-p) = 1-p$, however non-optimal upper bounds. Example 7.2 illustrates this asymmetry and the possible improvement through dissociation with a detailed simulation-based example.

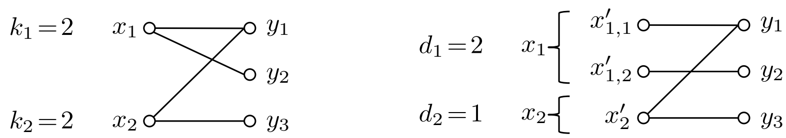
Bounds for hard probabilistic queries. Query evaluation on probabilistic databases reduces to the problem of computing the probability of its lineage expression which is a a monotone, k-partite Boolean DNF where k is fixed by the number of joins in the query. Computing the probability of the lineage is known to be #P-hard for some queries [Dalvi and Suciu 2007], hence we are interested in approximating these probabilities by computing dissociated Boolean expressions for the lineage. We have previously shown in [Gatterbauer et al. 2010] that every query plan for a query corresponds to one possible dissociation for its lineage expression. The results in this paper show how to best set the probabilities for the dissociated expressions in order to obtain both upper bounds and lower bounds. We further show that *all the computation* can be pushed inside a standard relational database engine with the help of SQL queries that use User-Defined-Aggregates and views that replace the probabilities of input tuples with their optimal symmetric lower bound explained in Sect. 4.4. We illustrate this approach in Sect. 6 and validate it on TPC/H data in Sect. 7.5.

Main contributions. (1) We introduce an algebraic framework for approximating the probability of Boolean functions by treating multiple occurrences of variables as independent and assigning them new individual probabilities. We call this approach *dissociation*; (2) we determine the optimal upper and lower bounds for conjunctive and disjunctive dissociations under the assumption of *oblivious* value assignments; (3) we show how existing relaxation-based and model-based approximations fit into the larger design space of dissociations, and show that for every model-based bound there is at least one dissociation-based bound which is as good or tighter; (4) we apply our general framework to both upper *and lower bound* hard probabilistic self-join free conjunctive queries in guaranteed PTIME by translating the query into a sequence of standard SQL queries; and (5) we illustrate and evaluate with several detailed examples the application of this technique. Note that this paper does *not* address the algorithmic complexities in determining alternative dissociations, in general.

Outline. Section 2 starts with some notational background, and Sect. 3 formally defines dissociation. Section 4 contains our main results on optimal oblivious bounds. Section 5 formalizes the connection between relaxation, model-based bounds and dissociation, and shows how both previous approaches can be unified under the framework of dissociation. Section 6 applies our framework to derive upper and lower bounds for hard probabilistic queries with standard relational database management systems. Section 7 gives detailed illustrations on the application of dissociation and oblivious bounds. Finally, Sect. 8 relates to previous work before Sect. 9 concludes.

2. GENERAL NOTATIONS AND CONVENTIONS

We use $[m]$ as short notation for $\{1, \ldots, m\}$, use the bar sign for the complement of an event or probability (e.g., $\bar{x} = \neg x$, and $\bar{p} = 1-p$), and use a bold notation for sets (e.g., $\mathbf{s} \subseteq [m]$) or vectors (e.g., $\mathbf{x} = \langle x_1, \ldots, x_m \rangle$) alike. We assume a set $\mathbf{x}$ of independent Boolean random variables, and assign to each variable x_i a primitive event which is true with probability $p_i = \mathbb{P}[x_i]$. We do not formally distinguish between the variable x_i and the event x_i that it is true. By default, all primitive events are assumed to be



(a) f (b) f'

Fig. 2. Example 3.2. (a): Bipartite primal graph for CNF representing f. (b): A dissociation f' where variable x_1 appearing $k_1 = 2$ times in f is dissociated into (replaced by) $d_1 = 2$ fresh variables in f'.

independent (e.g., $\mathbb{P}[x_1 \wedge x_2] = p_1 p_2$). We are interested in bounding the probability $\mathbb{P}[f]$ of a Boolean function f, i.e. the probability that the function is true if each of the variables is independently true or false with given probabilities. When no confusion arises, we blur the distinction between a Boolean expression φ and the Boolean function f_φ it represents (cf. [Crama and Hammer 2011, Sect. 1.2]) and write $\mathbb{P}[\varphi]$ instead of $\mathbb{P}[f_\varphi]$. We also use the words formula and expression interchangeably. We write $f(\mathbf{x})$ to indicate that $\mathbf{x}$ is the set of primitive events appearing in the function f, and $f[\mathbf{x}_1/\mathbf{x}]$ to indicate that $\mathbf{x}_1$ is substituted for $\mathbf{x}$ in f. We often omit the operator $\wedge$ and denote conjunction by mere juxtaposition instead.

3. DISSOCIATION OF BOOLEAN FUNCTIONS AND EXPRESSIONS

We define here dissociation formally. Let $f(\mathbf{x}, \mathbf{y})$ and $f'(\mathbf{x}', \mathbf{y})$ be two Boolean functions, where $\mathbf{x}, \mathbf{x}', \mathbf{y}$ are three disjoint sets of variables. Denote $|\mathbf{x}| = m$, $|\mathbf{x}'| = m'$, and $|\mathbf{y}| = n$. We restrict f and f' to be *positive* in $\mathbf{x}$ and $\mathbf{x}'$, respectively [Crama and Hammer 2011, Def. 1.24].

Definition 3.1 (*Dissociation*). We call a function f' a dissociation of f if there exists a substitution $\theta : \mathbf{x}' \to \mathbf{x}$ s.t. $f'[\theta] = f$.

Example 3.2 (*CNF Dissociation*). Consider two functions f and f' given by CNF expressions

$$f = (x_1 \vee y_1)(x_1 \vee y_2)(x_2 \vee y_1)(x_2 \vee y_3)$$
$$f' = (x'_{1,1} \vee y_1)(x'_{1,2} \vee y_2)(x'_2 \vee y_1)(x'_2 \vee y_3)$$

Then f' is a dissociation of f as $f'[\theta] = f$ for the substitution $\theta = \{(x'_{1,1}, x_1), (x'_{1,2}, x_1), (x'_2, x_2)\}$. Figure 2 shows the CNF expressions' primal graphs.[6] ■

In practice, to find a dissociation for a function $f(\mathbf{x}, \mathbf{y})$, one proceeds like this: Choose any expression $\varphi(\mathbf{x}, \mathbf{y})$ for f and thus $f = f_\varphi$. Replace the k_i distinct occurrences of variables x_i in φ with d_i fresh variables $x'_{i,1}, x'_{i,2}, \ldots, x'_{i,d_i}$, with $d_i \leq k_i$. The resulting expression φ' represents a function f' that is a dissociation of f. Notice that we may obtain different dissociations by deciding for which occurrences of x_i to use distinct fresh variables, and for which occurrences to use the same variable. We may further obtain more dissociations by starting with different, equivalent expressions φ for the function f. In fact, we may construct infinitely many dissociations this way. We also note that every dissociation of f can be obtained through the process outlined here. Indeed, let $f'(\mathbf{x}', \mathbf{y})$ be a dissociation of $f(\mathbf{x}, \mathbf{y})$ according to Definition 3.1, and let θ be the substitution for which $f'[\theta] = f$. Then, if φ' is any expression representing f', the expression $\varphi = \varphi'[\theta]$ represents f. We can thus apply the described dissociation process to a certain expression φ and obtain f'.

[6]The *primal graph* of a CNF (DNF) has one node for each variable and one edge for each pair of variables that co-occur in some clause (conjunct). This concept originates in constraint satisfaction and it is also varyingly called co-occurrence graph or variable interaction graph [Crama and Hammer 2011].

Example 3.3 (*Alternative Dissociations*). Consider the two expressions:

$$\varphi = (x \vee y_1)(x \vee y_2)(x \vee y_3)(y_4 \vee y_5)$$
$$\psi = xy_4 \vee xy_5 \vee y_1y_2y_3y_4 \vee y_1y_2y_3y_5$$

Both are equivalent ($\varphi \equiv \psi$) and thus represent the same Boolean function ($f_\varphi = f_\psi$). Yet each leads to a quite different dissociation in the variable x:

$$\varphi' = (x'_1 \vee y_1)(x'_2 \vee y_2)(x'_3 \vee y_3)(y_4 \vee y_5)$$
$$\psi' = x'_1y_4 \vee x'_2y_5 \vee y_1y_2y_3y_4 \vee y_1y_2y_3y_5$$

Here, φ' and ψ' represent different functions ($f_{\varphi'} \neq f_{\psi'}$) and are both dissociations of f for the substitutions $\theta_1 = \{(x'_1, x), (x'_2, x), (x'_3, x)\}$ and $\theta_2 = \{(x'_1, x), (x'_2, x)\}$, respectively. ■

Example 3.4 (*More alternative Dissociations*). Consider the AND-function $f(x, y) = xy$. It can be represented by the expressions xxy, or $xxxy$, etc., leading to the dissociations $x'_1x'_2y$, or $x'_1x'_2x'_3y$, etc. For even more dissociations, represent f using the expression $(x \vee x)y \vee xy$, which can dissociate to $(x'_1 \vee x'_2)y \vee x'_3y$, or $(x'_1 \vee x'_2)y \vee x'_1y$, etc. Note that several occurrences of a variable can be replaced by the same new variables in the dissociated expression. ■

4. OBLIVIOUS BOUNDS FOR DISSOCIATED EVENT EXPRESSIONS

Throughout this section, we fix two Boolean functions $f(\mathbf{x}, \mathbf{y})$ and $f'(\mathbf{x}', \mathbf{y})$ such that f' is a dissociation of f. We are given the probabilities $\mathbf{p} = \mathbb{P}[\mathbf{x}]$ and $\mathbf{q} = \mathbb{P}[\mathbf{y}]$. Our goal is to find probabilities $\mathbf{p}' = \mathbb{P}[\mathbf{x}']$ of the dissociated variables so that $\mathbb{P}[f']$ is an upper or lower bound for $\mathbb{P}[f]$. We first define oblivious bounds (Sect. 4.1), then characterize them, in general, through valuations (Sect. 4.2) and, in particular, for conjunctive and disjunctive dissociations (Sect. 4.3), then derive optimal bounds (Sect. 4.4), and end with illustrated examples for CNF and DNF dissociations (Sect. 4.5).

4.1. Definition of Oblivious Bounds

We use the subscript notation $\mathbb{P}_{\mathbf{p},\mathbf{q}}[f]$ and $\mathbb{P}_{\mathbf{p}',\mathbf{q}}[f']$ to emphasize that the probability space is defined by the probabilities $\mathbf{p} = \langle p_1, p_2, \ldots\rangle$, $\mathbf{q} = \langle q_1, q_2, \ldots\rangle$, and $\mathbf{p}' = \langle p'_1, p'_2, \ldots\rangle$, respectively. Given $\mathbf{p}$ and $\mathbf{q}$, our goal is thus to find $\mathbf{p}'$ such that $\mathbb{P}_{\mathbf{p}',\mathbf{q}}[f'] \geq \mathbb{P}_{\mathbf{p},\mathbf{q}}[f]$ or $\mathbb{P}_{\mathbf{p}',\mathbf{q}}[f'] \leq \mathbb{P}_{\mathbf{p},\mathbf{q}}[f]$.

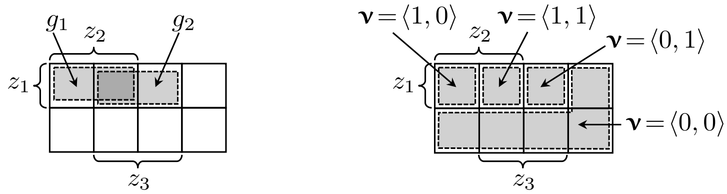
Definition 4.1 (*Oblivious Bounds*). Let f' be a dissociation of f and $\mathbf{p} = \mathbb{P}[\mathbf{x}]$. We call $\mathbf{p}'$ an *oblivious upper bound* for $\mathbf{p}$ and dissociation f' of f iff $\forall \mathbf{q} : \mathbb{P}_{\mathbf{p}',\mathbf{q}}[f'] \geq \mathbb{P}_{\mathbf{p},\mathbf{q}}[f]$. Similarly, $\mathbf{p}'$ is an *oblivious lower bound* iff $\forall \mathbf{q} : \mathbb{P}_{\mathbf{p}',\mathbf{q}}[f'] \leq \mathbb{P}_{\mathbf{p},\mathbf{q}}[f]$.

In other words, $\mathbf{p}'$ is an oblivious upper bound if the probability of the dissociated function f' is bigger than that of f *for every choice of* $\mathbf{q}$. Put differently, the probabilities of $\mathbf{x}'$ depend only on the probabilities of $\mathbf{x}$ and not on those of $\mathbf{y}$.

An immediate upper bound is given by $\mathbf{p}' = \mathbf{1}$, since f is monotone and $f'[\mathbf{1}/\mathbf{x}'] = f[\mathbf{1}/\mathbf{x}]$. Similarly, $\mathbf{p} = \mathbf{0}$ is a naïve lower bound. This proves that the set of upper and lower bounds is never empty. Our goal is to characterize all oblivious bounds and to then find optimal ones.

4.2. Characterization of Oblivious Bounds through Valuations

We will give a necessary and sufficient characterization of oblivious bounds, but first we need to introduce some notations. If $f(\mathbf{x}, \mathbf{y})$ is a Boolean function, let $\nu : \mathbf{y} \to \{0, 1\}$ be a truth assignment or valuation for $\mathbf{y}$. We use $\boldsymbol{\nu}$ for the vector $\langle \nu(y_1), \ldots, \nu(y_n)\rangle$, and denote with $f[\nu]$ the Boolean function obtained after applying the substitution ν. Note that $f[\nu]$ depends on variables $\mathbf{x}$ only. Furthermore, let $\mathbf{g}$ be n Boolean functions,



(a) $\mathbf{g} = \langle g_1, g_2\rangle$ (b) $\mathbf{g}^\nu$ for different valuations $\boldsymbol{\nu}$

Fig. 3. Example 4.2. Illustration of the valuation notation with Karnaugh maps. (a): Boolean functions $g_1 = z_1z_2$ and $g_2 = z_1z_3$. (b): Boolean functions $\mathbf{g}^\nu$ for all 4 possible valuations $\boldsymbol{\nu}$. For example, $\mathbf{g}^\nu = z_1\bar{z}_2z_3$ for $\boldsymbol{\nu} = \langle 0, 1\rangle$.

over variables $\mathbf{z}$. We denote with $\mathbf{g}^\nu$ the Boolean function $\mathbf{g}^\nu = \bigwedge_j g_j^\nu$, where $g_j^\nu = \bar{g}_j$ if $\nu(y_j) = 0$ and $g_j^\nu = g_j$ if $\nu(y_j) = 1$.

Example 4.2 (*Valuation Notation*). Assume $\mathbf{g} = \langle g_1, g_2\rangle$ with $g_1 = z_1z_2$ and $g_2 = z_1z_3$, and $\boldsymbol{\nu} = \langle 0, 1\rangle$. Then $\mathbf{g}^\nu = \neg(z_1z_2) \wedge z_1z_3 = z_1\bar{z}_2z_3$. Figure 3 illustrates our notation for this simple example with the help of Karnaugh maps. We encourage the reader to take a moment and study carefully the correspondences between $\mathbf{g}$, ν, and $\mathbf{g}^\nu$. ■

Then, any function $f(\mathbf{x}, \mathbf{y})$ admits the following expansion by the $\mathbf{y}$-variables:

$$f(\mathbf{x}, \mathbf{y}) = \bigvee_\nu \Big(f[\nu] \wedge \mathbf{y}^\nu\Big) \tag{3}$$

Note that any two expressions in the expansion above are logically contradictory, a property called determinism by Darwiche and Marquis [2002], and that it can be seen as the result of applying Shannon's expansion to all variables of $\mathbf{y}$.

Example 4.3 (*Valuation Notation continued*). Consider the function $f = (x\vee y_1)(x\vee y_2)$. For the example valuation $\boldsymbol{\nu} = \langle 0, 1\rangle$, we have $f[\nu] = (x\vee 0)(x\vee 1) = x$ and $\mathbf{y}^\nu = \bar{y}_1y_2$. Equation 3 gives us an alternative way to write f as disjunction over all 2^2 valuations of $\mathbf{y}$ as $f = x(\bar{y}_1\bar{y}_2) \vee x(y_1\bar{y}_2) \vee x(\bar{y}_1y_2) \vee y_1y_2$. ■

The following proposition is a necessary and sufficient condition for oblivious upper and lower bounds, based on valuations.

PROPOSITION 4.4 (OBLIVIOUS BOUNDS AND VALUATIONS). *Fix two Boolean functions $f(\mathbf{x}, \mathbf{y})$, $f'(\mathbf{x}', \mathbf{y})$ s.t. f' is a dissociation of f, and let $\mathbf{p}$ and $\mathbf{p}'$ denote the probabilities of the variables $\mathbf{x}$ and $\mathbf{x}'$, respectively. Then $\mathbf{p}'$ is an oblivious upper bound iff $\mathbb{P}_{\mathbf{p}'}[f'[\nu]] \geq \mathbb{P}_{\mathbf{p}}[f[\nu]]$ for every valuation ν for $\mathbf{y}$. The proposition holds similarly for oblivious lower bounds.*

PROOF. Remember that any two events in Eq. 3 are disjoint. The total probability theorem thus allows us to sum over the probabilities of all conjuncts:

$$\mathbb{P}_{\mathbf{p},\mathbf{q}}[f(\mathbf{x}, \mathbf{y})] = \sum_\nu \Big(\mathbb{P}_{\mathbf{p}}[f[\nu]] \cdot \mathbb{P}_{\mathbf{q}}[\mathbf{y}^\nu]\Big)$$

$$\mathbb{P}_{\mathbf{p}',\mathbf{q}}[f'(\mathbf{x}, \mathbf{y})] = \sum_\nu \Big(\mathbb{P}_{\mathbf{p}'}[f'[\nu]] \cdot \mathbb{P}_{\mathbf{q}}[\mathbf{y}^\nu]\Big)$$

The "if" direction follows immediately. For the "only if" direction, assume that $\mathbf{p}'$ is an oblivious upper bound. By definition, $\mathbb{P}_{\mathbf{p}',\mathbf{q}}[f'] \geq \mathbb{P}_{\mathbf{p},\mathbf{q}}[f]$ for every $\mathbf{q}$. Fix any valuation $\nu : \mathbf{y} \to \{0, 1\}$, and define the following probabilities $\mathbf{q}$: $q_i = 0$ when $\nu(y_i) = 0$, and $q_i = 1$ when $\nu(y_i) = 1$. It is easy to see that $\mathbb{P}_{\mathbf{p},\mathbf{q}}[f] = \mathbb{P}_{\mathbf{p}}[f[\nu]]$, and similarly, $\mathbb{P}_{\mathbf{p}',\mathbf{q}}[f'] = \mathbb{P}_{\mathbf{p}'}[f'[\nu]]$, which proves $\mathbb{P}_{\mathbf{p}'}[f'[\nu]] \geq \mathbb{P}_{\mathbf{p}}[f[\nu]]$. □

A consequence of choosing $\mathbf{p}'$ obliviously is that it remains a bound even if we allow the variables $\mathbf{y}$ to be arbitrarily correlated. More precisely:

COROLLARY 4.5 (OBLIVIOUS BOUNDS AND CORRELATIONS). *Let $f'(\mathbf{x}', \mathbf{y})$ be a dissociation of $f(\mathbf{x}, \mathbf{y})$, let $\mathbf{p}'$ be an oblivious upper bound for $\mathbf{p}$, and let $\mathbf{g} = \langle g_1, \ldots, g_{|\mathbf{y}|}\rangle$ be Boolean functions in some variables $\mathbf{z}$ with probabilities $\mathbf{r} = \mathbb{P}[\mathbf{z}]$. Then: $\mathbb{P}_{\mathbf{p}',\mathbf{r}}[f'(\mathbf{x}', \mathbf{g}(\mathbf{z}))] \geq \mathbb{P}_{\mathbf{p},\mathbf{r}}[f(\mathbf{x}, \mathbf{g}(\mathbf{z}))]$. The result for oblivious lower bounds is similar.*

The intuition is that, by substituting the variables $\mathbf{y}$ with functions $\mathbf{g}$ in $f(\mathbf{x}, \mathbf{y})$, we make $\mathbf{y}$ correlated. The corollary thus says that an oblivious upper bound remains an upper bound even if the variables $\mathbf{y}$ are correlated. This follows from folklore that any correlation between the variables $\mathbf{y}$ can be captured by general Boolean functions $\mathbf{g}$. For completeness, we include the proof in Appendix B.

PROOF OF COROLLARY 4.5. We derive the probabilities of f and f' from Eq. 3:

$$\mathbb{P}_{\mathbf{p},\mathbf{r}}[f(\mathbf{x}, \mathbf{g})] = \sum_{\nu} \Big(\mathbb{P}_{\mathbf{p}}[f[\nu]] \cdot \mathbb{P}_{\mathbf{r}}[\mathbf{g}^{\nu}]\Big)$$
$$\mathbb{P}_{\mathbf{p}',\mathbf{r}}[f'(\mathbf{x}, \mathbf{g})] = \sum_{\nu} \Big(\mathbb{P}_{\mathbf{p}'}[f'[\nu]] \cdot \mathbb{P}_{\mathbf{r}}[\mathbf{g}^{\nu}]\Big)$$

The proof follows now immediately from Prop. 4.4. □

4.3. Oblivious Bounds for Unary Conjunctive and Disjunctive Dissociations

A dissociation $f'(\mathbf{x}', \mathbf{y})$ of $f(\mathbf{x}, \mathbf{y})$ is called *unary* if $|\mathbf{x}| = 1$, in which case we write the function as $f(x, \mathbf{y})$. We next focus on unary dissociations, and establish a necessary and sufficient condition for probabilities to be oblivious upper or lower bounds for the important classes of conjunctive and disjunctive dissociations. The criterion also extends as a sufficient condition to non-unary dissociations, since these can be obtained as a sequence of unary dissociations.[7]

Definition 4.6 (*Conjunctive and Disjunctive Dissociation*). Let $f'(\mathbf{x}', \mathbf{y})$ be a Boolean function in variables $\mathbf{x}', \mathbf{y}$. We say that the variables $\mathbf{x}'$ are *conjunctive* in f' if $f'(\mathbf{x}', \mathbf{y}) = \bigwedge_{j \in [d]} f_j(x'_j, \mathbf{y})$, $d = |\mathbf{x}'|$. We say that a dissociation $f'(\mathbf{x}', \mathbf{y})$ of $f(x, \mathbf{y})$ is conjunctive if $\mathbf{x}'$ are conjunctive in f'. Similarly, we say that $\mathbf{x}'$ are *disjunctive* in f' if $f'(\mathbf{x}', \mathbf{y}) = \bigvee_{j \in [d]} f_j(x'_j, \mathbf{y})$, and a dissociation is disjunctive if $\mathbf{x}'$ is disjunctive in f'.

Thus in a conjunctive dissociation, each dissociated variable x'_j occurs in exactly one Boolean function f_j and these functions are combined by $\wedge$ to obtain f'. In practice, we start with f written as a conjunction, then replace x with a fresh variable in each conjunct:

$$f(x, \mathbf{y}) = \bigwedge f_j(x, \mathbf{y})$$
$$f'(\mathbf{x}', \mathbf{y}) = \bigwedge f_j(x'_j, \mathbf{y})$$

Disjunctive dissociations are similar.

Note that if $\mathbf{x}'$ is conjunctive in $f'(\mathbf{x}', \mathbf{y})$, then for any substitution $\nu : \mathbf{y} \to \{0, 1\}$, $f'[\nu]$ is either 0, 1, or a conjunction of variables in $\mathbf{x}'$: $f'[\nu] = \bigwedge_{j \in \mathbf{s}} x'_j$, for some set $\mathbf{s} \subseteq [d]$, where $d = |\mathbf{x}'|$. Similarly, if $\mathbf{x}'$ is disjunctive, then $f'[\nu]$ is 0, 1, or $\bigvee_{j \in \mathbf{s}} x'_j$.[8]

[7]Necessity does not always extend to non-unary dissociations. The reason is that an oblivious dissociation for $\mathbf{x}$ may set the probability of a fresh variable by examining *all* variables $\mathbf{x}$, while a in sequence of oblivious dissociations each new probability $\mathbb{P}[x'_{i,j}]$ may depend only on the variable x_i currently being dissociated.
[8] Note that for $\mathbf{s} = \emptyset$: $f'[\nu] = \bigwedge_{j \in \mathbf{s}} x'_j = 1$ and $f'[\nu] = \bigvee_{j \in \mathbf{s}} x'_j = 0$.

We need one more definition before we state the main result in our paper.

Definition 4.7 (*Cover*). Let $\mathbf{x}'$ be conjunctive in $f'(\mathbf{x}', \mathbf{y})$. We say that f' *covers the set* $\mathbf{s} \subseteq [d]$ if there exists a substitution ν s.t. $f'[\nu] = \bigwedge_{j \in \mathbf{s}} x'_j$. Similarly, if $\mathbf{x}'$ is disjunctive, then we say that f' covers $\mathbf{s}$ if there exists ν s.t. $f'[\nu] = \bigvee_{j \in \mathbf{s}} x'_j$.

THEOREM 4.8 (OBLIVIOUS BOUNDS). *Let $f'(\mathbf{x}', \mathbf{y})$ be a conjunctive dissociation of $f(x, \mathbf{y})$, and let $p = \mathbb{P}[x]$, $\mathbf{p}' = \mathbb{P}[\mathbf{x}']$ be probabilities of x and $\mathbf{x}'$, respectively. Then:*

(*1*) *If $p'_j \leq p$ for all j, then $\mathbf{p}'$ is an oblivious lower bound for p, i.e. $\forall \mathbf{q} : \mathbb{P}_{\mathbf{p}',\mathbf{q}}[f'] \leq \mathbb{P}_{p,\mathbf{q}}[f]$. Conversely, if $\mathbf{p}'$ is an oblivious lower bound for p and f' covers all singleton sets $\{j\}$ with $j \in [d]$, then $p'_j \leq p$ for all j.*

(*2*) *If $\prod_j p'_j \geq p$, then $\mathbf{p}'$ is an oblivious upper bound for p, i.e. $\forall \mathbf{q} : \mathbb{P}_{\mathbf{p}',\mathbf{q}}[f'] \geq \mathbb{P}_{p,\mathbf{q}}[f]$. Conversely, if $\mathbf{p}'$ is an oblivious upper bound for p and f' covers the set $[d]$, then $\prod_j p'_j \geq p$.*

Similarly, the dual result holds for disjunctive dissociations $f'(\mathbf{x}', \mathbf{y})$ of $f(x, \mathbf{y})$:

(*1*) *If $p'_j \geq p$ for all j, then $\mathbf{p}'$ is an oblivious upper bound for p. Conversely, if $\mathbf{p}'$ is an oblivious upper bound for p and f' covers all singleton sets $\{j\}$, $j \in [d]$, then $p'_j \geq p$.*

(*2*) *If $\prod_j (1 - p'_j) \leq 1 - p$, then $\mathbf{p}'$ is an oblivious lower bound for p. Conversely, if $\mathbf{p}'$ is an oblivious lower bound for p and f' covers the set $[d]$, then $\prod_j (1 - p'_j) \leq 1 - p$.*

PROOF. We make repeated use of Prop. 4.4. We give here the proof for conjunctive dissociations only; the proof for disjunctive dissociations is dual and similar.

(1) We need to check $\mathbb{P}_{\mathbf{p}'}[f'[\nu]] \leq \mathbb{P}_p[f[\nu]]$ for every ν. Since the dissociation is unary, $f[\nu]$ can be only 0, 1, or x, while $f'[\nu]$ is 0, 1, or $\bigwedge_{j \in \mathbf{s}} x'_j$ for some set $\mathbf{s} \subseteq [d]$.

Case 1: $f[\nu] = 0$. We will show that $f'[\nu] = 0$, which implies $\mathbb{P}_{\mathbf{p}'}[f'[\nu]] = \mathbb{P}_p[f[\nu]] = 0$. Recall that, by definition, $f'(\mathbf{x}', \mathbf{y})$ becomes $f(x, \mathbf{y})$ if we substitute x for all variables x'_j. Therefore, $f'[\nu][x/x'_1, \ldots, x/x'_d] = 0$, which implies $f'[\nu] = 0$ because f' is monotone in the variables $\mathbf{x}'$.

Case 2: $f[\nu] = 1$. Then $\mathbb{P}_{\mathbf{p}'}[f'[\nu]] \leq \mathbb{P}_p[f[\nu]]$ holds trivially.

Case 3: $f[\nu] = x$. Then $\mathbb{P}_p[f[\nu]] = p$, while $\mathbb{P}_{\mathbf{p}'}[f'[\nu]] = \prod_{j \in \mathbf{s}} p'_j$. We prove that $\mathbf{s} \neq \emptyset$: this implies our claim, because $\prod_{j \in \mathbf{s}} p'_j \leq p'_j \leq p$, for any choice of $j \in \mathbf{s}$. Suppose otherwise, that $\mathbf{s} = \emptyset$, hence $f'[\nu] = 1$. Substituting all variables $\mathbf{x}'$ with x transforms f' to f, which implies $f[\nu] = 1$, contradiction.

For the converse, assume that $\mathbf{p}'$ is an oblivious lower bound. Since f' covers $\{j\}$, there exists a substitution ν s.t. $f'[\nu] = x'_j$, and therefore $f[\nu] = x$. By Prop. 4.4 we have $p'_j = \mathbb{P}_{\mathbf{p}'}[f'[\nu]] \leq \mathbb{P}_p[f[\nu]] = p$, proving the claim.

(2) Here we need to check $\mathbb{P}_{\mathbf{p}'}[f'[\nu]] \geq \mathbb{P}_p[f[\nu]]$ for every ν. The cases when $f[\nu]$ is either 0 or 1 are similar to the cases above, so we only consider the case when $f[\nu] = x$. Then $f'[\nu] = \bigwedge_{j \in \mathbf{s}} x'_j$ and $\mathbb{P}_{\mathbf{p}'}[f'[\nu]] = \prod_{j \in \mathbf{s}} p'_j \geq \prod_j p_j \geq p = \mathbb{P}_{\mathbf{p}}[f[\nu]]$.

For the converse, assume $\mathbf{p}'$ is an oblivious upper bound, and let ν be the substitution for which $f'[\nu] = \bigwedge_j x'_j$ (which exists since f' is covers $[d]$). Then $\mathbb{P}_{\mathbf{p}'}[f'[\nu]] \geq \mathbb{P}_{\mathbf{p}}[f[\nu]]$ implies $p \leq \prod_j p_j$. $\square$

4.4. Optimal Oblivious Bounds for Unary Conjunctive and Disjunctive Dissociations

We are naturally interested in the "best possible" oblivious bounds. Call a dissociation f' *non-degenerate* if it covers all singleton sets $\{j\}$, $j \in [d]$ and the complete set $[d]$. Theorem 4.8 then implies:

COROLLARY 4.9 (OPTIMAL OBLIVIOUS BOUNDS). *If f' is a conjunctive dissociation of f and f' is non-degenerate, then the optimal oblivious lower bound is $p'_1 = p'_2 = \ldots = p$, while the optimal oblivious upper bounds are obtained whenever $p'_1 p'_2 \cdots = p$. Similarly, if f' is a disjunctive dissociation of f and f' is non-degenerate, then the optimal oblivious upper bound is $p'_1 = p'_2 = \ldots = p$, while the optimal oblivious lower bounds are obtained whenever $(1-p'_1)\cdot(1-p'_2)\cdots = 1-p$.*

Notice that while optimal lower bounds for conjunctive dissociations and optimal upper bounds for disjunctive dissociations are uniquely defined with $p'_j = p$, there are infinitely many optimal bounds for the other directions (see Fig. 1). Let us call bounds *symmetric* if all dissociated variable have the same probability. Then optimal symmetric upper bounds for conjunctive dissociations are $p'_j = \sqrt[d]{p}$, and optimal symmetric lower bounds for disjunctive dissociations $p'_j = 1 - \sqrt[d]{1-p}$.

We give two examples of degenerate dissociations. First, the dissociation $f' = (x'_1 y_1 \vee y_3) \wedge (x'_2 y_2 \vee y_3)$ does not cover either $\{1\}$ nor $\{2\}$: no matter how we substitute y_1, y_2, y_3, we can never transform f' to x'_1. For example, $f'[1/y_1, 0/y_2, 0/y_3] = 0$ and $f'[1/y_1, 0/y_2, 1/y_3] = 1$. But f' does cover the set $\{1, 2\}$ because $f'[1/y_1, 1/y_2, 0/y_3] = x_1 x_2$. Second, the dissociation $f' = (x'_1\, y_1 \vee y_2) \wedge (x'_2\, y_2 \vee y_1)$ covers both $\{1\}$ and $\{2\}$, but does not cover the entire set $\{1, 2\}$. In these cases the oblivious upper or lower bounds in Theorem 4.8 still hold, but are not necessarily optimal.

However, most cases of practical interest result in dissociations that are non-degenerate, in which case the bounds in Theorem 4.8 are tight. We explain this here. Consider the original function, pre-dissociation, written in a conjunctive form:

$$f(x, \mathbf{y}) = g_0 \wedge \bigwedge_{j \in [d]} (x \vee g_j) = g_0 \wedge \bigwedge_{j \in [d]} f_j \tag{4}$$

where each g_j is a Boolean function in the variables $\mathbf{y}$, and where we denoted $f_j = x \vee g_j$. For example, if f is a CNF expression, then each f_j is a clause containing x, and g_0 is the conjunction of all clauses that do not contain x. Alternatively, we may start with a CNF expression, and group the clauses containing x in equivalence classes, such that each f_j represents one equivalence class. For example, starting with four clauses, we group into two functions $f = [(x \vee y_1)(x \vee y_2)] \wedge [(x \vee y_3)(x \vee y_4)] = (x \vee y_1 y_2) \wedge (x \vee y_3 y_4) = f_1 \wedge f_2$. Our only assumption about Eq. 4 is that it is *non-redundant*, meaning that none of the expressions g_0 or f_j may be dropped. Then we prove:

PROPOSITION 4.10 (NON-DEGENERATE DISSOCIATION). *Suppose the function f in Eq. 4 is non-redundant. Define $f' = g_0 \wedge \bigwedge_j (x'_j \vee g_j)$. Then f' covers every singleton set $\{j\}$. Moreover, if the implication $g_0 \Rightarrow \bigvee_j g_j$ does not hold, then f' also covers the set $[d]$. Hence f' is non-degenerate. A similar result holds for disjunctive dissociations if the dual implication $g_0 \Leftarrow \bigwedge_j g_j$ does not hold.*

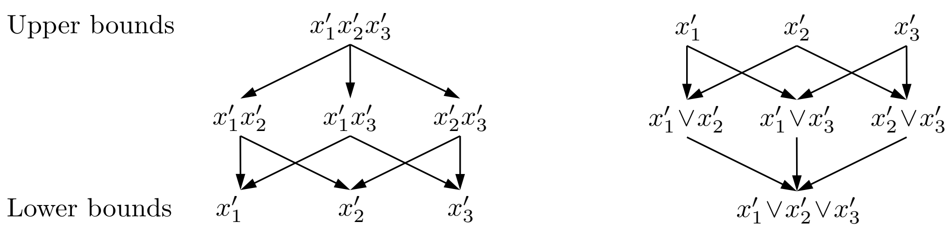
PROOF. We give here the proof for conjunctive dissociations only; the proof for disjunctive dissociations follows from duality. We first prove that f' covers any singleton set $\{j\}$, for $j \in [d]$. We claim that the following logical implication does not hold:

$$g_0 \wedge \bigwedge_{i \neq j} g_i \Rightarrow g_j \tag{5}$$

Indeed, suppose the implication holds for some j. Then the following implication also holds: $g_0 \wedge \bigwedge_{i \neq j} (x \vee g_i) \Rightarrow (x \vee g_j)$, since for $x = 0$ it is the implication above, while for $x = 1$ it is a tautology. Therefore, the function f_j is redundant in Eq. 4, which contradicts our assumption. Hence, the implication Eq. 5 does not hold. Let ν be any

Valuation		CNF dissociation					DNF dissociation				
#	ν	$f'_c[\nu]$	$f_c[\nu]$	$\mathbb{P}[f'_c[\nu]]$	$\geq$	$\mathbb{P}[f_c[\nu]]$	$f'_d[\nu]$	$f_d[\nu]$	$\mathbb{P}[f'_d[\nu]]$	$\geq$	$\mathbb{P}[f_d[\nu]]$
1.	$\langle 0,0,0,0\rangle$	0	0	0	$\geq$	0	0	0	0	$\geq$	0
2.	$\langle 1,0,0,0\rangle$	0	0	0	$\geq$	0	x'_1	x	p'_1	$\geq$	p
3.	$\langle 0,1,0,0\rangle$	0	0	0	$\geq$	0	x'_2	x	p'_2	$\geq$	p
4.	$\langle 0,0,1,0\rangle$	0	0	0	$\geq$	0	x'_3	x	p'_3	$\geq$	p
5.	$\langle 0,0,0,1\rangle$	$x'_1x'_2x'_3$	x	$p'_1p'_2p'_3$	$\geq$	p	1	1	1	$\geq$	1
6.	$\langle 1,1,0,0\rangle$	0	0	0	$\geq$	0	$x'_1 \vee x'_2$	x	$\bar{p}'_1\bar{p}'_2$	$\geq$	$\bar{p}$
7.	$\langle 1,0,1,0\rangle$	0	0	0	$\geq$	0	$x'_1 \vee x'_3$	x	$\bar{p}'_1\bar{p}'_3$	$\geq$	$\bar{p}$
8.	$\langle 0,1,1,0\rangle$	0	0	0	$\geq$	0	$x'_2 \vee x'_3$	x	$\bar{p}'_2\bar{p}'_3$	$\geq$	$\bar{p}$
9.	$\langle 1,0,0,1\rangle$	$x'_2x'_3$	x	$p'_2p'_3$	$\geq$	p	1	1	1	$\geq$	1
10.	$\langle 0,1,0,1\rangle$	$x'_1x'_3$	x	$p'_1p'_3$	$\geq$	p	1	1	1	$\geq$	1
11.	$\langle 0,0,1,1\rangle$	$x'_1x'_2$	x	$p'_1p'_2$	$\geq$	p	1	1	1	$\geq$	1
12.	$\langle 1,1,1,0\rangle$	0	0	0	$\geq$	0	$x'_1 \vee x'_2 \vee x'_3$	x	$\bar{p}'_1\bar{p}'_2\bar{p}'_3$	$\geq$	$\bar{p}$
13.	$\langle 1,1,0,1\rangle$	x'_3	x	p'_3	$\geq$	p	1	1	1	$\geq$	1
14.	$\langle 1,0,1,1\rangle$	x'_2	x	p'_2	$\geq$	p	1	1	1	$\geq$	1
15.	$\langle 0,1,1,1\rangle$	x'_1	x	p'_1	$\geq$	p	1	1	1	$\geq$	1
16.	$\langle 1,1,1,1\rangle$	1	1	1	$\geq$	1	1	1	1	$\geq$	1

(a) Comparing 2^4 valuations for determining oblivious bounds.



(b) Non-trivial valuations $f'_c[x]$ (c) Non-trivial valuations $f'_d[x]$

Fig. 4. Example 4.11 (CNF f_c) and Example 4.12 (DNF f_d). (a): Determining oblivious bounds by ensuring that bounds hold for all valuations. (b), (c): Partial order of implication ($\Rightarrow$) for the non-trivial valuations $f_c[\nu]'$ and $f_d[\nu]'$, e.g.: from $x'_1x'_2 \Rightarrow x'_1$ it follows that $p'_1p'_2 \geq p \Rightarrow p'_1 \geq p$. Note that $f_c \neq f_d$.

assignment that causes Eq. 5 to fail: thus, for all $j \in \{0,\dots,d\}$, $j \neq i$, $g_i[\nu] = 1$ and $g_j[\nu] = 0$. Therefore $f'[\nu] = x_j$, proving that it covers $\{j\}$.

Next, assume that $g_0 \Rightarrow \bigvee_j g_j$ does not hold. We prove that f' covers $[d]$. Let ν be any substitution that causes the implication to fail: $g_0[\nu] = 1$ and $g_j[\nu] = 0$ for $j \in [d]$. Then $f'[\nu] = \bigwedge_{j\in[d]} x'_j$. □

4.5. Illustrated Examples for Optimal Oblivious Bounds

We next give two examples that illustrate optimal oblivious bounds for conjunctive and disjunctive dissociations in some detail.

Example 4.11 (*CNF Dissociation*). Consider the function f_c given by an CNF expression and its dissociation f'_c:

$$f_c = (x \vee y_1)(x \vee y_2)(x \vee y_3)y_4$$
$$f'_c = (x'_1 \vee y_1)(x'_2 \vee y_2)(x'_3 \vee y_3)y_4$$

There are $2^4 = 16$ valuations for $\mathbf{y} = \langle y_1, y_2, y_3, y_4\rangle$. Probabilities $\mathbf{p}' = \langle p'_1, p'_2, p'_3\rangle$ are thus an oblivious *upper* bound exactly if they satisfy the 16 inequalities given under "CNF dissociation" in Fig. 4a. For valuations with $\nu_4 = 0$ (and thus $f_c[\nu] = 0$) or all

$\nu_j = 1$ (and thus $f'_c[\nu] = 1$) the inequalities trivially hold. For the remaining 7 non-trivial inequalities, $p'_1 p'_2 p'_3 \geq p$ implies all others. Figure 4b shows the partial order between the non-trivial valuations, with $x'_1 x'_2 x'_3$ implying all others. Since f_c and f'_c are positive in x and x′, respectively, it follows that optimal oblivious upper bounds are given by $p'_1 p'_2 p'_3 = p$, e.g., by setting $p'_i = \sqrt[3]{p}$ for symmetric bounds.

Oblivious *lower* bounds are given by the 16 inequalities after inverting the inequality sign. Here we see that the three inequalities $p'_j \leq p$ together imply the others. Hence, oblivious lower bounds are those that satisfy all three inequalities. The only optimal oblivious upper bounds are then given by $p'_j = p$. ■

Example 4.12 (*DNF Dissociation*). Consider the function f_d given by an DNF expression and its dissociation f'_d:

$$f_d = xy_1 \vee xy_2 \vee xy_3 \vee y_4$$
$$f'_d = x'_1 y_1 \vee x'_2 y_2 \vee x'_3 y_3 \vee y_4$$

An oblivious upper bound $\mathbf{p}' = \langle p'_1, p'_2, p'_3 \rangle$ must thus satisfy the 16 inequalities[9] given under "DNF dissociation" in Fig. 4a. For valuations with $\nu_4 = 1$ (and thus $f'_d[\nu] = 1$) or $\nu_j = 0$ (and thus $f_d[\nu] = 0$) the inequalities trivially hold. For the remaining inequalities we see that the elements of set $\{x'_1, x'_2, x'_3\}$ together imply all others, and that $x'_1 \vee x'_2 \vee x'_3$ is implied by all others (Fig. 4c shows the partial order between the non-trivial valuations). Thus, an oblivious upper bound must satisfy $p'_j \geq p$, and the optimal one is given by $p'_j = p$. Analogously, an oblivious lower bound must satisfy $\bar{p}'_1 \bar{p}'_2 \bar{p}'_3 \leq \bar{p}$. Optimal ones are given for $\bar{p}'_1 \bar{p}'_2 \bar{p}'_3 = \bar{p}$, e.g., by setting $p'_j = 1 - \sqrt[3]{\bar{p}}$. ■

5. RELAXATION AND MODEL-BASED BOUNDS AS DISSOCIATION

This section formalizes the connection between relaxation, model-based bounds and dissociation that was outlined in the introduction. In other words, we show how both previous approaches can be unified under the framework of dissociation.

5.1. Relaxation & Compensation

The following proposition shows relaxation & compensation as conjunctive dissociation and was brought to our attention by Choi and Darwiche [2011].

PROPOSITION 5.1 (COMPENSATION AND CONJUNCTIVE DISSOCIATION). *Let f_1, f_2 be two monotone Boolean functions which share only one single variable x. Let f be their conjunction, and f' be the dissociation of f on x, i.e.*

$$f = f_1 \wedge f_2$$
$$f' = f_1[x'_1/x] \wedge f_2[x'_2/x]$$

Then $\mathbb{P}[f] = \mathbb{P}[f']$ for $\mathbb{P}[x'_1] = \mathbb{P}[x]$ and $\mathbb{P}[x'_2] = \mathbb{P}[x|f_1]$.

PROOF OF PROP. 5.1. First, note that $\mathbb{P}[f] = \mathbb{P}[f_1]\mathbb{P}[f_2|f_1]$. On the other hand, $\mathbb{P}[f'] = \mathbb{P}[f'_1]\mathbb{P}[f'_2]$ as f'_1 and f'_2 are independent after dissociating on the only shared variable x. We also have $\mathbb{P}[f_1] = \mathbb{P}[f'_1]$ since $\mathbb{P}[x] = \mathbb{P}[x'_1]$. It remains to be shown that

[9] Remember that the probability of a disjunction of two independent events is $\mathbb{P}[x'_1 \vee x'_2] = 1 - \bar{p}'_1 \bar{p}'_2$.

$\mathbb{P}[f_2'] = \mathbb{P}[f_2|f_1]$. Indeed:

$$\begin{aligned}\mathbb{P}[f_2'] &= \mathbb{P}[x_2' \cdot f_2'[1/x_2'] \vee \bar{x}_2' \cdot f_2'[0/x_2']]\\ &= \mathbb{P}[x_2'] \cdot \mathbb{P}[f_2'[1/x_2']] + \mathbb{P}[\bar{x}_2'] \cdot \mathbb{P}[f_2'[0/x_2']]\\ &= \mathbb{P}[x|f_1] \cdot \mathbb{P}[f_2[1/x]] + \mathbb{P}[\bar{x}|f_1] \cdot \mathbb{P}[f_2[0/x]]\\ &= \mathbb{P}[x|f_1] \cdot \mathbb{P}[f_2[1/x]|f_1] + \mathbb{P}[\bar{x}|f_1] \cdot \mathbb{P}[f_2[0/x]|f_1]\\ &= \mathbb{P}[f_2|f_1]\end{aligned}$$

which proves the claim. □

Note that compensation is not oblivious, since the probability p_2' depends on the other variables occurring in φ_1. Further note that, in general, φ_1, φ_2 have more than one variable in common, and in this case we have $\mathbb{P}[\varphi'] \neq \mathbb{P}[\varphi]$ for the same compensation. Thus in general, compensation is applied as a heuristics, and it is then not known whether it provides an upper or lower bound.

The dual result for disjunctions holds by replacing f_1 with its negation $\bar{f}_1$ in $\mathbb{P}[x_2'] = \mathbb{P}[x|\bar{f}_1]$. This result is not immediately obvious from the previous one and has, to our best knowledge, not been stated or applied anywhere before.

PROPOSITION 5.2 ("DISJUNCTIVE COMPENSATION"). *Let f_1, f_2 be two monotone Boolean functions which share only one single variable x. Let f be their disjunction, and f' be the dissociation of f on x, i.e. $f = f_1 \vee f_2$, and $f' = f_1[x_1'/x] \vee f_2[x_2'/x]$. Then $\mathbb{P}[f] = \mathbb{P}[f']$ for $\mathbb{P}[x_1'] = \mathbb{P}[x]$ and $\mathbb{P}[x_2'] = \mathbb{P}[x|\bar{f}_1]$.*

PROOF OF PROP. 5.2. Let $g = \bar{f}$, $g_1 = \bar{f}_1$, and $g_2 = \bar{f}_2$. Then $f = f_1 \vee f_2$ is equivalent to $g = g_1 \wedge g_2$. From Prop. 5.1, we know that $\mathbb{P}[g] = \mathbb{P}[g']$, and thus $\mathbb{P}[f] = \mathbb{P}[f']$, for $\mathbb{P}[x_1'] = \mathbb{P}[x]$ and $\mathbb{P}[x_2'] = \mathbb{P}[x|g_1] = \mathbb{P}[x|\bar{f}_1]$. □

5.2. Model-based Approximation

The following proposition shows that all model-based bounds can be derived by repeated dissociation. However, not all dissociation-bounds can be explained as models since dissociation is in its essence an *algebraic* and not a model-based technique (dissociation creates more variables and thus changes the probability space). Therefore, dissociation can improve any existing model-based approximation approach. Example 7.2 will illustrate this with a detailed simulation-based example.

PROPOSITION 5.3 (MODEL-BASED BOUNDS AS DISSOCIATIONS). *Let f, f_U be two monotone Boolean functions over the same set of variables, and for which the logical implication $f \Rightarrow f_U$ holds. Then: (a) there exists a sequence of optimal conjunctive dissociations that transform f to f_U, and (b) there exists a sequence of non-optimal disjunctive dissociations that transform f to f_U. The dual result holds for the logical implication $f_L \Rightarrow f$: (c) there exists a sequence of optimal disjunctive dissociations that transform f to f_L, and (d) there exists a sequence of non-optimal conjunctive dissociations that transform f to f_L.*

PROOF OF PROP. 5.3. We focus here on the implication $f \Rightarrow f_U$. The proposition for the results $f_L \Rightarrow f$ then follows from duality.

(a) The implication $f \Rightarrow f_U$ holds iff there exists a positive function f_2 such that $f = f_U \wedge f_2$. Pick a set of variables $\mathbf{x}$ s.t. $f_2[\mathbf{1}/\mathbf{x}] = 1$, and dissociate f on $\mathbf{x}$ into $f' = f_U[\mathbf{x}_1'/\mathbf{x}] \wedge f_2[\mathbf{x}_2'/\mathbf{x}]$. By setting the probabilities of the dissociated variables to $\mathbf{p}_1' = \mathbf{p}$ and $\mathbf{p}_2' = \mathbf{1}$, the bounds become optimal ($p_1'p_2' = p$). Further more, f_U remains unchanged (except for the renaming of $\mathbf{x}$ to $\mathbf{x}_1'$), whereas f_2 becomes true. Hence, we get $f' = f_U$. Thus, all model-based upper bounds can be obtained by conjunctive dissociation and choosing optimal oblivious bounds at each dissociation step.

(b) The implication $f \Rightarrow f_U$ also holds iff there exists a positive function f_d such that $f_U = f \vee f_d$. Let m be the positive minterm or elementary conjunction involving all variables of f. The function f_d can be then written as DNF $f_d = c_1 \vee c_2 \vee \ldots$, with products $c_i \subseteq m$. Since f is monotone, we know $m \Rightarrow f$, and thus also $mf_d \Rightarrow f$. We can therefore write $f = f \vee mf_d$ or as

$$f = f \vee mc_1 \vee mc_2 \vee \ldots$$

Let $\mathbf{x}_i$ be the set of all variables in m that do not occur in c_i and denote with m_i the conjunction of $\mathbf{x}_i$. Then then each mc_i can instead be written as $m_i c_i$ and thus:

$$f = f \vee m_1 c_1 \vee m_2 c_2 \vee \ldots$$

WLOG, we now separate one particular conjunct $m_i c_i$ and dissociate on the set $\mathbf{x}_i$

$$f' = \underbrace{f \vee m_1 c_1 \vee m_2 c_2 \vee \ldots}_{f_1}[\mathbf{x}'_1/\mathbf{x}_i] \vee \underbrace{m_i c_i}_{f_2}[\mathbf{x}'_2/\mathbf{x}_i]$$

By setting the probabilities of the dissociated variables to the non-optimal upper bounds $\mathbf{p}'_1 = \mathbf{p}$ and $\mathbf{p}'_2 = \mathbf{1}$, f_1 remains unchanged (except for the renaming of $\mathbf{x}_i$ to $\mathbf{x}'_1$), whereas f_2 becomes c_i. Hence, we get $f' = f \vee m_1 c_1 \vee m_2 c_2 \vee \cdots \vee c_i$. We can now repeat the same process for all conjuncts mc_i and receive after a finite number of dissociation steps

$$f'' = f \vee (c_1 \vee c_2 \vee \ldots) = f \vee f_d$$

Hence $f'' = f_U$. Thus, all model-based upper bounds can be obtained by disjunctive dissociation and choosing non-optimal bounds at each dissociation step. □

6. QUERY-CENTRIC DISSOCIATION BOUNDS FOR PROBABILISTIC QUERIES

Our previous work [Gatterbauer et al. 2010] has shown how to *upper bound* the probability of conjunctive queries without self-joins by issuing a sequence of SQL statements over a standard relational DBMS. This section illustrates such dissociation-based upper bounds and also complements them with *new lower bounds*. We use the Boolean query $Q \,{:}{-}\, R(X), S(X,Y), T(Y)$, for which the probability computation problem is known to be #P-hard, over the following database instance D:

R	A
x_1	1
x_2	2

S	A	B
z_1	1	1
z_2	2	1
z_3	2	2

T	B
y_1	1
y_2	2

Thus, relation S has three tuples $(1,1)$, $(2,1)$, and $(2,2)$ and both R and T have two tuples (1) and (2). Each tuple is annotated with a Boolean variable $x_1, x_2, z_1, \ldots, y_2$, which represents the independent event that the corresponding tuple is present in the database. The *lineage expression* φ is then the propositional formula that states which input tuples must be present in order for the query q to be true:

$$\varphi = x_1 z_1 y_1 \vee x_2 z_2 y_1 \vee x_2 z_3 y_2$$

Calculating $\mathbb{P}[\varphi]$ for a general database instance is #P-hard. However, if we treat *each* occurrence of a variable x_i as different (in other words, we dissociate φ eagerly on *all* tuple variables x_i from table R), then we get a read-once expression

$$\begin{aligned}\varphi' &= x_1 z_1 y_1 \vee x'_{2,1} z_2 y_1 \vee x'_{2,2} z_3 y_2 \\ &= (x'_1 z_1 \vee x'_{2,1} z_2) y_1 \vee x'_{2,2} z_3 y_2\end{aligned}$$

```
select iOR(Q3.P) as P
from
  (select T.B, T.P*Q2.P as P
  from T,
    (select Q1.B, iOR(Q1.P) as P
    from (select S.A, S.B, S.P*R.P as P
      from R,S
      where R.A = S.A) as Q1
    group by Q1.B) as Q2
  where T.B = Q2.B) as Q3
```

(a) SQL query P_R

```
create view VR as
select R.A,
  1-power(1-R.P,1e0/count(*)) as P
from R, S, T
where R.A=S.A
and S.B=T.B
group by R.A, R.P
```

(b) View V_R for lower bounds with P_R

Fig. 5. (a): SQL query corresponding to plan P_R for deriving an upper bound for the hard probabilistic Boolean query $Q :- R(X), S(X,Y), T(Y)$. Table R needs to be replaced with the view V_R from (b) for deriving a lower bound. **iOR** is a user-defined aggregate explained in the text and stated in Appendix C.

Writing p_i, q_i, r_i for the probabilities of variables x_i, y_i, z_i, respectively, we can calculate

$$\mathbb{P}[\varphi'] = \Big(\big((p'_1 \cdot r_1) \otimes (p'_{2,1} \cdot r_2)\big) \cdot q_1\Big) \otimes (p'_{2,2} \cdot r_3 \cdot q_2)$$

where "$\cdot$" stands for multiplication and "$\otimes$" for independent-or.[10]

We know from Theorem 4.8 that $\mathbb{P}[\varphi']$ gives us an upper bound to $\mathbb{P}[\varphi]$ by assigning the original probabilities to the dissociated variables. Furthermore, as we have shown in [Gatterbauer et al. 2010], $\mathbb{P}[\varphi]$ can be calculated with a probabilistic query plan

$$P_R = \pi^p_\emptyset \bowtie^p_Y \big[\pi^p_Y \bowtie^p_X \big[R(X), S(X,Y)\big], T(Y)\big]$$

where the *probabilistic join operator* in prefix notation $\bowtie^p \big[\dots\big]$ multiplies the tuple probabilities $\prod_{i \in [k]} p_i$ for k tuples that are joined, and the *probabilistic project operator* with duplicate elimination π^p computes the probability with the independent-or between tuples, i.e. $1 - \prod_{i \in [k]} \bar{p}_i$ for k tuples with the same projection attributes [Fuhr and Rölleke 1997]. This connection is not immediately obvious but explained in detail in our prior work. We write here P_R to emphasize that this plan dissociates tuples in table R and give the corresponding SQL statement in Fig. 5a, assuming that each of the input tables has one additional attribute P for the probability of a tuple. The query deploys a user-defined aggregate (UDA) **iOR** that calculates the independent-or for the probabilities of the tuples grouped together, i.e. **iOR**$(p_1, p_2, \dots, p_n) = 1 - \bar{p}_1 \bar{p}_2 \cdots \bar{p}_n$. Appendix C states the UDA definition for PostgreSQL.

Similarly, we know from Theorem 4.8 that $\mathbb{P}[\varphi']$ gives us a lower bound to $\mathbb{P}[\varphi]$ by assigning *new probabilities* $1 - \sqrt[2]{1 - p_2}$ to $x'_{2,1}$ and $x'_{2,2}$ (or more generally, any probabilities $p'_{2,1}$ and $p'_{2,2}$ with $\bar{p}'_{2,1} \cdot \bar{p}'_{2,2} = \bar{p}_2$). Because of the connection between the read-once expression φ' and the query plan P_R, we can thus calculate the lower bound *by using the same SQL query* from Fig. 5a, but after replacing the table R with a view V_R that has the probability p_i of a tuple x_i replaced with $1 - \sqrt[d_i]{1 - p_i}$, where d_i is the number of times that x_i appears in the lineage of query Q. Figure 5b shows the view definition for V_R in SQL: it joins tables R, S and T, groups the original input tuples x_i from R, and assigns each x_i a new probability $1 - \sqrt[d_i]{1 - p_i}$, calculated as 1-power(1-T.P,1e0/count(*)).

Alternatively to φ', if we treat each occurrence of a variable y_j in φ as different, (in other words, we dissociate φ eagerly on all tuple variables y_j from table T), then we

[10]The *independent-or* combines two probabilities as if calculating the disjunction between two independent events. It is defined as $p_1 \otimes p_2 := 1 - \bar{p}_1 \bar{p}_2$.

get another read-once expression

$$\begin{aligned}\varphi'' &= x_1 z_1 y'_{1,1} \vee x_2 z_2 y'_{1,2} \vee x_2 z_3 y_2 \\ &= x_1 z_1 y'_{1,1} \vee x_2 (z_2 y'_{1,2} \vee z_3 y_2)\end{aligned}$$

$\mathbb{P}[\varphi'']$ gives us an upper bound to $\mathbb{P}[\varphi]$ by assigning the original probabilities to the dissociated variables. In turn, $\mathbb{P}[\varphi'']$ can be calculated with another probabilistic query plan that dissociates all tuple variables from table T instead of R:

$$P_T = \pi^p_\emptyset \bowtie^p_X \big[R(X), \pi^p_X \bowtie^p_Y \big[S(X,Y), T(Y)\big]\big]$$

Similarly to before, $\mathbb{P}[\varphi'']$ gives us a lower bound to $\mathbb{P}[\varphi]$ by assigning new probabilities $1 - \sqrt[2]{1-q_1}$ to $y'_{1,1}$ and $y'_{1,2}$. And we can calculate this lower bound with query plan P_T by replacing T with a view V_T that has the probability q_j of a tuple y_j replaced with $1 - \sqrt[d_j]{1-q_j}$, where d_j is the number of tuples in S that join with tuple y_j in T.

Note that both query plans will calculate upper and lower bounds to query q over *any database instance* D. In fact, all possible query plans give upper bounds to the true query probability [Gatterbauer et al. 2010]. And as we have illustrated here, by replacing the input tables with appropriate views, we can use the same query plans to derive lower bounds. We refer the reader to [Gatterbauer et al. 2010] where we develop the theory of the partial dissociation order among all possible query plans, including a sound and complete algorithm that returns a set of query plans which are guaranteed to give the tightest bounds possible in a query-centric way for *any conjunctive query without self-joins*. For our example hard query of this section, plans P_R and P_T are the best possible plans. We further refer to [Gatterbauer and Suciu 2013] for more details and an extensive discussion on how to speed up the resulting multi-query evaluation.

Also note that these upper and lower bounds can be derived with the help of *any standard relational database*, even cloud-based databases which commonly do not allow users to define their own UDAs.[11] To our best knowledge, this is the first technique that allows to upper and lower bound hard probabilistic queries *without any modifications* to the database engine nor performing any calculations outside the database.

7. ILLUSTRATIONS OF OBLIVIOUS BOUNDS

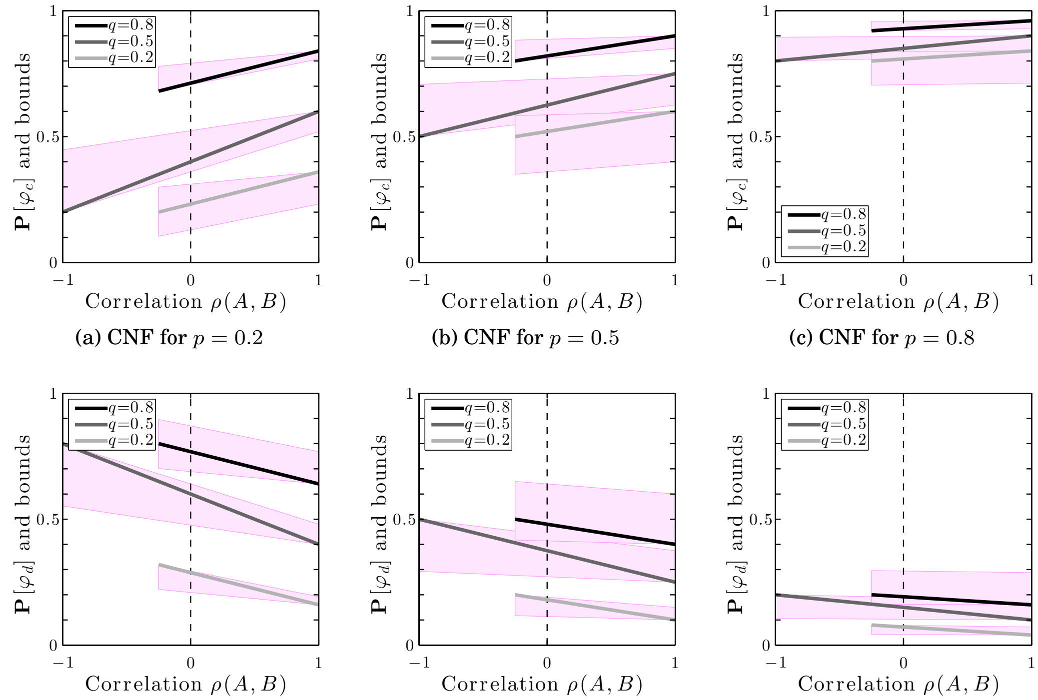
In this section, we study the quality of oblivious bounds across varying scenarios: We study the bounds as function of correlation between non-dissociated variables (Sect. 7.1), compare dissociation-based with model-based approximations (Sect. 7.2), illustrate a fundamental asymmetry between optimal upper and lower bounds (Sect. 7.3), show that increasing the number of simultaneous dissociations does not necessarily worsen the bounds (Sect. 7.4), and apply our framework to approximate hard probabilistic queries over TPC-H data with a standard relational database management system (Sect. 7.5).

7.1. Oblivious Bounds as Function of Correlation between Variables

Example 7.1 (*Oblivious Bounds and Correlations*). Here we dissociate the DNF $\varphi_d = xA \vee xB$ and the analogous CNF $\varphi_c = (x \vee A)(x \vee B)$ on x and study the error of the optimal oblivious bounds as function of the correlation between A and B.[12] Clearly, the bounds also depend on the probabilities of the variables x, A, and B. Let

[11] The UDA **iOR** can be expressed with standard SQL aggregates, e.g., "**iOR**(Q3.P)" can be evaluated with "1-exp(sum(log(case Q3.P when 1 then 1E-307 else 1-Q3.P end)))" on Microsoft SQL Azure.

[12] Note that this simplified example also illustrates the more general case $\psi_d = xA \vee xB \vee C$ when C is independent of A and B, and thus $\mathbb{P}[\psi_d] = \mathbb{P}[\varphi_d](1 - \mathbb{P}[C]) + \mathbb{P}[C]$. As a consequence, the graphs in Fig. 6 for $\mathbb{P}[C] \neq 0$ would be vertically compressed and the bounds tighter in absolute terms.



(a) CNF for $p = 0.2$ (b) CNF for $p = 0.5$ (c) CNF for $p = 0.8$

(d) DNF for $p = 0.8$ (e) DNF for $p = 0.5$ (f) DNF for $p = 0.2$

Fig. 6. Example 7.1. Probabilities of CNF $\varphi_c = (x \vee A)(x \vee B)$ and DNF $\varphi_d = xA \vee xB$ together with their symmetric optimal upper and lower oblivious bounds (borders of shaded areas) as function of the correlation $\rho(A, B)$ between A and B, and parameters $p = \mathbb{P}[x]$ and $q = \mathbb{P}[A] = \mathbb{P}[B]$. For every choice of p, there are some A and B for which the upper or lower bound becomes tight.

$p = \mathbb{P}[x]$ and assume A and B have the same probability $q = \mathbb{P}[A] = \mathbb{P}[B]$. We set $p' = \mathbb{P}[x'_1] = \mathbb{P}[x'_2]$ according to the optimal symmetric bounds from Corollary 4.9.

In a few steps, one can calculate the probabilities as

$$\mathbb{P}[\varphi_d] = 2pq - p\mathbb{P}[AB]$$
$$\mathbb{P}[\varphi'_d] = 2p'q - {p'}^2\mathbb{P}[AB]$$
$$\mathbb{P}[\varphi_c] = p + (1-p)\mathbb{P}[AB]$$
$$\mathbb{P}[\varphi'_c] = 2p'q + {p'}^2(1-2q) + (1-{p'}^2)\mathbb{P}[AB]$$

Results: Figure 6 shows the probabilities of the expressions $\mathbb{P}[\varphi]$ (full lines) and those of their dissociations $\mathbb{P}[\varphi']$ (border of shaded areas) for various values of p, q and as function of the correlation $\rho(A, B)$.[13] For example, Fig. 6d shows the answer when $\mathbb{P}[x]$

[13]The *correlation* $\rho(A, B)$ between Boolean events A and B is defined as $\rho(A, B) = \frac{\mathrm{cov}(A,B)}{\sqrt{\mathrm{var}(A)\mathrm{var}(B)}}$ with covariance $\mathrm{cov}(A, B) = \mathbb{P}[AB] - \mathbb{P}[A]\mathbb{P}[B]$ and variance $\mathrm{var}(A) = \mathbb{P}[A] - (\mathbb{P}[A])^2$ [Feller 1968]. Notice that $\rho(A, B) = \frac{\mathbb{P}[AB]-q^2}{q-q^2}$ and, hence: $\mathbb{P}[AB] = \rho(A, B)\cdot(q-q^2)+q^2$. Further, $\mathbb{P}[AB] = 0$ (i.e. disjointness between A and B) is not possible for $q > 0.5$, and from $\mathbb{P}[A \vee B] \leq 1$, one can derive $\mathbb{P}[AB] \geq 2p - 1$. In turn, $\rho = -1$ is not possible for $q < 0.5$, and it must hold $\mathbb{P}[AB] \geq 0$. From both together, one can derive the condition $\rho_{\min}(q) = \max(-\frac{q}{1-q}, -\frac{1+q^2-2q}{q-q^2})$ which gives the minimum possible value for ρ, and which marks the left starting point of the graphs in Fig. 6 as function of q.

is $p = 0.8$ and A, B have the same probability q of either 0.8, 0.5, or 0.2. When A, B are not correlated at all ($\rho = 0$), then the upper bounds seem a better approximation, especially when q is small. On the other hand, if A, B are not correlated, then there is no need to dissociate the two instances of x. The more interesting case is when A, B are positively correlated ($\mathbb{P}[AB] \geq \mathbb{P}[A]\mathbb{P}[B]$, e.g., positive Boolean functions of other independent variables $\mathbf{z}$, such as the provenance for probabilistic conjunctive queries). The right of the vertical dashed line of Fig. 6d shows that, in this case, dissociation offers very good upper and lower bounds, especially when the formula has a low probability. The graph also shows the effect of dissociation when A, B are negatively correlated (left of dashed line). Notice that the correlation cannot always be -1 (e.g., two events, each with probability > 0.5, can never be disjunct). The graphs also illustrate why these bounds are obliviously optimal, i.e. without knowledge of A, B: for every choice of p, there are some A, B for which the upper or lower bound becomes tight. ■

7.2. Oblivious Bounds versus Model-based Approximations

Example 7.2 (*Dissociation and Models*). This example compares the approximation of our dissociation-based approach with the model-based approach by Fink and Olteanu [2011] and illustrates how dissociation-based bounds are tighter, in general, than model-based approximations. For this purpose, we consider again the hard Boolean query $Q \text{:-}\, R(X), S^d(X, Y), T(Y)$ over the database D from Sect. 6. We now only assume that the table S is deterministic, as indicated by the superscript d in S^d. The query-equivalent lineage formula is then

$$\varphi = x_1y_1 \vee x_2y_1 \vee x_2y_2$$

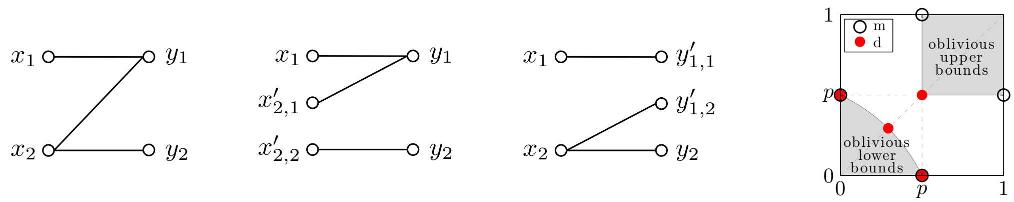
for which Fig. 7a shows the bipartite primal graph. We use this instance as its primal graph forms a P_4, which is the simplest 2-partite lineage that is not read-once.[14] In order to compare the approximation quality, we need to limit ourselves to an example which is tractable enough so we can generate the whole probability space. In practice, we allow each variable to have any of 11 discrete probabilities $D = \{0, 0.1, 0.2, \ldots, 1\}$ and consider all $11^4 = 14641$ possible probability assignments $\nu : \langle p_1, p_2, q_1, q_2\rangle \to D^4$ with $\mathbf{p} = \mathbb{P}[\mathbf{x}]$ and $\mathbf{q} = \mathbb{P}[\mathbf{y}]$. For each ν, we calculate both the absolute error $\delta^* = \mathbb{P}[\varphi^*] - \mathbb{P}[\varphi]$ and the relative error $\varepsilon^* = \frac{\delta^*}{\mathbb{P}[\varphi]}$, where $\mathbb{P}[\varphi^*]$ stands for any of the approximations, and the exact probability $\mathbb{P}[\varphi]$ is calculated by the Shannon expansion on y_1 as $\varphi \equiv y_1(x_1 \vee x_2) \vee \neg y_1(x_2y_2)$ and thus $\mathbb{P}_{\mathbf{p},\mathbf{q}}[\varphi] = (1-(1-p_1)(1-p_2))q_1 + (1-q_1)p_2q_2$.

Models: We use the model-based approach by Fink and Olteanu [2011] to approximate φ with lowest upper bound (LUB) formulas φ_{Ui} and greatest lower bound (GLB) formulas φ_{Li}, for which $\varphi_{Li} \Rightarrow \varphi$ and $\varphi \Rightarrow \varphi_{Ui}$, and neither of the upper (lower) bounds implies another upper (lower) bound. Among all models considered, we focus on only read-once formulas. Given lineage φ, the 4 LUBs are $\varphi_{U1} = x_1y_1 \vee x_2$, $\varphi_{U2} = y_1 \vee x_2y_2$, $\varphi_{U3} = (x_1 \vee x_2)y_1 \vee y_2$, and $\varphi_{U4} = x_1 \vee x_2(y_1 \vee y_2)$. The 3 GLBs are $\varphi_{L1} = (x_1 \vee x_2)y_1$, $\varphi_{L2} = x_1(y_1 \vee y_2)$, and $\varphi_{L3} = x_1y_1 \vee x_2y_2$. For each ν, we choose $\min_i(\mathbb{P}[\varphi_{Ui}])$ and $\max_i(\mathbb{P}[\varphi_{Li}])$ as the best upper and lower model-based bounds, respectively.

Dissociation: Analogously, we consider the possible dissociations into read-once formulas. For our given φ, those are $\varphi'_1 = (x_1 \vee x'_{2,1})y_1 \vee x'_{2,2}y_2$ and $\varphi'_2 = x_1y'_{1,1} \vee x_2(y'_{1,2} \vee y_2)$, with Fig. 7b and Fig. 7c illustrating the dissociated read-once primal graphs.[15] From Corollary 4.9, we know that $\mathbb{P}_{\mathbf{p}',\mathbf{q}}[\varphi'_1] \geq \mathbb{P}_{\mathbf{p},\mathbf{q}}[\varphi]$ for the only optimal oblivious upper bounds $p'_{2,1} = p'_{2,2} = p_2$ and $\mathbb{P}_{\mathbf{p}',\mathbf{q}}[\varphi'_1] \leq \mathbb{P}_{\mathbf{p},\mathbf{q}}[\varphi]$ for any $\mathbf{p}'_2$ with $\bar{p}'_{2,1}\bar{p}'_{2,2} = \bar{p}_2$. In particular, we choose 3 alternative optimal oblivious lower bounds $\mathbf{p}'_2 \in \{\langle p_2, 0\rangle, \langle 1 -$

---

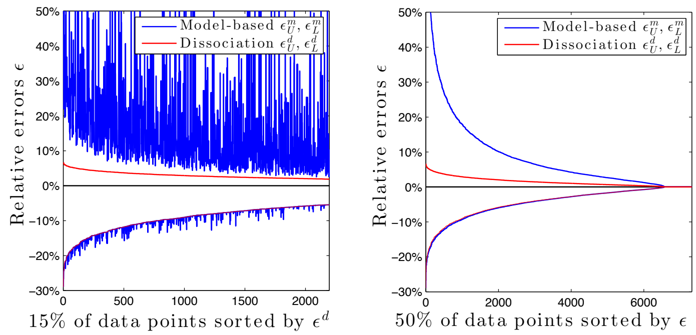
[14] A path P_n is a graph with vertices $\{x_1, \ldots, x_n\}$ and edges $\{x_1x_2, x_2x_3, \ldots, x_{n1}x_n\}$.
[15] Note that we consider here dissociation on both x- and y-variables, thus do not treat them as distinct.



(a) φ (b) φ'_1 (c) φ'_2 (d) $\mathbf{p}'$ for DNF

			average	worst
absolute	m	δ_U^m	1.54%	15.8%
		δ_L^m	−0.98%	−10.5%
	d	δ_U^d	0.35%	3.7%
		δ_L^d	−0.94%	−8.7%
relative	m	ε_U^m	4.55%	289.3%
		ε_L^m	−2.23%	−28.9%
	d	ε_U^d	0.73%	6.7%
		ε_L^d	−2.17%	−28.9%

(e) Errors



(f) (g)

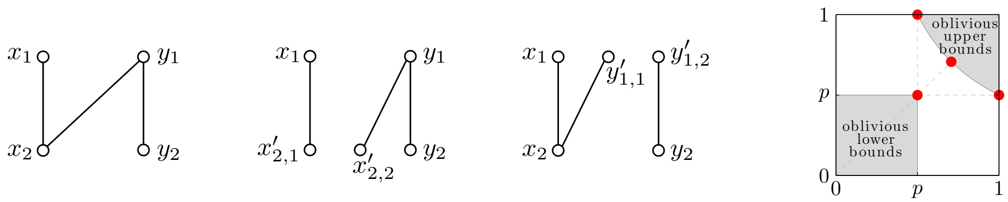
Fig. 7. Example 7.2. (a)-(c): Bipartite primal graphs for DNF φ and two dissociations. Notice that the primal graphs of φ'_1 and φ'_2 are forests and thus correspond to read-once expressions. (d): For a given disjunctive dissociation d, there is only one optimal oblivious upper bound but infinitely many optimal lower bounds. We evaluate $\mathbb{P}[\varphi']$ for three of the latter (two of which coincide with models m) and keep the maximum as the best oblivious lower bound. (e): In comparison, dissociation gives substantially better *upper* bounds than model-based bounds (0.73% vs. 4.55% average and 6.7% vs. 289.3% worst-case relative errors), yet *lower* bounds are only slightly better. (f): Relative errors of 4 approximations for individual data points sorted by the dissociation error for upper bounds and for lower bounds separately; this is why the dissociation errors show up as smooth curves (red) while the model based errors are unsorted and thus ragged (blue). (g): Here we sorted the errors for each approximation individually; this is why all curves are smooth.

$\sqrt{1-p_2}, 1-\sqrt{1-p_2}\rangle, \langle 0, p_2\rangle\}$ (see Fig. 7d). Analogously $\mathbb{P}_{\mathbf{p},\mathbf{q}'}[\varphi'_2] \geq \mathbb{P}_{\mathbf{p},\mathbf{q}}[\varphi]$ for $q'_{1,1} = q'_{1,2} = q_1$ and $\mathbb{P}_{\mathbf{p},\mathbf{q}'}[\varphi'_2] \leq \mathbb{P}_{\mathbf{p},\mathbf{q}}[\varphi]$ for $\mathbf{q}'_1 \in \{\langle q_1, 0\rangle, \langle 1-\sqrt{1-q_1}, 1-\sqrt{1-q_1}\rangle, \langle 0, q_1\rangle\}$. For each ν, we choose the minimum among the 2 upper bounds and the maximum among the 6 lower bounds as the best upper and lower dissociation-based bounds, respectively.

Results: Figures 7e-g show that dissociation-based bounds are always better or equal to model-based bounds. The reason is that all model-based bounds are a special case of oblivious dissociation bounds. Furthermore, dissociation gives far better upper bounds, but only slighter better lower bounds. The reason is illustrated in Fig. 7d: the single dissociation-based upper bound $\mathbf{p}' = \langle p, p\rangle$ always dominates the two model-based upper bounds, whereas the two model-based lower bounds are special cases of infinitely many optimal oblivious lower dissociation-based bounds. For our example, we evaluate three oblivious lower bounds, two of which coincide with models. ■

7.3. Conjunctive versus Disjunctive Dissociation

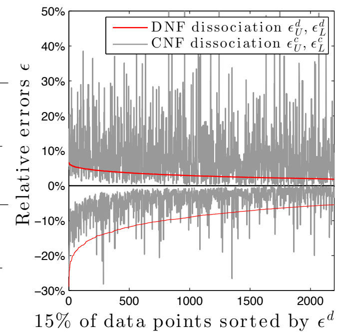
Example 7.3 (*Disjunctive and Conjunctive Dissociation*). This example illustrates an interesting asymmetry: optimal upper bounds for disjunctive dissociations and optimal lower bounds for conjunctive dissociations are not only unique but also generally better than their counterparts (see Figure 1). We show this by comparing the approximation of a function by either dissociating a conjunctive or a disjunctive expression.



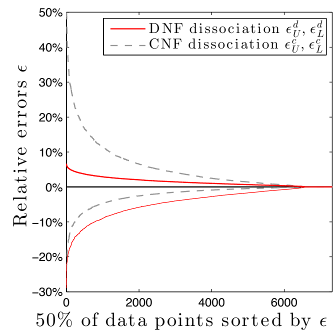
(a) ψ (b) ψ'_1 (c) ψ'_2 (d) $\mathbf{p}'$ for CNF

			average	worst
absolute	DNF	δ^d_U	0.35%	3.7%
		δ^d_L	−0.94%	−8.7%
	CNF	δ^c_U	0.94%	8.7%
		δ^c_L	−0.35%	−3.7%
relative	DNF	ε^d_U	0.73%	6.7%
		ε^d_L	−2.17%	−28.9%
	CNF	ε^c_U	2.69%	54.5%
		ε^c_L	−1.09%	−28.2%

(e) Errors



(f)



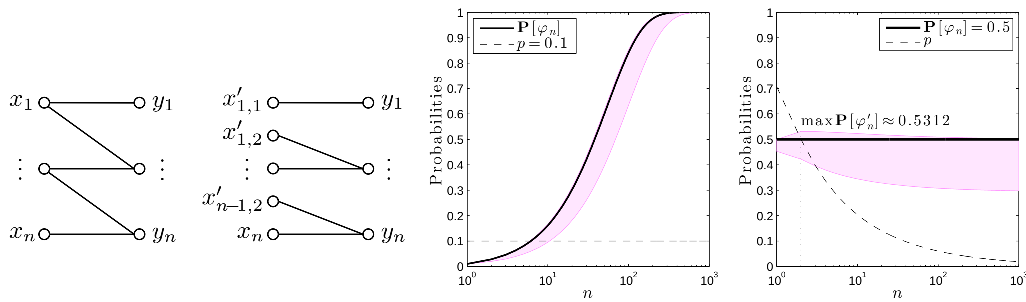
(g)

Fig. 8. Example 7.3. (a)-(c): Bipartite primal graphs for CNF ψ and two dissociations. (d): For a given conjunctive dissociation c, we use the only optimal oblivious lower bound and three of infinitely many optimal oblivious upper bounds. (e): In comparison, disjunctive dissociation gives better *upper* bounds than conjunctive dissociation (0.73% vs. 2.69% average and 6.7% vs. 54.5% worst-case relative errors), and v.v. for lower bounds. (f): Relative errors of 4 approximations for individual data points sorted by the disjunctive dissociation error ε^d for upper bounds and for lower bounds separately; this is why the DNF dissociation errors show up as smooth curves (red) while the CNF dissociation errors are unsorted and thus ragged (gray). (g): Here we sorted the errors for each approximation individually; this is why all curves are smooth.

We re-use the setup from Example 7.2 where we had a function expressed by a disjunctive expression φ. Our DNF φ can be written as CNF $\psi = (x_1 \vee x_2)(y_1 \vee x_2)(y_1 \vee y_2)$ with $f_\varphi = f_\psi$[16], and two conjunctive dissociations $\psi'_1 = (x_1 \vee x'_{2,1})(y_1 \vee x'_{2,2})(y_1 \vee y_2)$ and $\psi'_2 = (x_1 \vee x_2)(y'_{1,1} \vee x_2)(y'_{1,2} \vee y_2)$ (Figures 8a-c shows the primal graphs). Again from Corollary 4.9, we know that $\mathbb{P}_{\mathbf{p}',\mathbf{q}}[\varphi'_1] \leq \mathbb{P}_{\mathbf{p},\mathbf{q}}[\varphi]$ for the only optimal oblivious lower bounds $p'_{2,1} = p'_{2,2} = p_2$ and $\mathbb{P}_{\mathbf{p}',\mathbf{q}}[\varphi'_1] \geq \mathbb{P}_{\mathbf{p},\mathbf{q}}[\varphi]$ for any $\mathbf{p}'_2$ with $p'_{2,1}p'_{2,2} = p_2$. In particular, we choose 3 alternative optimal oblivious lower bounds $\mathbf{p}'_2 \in \{\langle p_2, 1\rangle, \langle\sqrt{p_2}, \sqrt{p_2}\rangle, \langle 1, p_2\rangle\}$ (see Fig. 7d). Analogously $\mathbb{P}_{\mathbf{p},\mathbf{q}'}[\varphi'_2] \leq \mathbb{P}_{\mathbf{p},\mathbf{q}}[\varphi]$ for $q'_{1,1} = q'_{1,2} = q_1$ and $\mathbb{P}_{\mathbf{p},\mathbf{q}'}[\varphi'_2] \geq \mathbb{P}_{\mathbf{p},\mathbf{q}}[\varphi]$ for $\mathbf{q}'_1 \in \{\langle q_1, 1\rangle, \langle\sqrt{q_1}, \sqrt{q_1}\rangle, \langle 1, q_1\rangle\}$. For each ν, we choose the maximum among the 2 lower bounds and the minimum among the 6 upper bounds as the best upper and lower conjunctive dissociation-based bounds, respectively. We then compare with the approximations from the DNF φ in Example 7.2.

Results: Figures 8e-g show that optimal disjunctive upper bounds are, in general but not consistently, better than optimal conjunctive upper bounds for the same function ($\approx 83.5\%$ of those data points with different approximations are better for conjunctive dissociations). The dual result holds for lower bounds. This duality can be best seen in the correspondences of absolute errors between upper and lower bounds. ■

[16]Notice that this transformation from DNF to CNF is hard, in general. We not do not focus on algorithmic aspects in this paper, but rather show the potential of this new approach.



(a) φ_n (b) φ'_n (c) $p = \text{const}$ (d) $\mathbb{P}[\varphi_n] = \text{const}$

Fig. 9. Example 7.4. (a), (b): Primal graphs for path P_n DNF φ_n and its dissociation φ'_n. (c), (d): $\mathbb{P}[\varphi_n]$ together with their symmetric optimal upper and lower oblivious bounds (borders of shaded areas) as function of n. (c) keeps $p = 0.1$ constant, whereas (d) varies p so as to keep $\mathbb{P}[\varphi] = 0.5$ constant for increasing n. The upper bounds approximate the probability of the DNF very well and even become tight for $n \to \infty$.

7.4. Multiple Dissociations at Once

Here we investigate the influence of the primal graph and number of dissociations on the tightness of the bounds. Both examples correspond to the lineage of the standard unsafe query $Q :- R(X), S(X,Y), T(Y)$ over two different database instances.

Example 7.4 (*Path P_n as Primal Graph*). This example considers a DNF expression whose primal graph forms a P_n, i.e. a path of length n (see Fig. 9a). Note that this is a generalization of the path P_4 from Example 7.2 and corresponds to the lineage of the same unsafe query over larger database instance with $2n-1$ tuples:

$$\varphi_n = x_1 y_1 \lor x_1 y_2 \lor x_2 y_2 \lor \ldots \lor x_{n-1} y_n \lor x_n y_n$$
$$\varphi'_n = x'_{1,1} y_1 \lor x'_{1,2} y_2 \lor x'_{2,1} y_2 \lor \ldots \lor x'_{n-1,2} y_n \lor x'_n y_n$$

Exact: In the following, we assume the probabilities of all variables to be p and use the notation $p_n := \mathbb{P}[\varphi_n]$ and $p^*_n := \mathbb{P}[\varphi^*_n]$, where φ^*_n corresponds to the formula φ_n without the last conjunct $x_n y_n$. We can then express p_n as function of p^*_n, p_{n-1} and p^*_{n-1} by recursive application of Shannon's expansion to x_n and y_n:

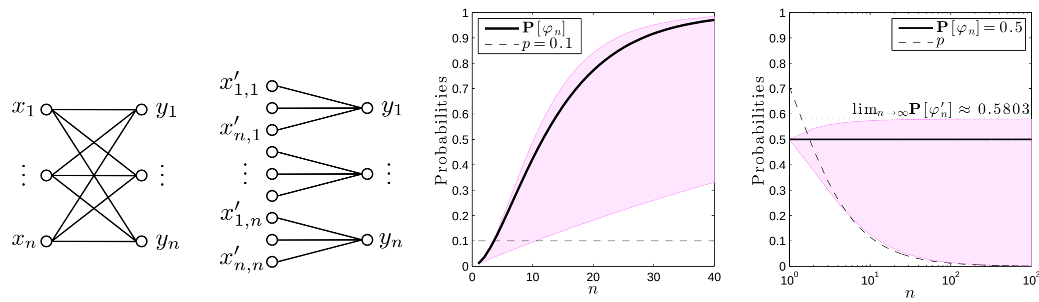
$$p_n = \mathbb{P}[x_n]\big(\mathbb{P}[y_n] + \mathbb{P}[\bar{y}_n] p_{n-1}\big) + \mathbb{P}[\bar{x}_n] p^*_n$$
$$p^*_n = \mathbb{P}[y_n]\big(\mathbb{P}[x_{n-1}] + \mathbb{P}[\bar{x}_{n-1}] p^*_{n-1}\big) + \mathbb{P}[\bar{y}_n] p_{n-1}$$

We thus get the linear recurrence system

$$p_n = A_1 p_{n-1} + B_1 p^*_{n-1} + C_1$$
$$p^*_n = A_2 p_{n-1} + B_2 p^*_{n-1} + C_2$$

with $A_1 = \bar{p}$, $B_1 = p\bar{p}^2$, $C_1 = p^2(2-p)$, $A_2 = \bar{p}$, $B_2 = p\bar{p}$, and $C_2 = p^2$. With a few manipulations, this recurrence system can be transformed into a linear non-homogenous recurrence relation of second order $p_n = A p_{n-1} + B p_{n-2} + C$ where $A = A_1 + B_2 = \bar{p}(1+p)$, $B = A_2 B_1 - A_1 B_2 = -p^2 \bar{p}^2$, and $C = B_1 C_2 + C_1(1 - B_2) = p^2(p\bar{p}^2 + (2-p)(1-p\bar{p}))$. Thus we can recursively calculate $\mathbb{P}[\varphi_n]$ for any probability assignment p starting with initial values $p_1 = p^2$ and $p_2 = 3p^2 - 2p^3$.

Dissociation: Figure 9b shows the primal graph for the dissociation φ'_n. Variables x_1 to x_{n-1} are dissociated into two variables with same probability p', whereas x_n into one with original probability p. In other words, with increasing n, there are more variables dissociated into two fresh ones each. The probability $\mathbb{P}[\varphi'_n]$ is then equal to



(a) φ_n (b) φ'_n (c) $p = \text{const}$ (d) $\mathbb{P}[\varphi_n] = \text{const}$

Fig. 10. Example 7.5. (a), (b): Primal graphs for complete bipartite DNF φ_n and its dissociation φ'_n. (c), (d): $\mathbb{P}[\varphi_n]$ together with their symmetric optimal upper and lower oblivious bounds (borders of shaded areas) as function of n. (d) varies p so as to keep $\mathbb{P}[\varphi] = 0.5$ constant for increasing size n. The oblivious upper bound is ultimately bounded despite having n^2 fresh variables in the dissociated DNF for increasing n.

the probability that at least one variable x_i is connected to one variable y_j:

$$\mathbb{P}[\varphi'_n] = 1 - (1 - pp')\big(1 - p(1 - \bar{p}'^2)\big)^{n-2}\big(1 - p(1 - \bar{p}\bar{p}')\big)$$

We set $p' = p$ for upper bounds, and $p' = 1 - \sqrt{1-p}$ for lower bounds.

Results: Figures 9c-d shows the interesting result that the disjunctive upper bounds become tight for increasing size of the primal graph, and thus increasing number of dissociations. This can be best seen in Fig. 9d for which p is chosen as to keep $\mathbb{P}[\varphi] = 0.5$ constant for varying n and we have $\lim_{n\to\infty} \mathbb{P}[\varphi'_n] = \mathbb{P}[\varphi_n] = 0.5$ for upper bounds. In contrast, the disjunctive lower bounds become weaker but still have a limit value $\lim_{n\to\infty} \mathbb{P}[\varphi'_n] \approx 0.2929$ (derived numerically). ■

Example 7.5 (*Complete bipartite graph $K_{n,n}$ as Primal Graph*). This example considers a DNF whose primal graph forms a complete bipartite graph of size n, i.e. each variable x_i is appearing in one clause with each variable y_j (see Fig. 10a). Note that this example corresponds to lineage for the standard unsafe query over a database instance with $\mathcal{O}(n^2)$ tuples:

$$\varphi_n = \bigvee_{(i,j)\,\in\,[n]^2} x_i y_j$$

$$\varphi'_n = \bigvee_{j\in[n]} \Big(y_j \bigvee_{i\in[n]} x'_{i,j}\Big)$$

Exact: We again assume that all variables have the same probability $p = \mathbb{P}[x_i] = \mathbb{P}[y_i]$. $\mathbb{P}[\varphi_n]$ is then equal to the probability that there is at least one tuple x_i and at least one tuple y_i:

$$\mathbb{P}[\varphi_n] = \big(1 - (1-p)^n\big)^2 \tag{6}$$

Dissociation: Figure 10b shows the primal graph for the dissociation φ'_n. Each variable x_i is dissociated into n fresh variables with same probability p', i.e. there are n^2 fresh variables in total. The probability $\mathbb{P}[\varphi'_n]$ is then equal to the probability that at least one variable y_i is connected to one variables $x_{i,j}$:

$$\mathbb{P}[\varphi'_n] = 1 - \Big(1 - p\big(1 - (1-p')^n\big)\Big)^n$$

```
select distinct s_nationkey
from Supplier, Partsupp, Part
where s_suppkey = ps_suppkey
and ps_partkey = p_partkey
and s_suppkey <= $1
and p_name like $2
```

(a) Deterministic SQL

```
Supplier(s_suppkey, s_nationkey)
PartSupp(ps_suppkey, ps_partkey)
Part(p_partkey, p_name)
```

(b) TPC-H schema

Fig. 11. Example 7.6. Parameterized SQL query and relevant portion of the TPC-H schema.

We will again choose p as to keep $r := \mathbb{P}[\varphi_n]$ constant with increasing n, and then calculate $\mathbb{P}[\varphi'_n]$ as function of r. From Eq. 6, we get $p = 1 - \sqrt[n]{1 - \sqrt{r}}$ and then set $p' = p$ for upper bounds, and $p' = 1 - \sqrt[n]{1 - p}$ for lower bounds as each dissociated variable is replaced by n fresh variables. It can then be shown that $\mathbb{P}[\varphi'_n]$ for the upper bound is monotonically increasing for n and bounded below 1 with the limit value:

$$\lim_{n \to \infty} \mathbb{P}[\varphi'_n] = 1 - (1 - \sqrt{r})^{\sqrt{r}}$$

Results: Figure 10d keeps $\mathbb{P}[\varphi] = 0.5$ constant (by decreasing p for increasing n) and shows the interesting result that the optimal upper bound is itself upper bounded and reaches a limit value, although there are more variables dissociated, and each variable is dissociated into more fresh ones. This limit value is 0.5803 for $r = 0.5$. However, lower bounds are not useful in this case. ■

7.5. Dissociation with a Standard Relational Database Management System

Example 7.6 (*TPC-H*). Here we apply the theory of dissociation to bound hard probabilistic queries with the help of PostgreSQL 9.2, an open-source relational database management systems.[17] We use the TPC-H DBGEN data generator[18] to generate a 1GB database. We then add a column P to each table, and assign to each tuple either (i) a random probability $p \in \{0.01, 0.02, \ldots, 0.5\}$ ("$p =$ rand 0.5"), or (i) the same probability $p = 0.5$, or (iii) the same probability $p = 0.1$. Choosing a small tuple probability $p = 0.1$ helps avoid floating-point errors for queries with large lineage and the answer tuple probabilities too close to 1 (see Fig. 13b for how close our results come to 1). We then consider the following parameterized query (also see Fig. 11):

$$Q(a) :- S(\underline{s}, a), PS(s, u), P(\underline{u}, n), s \leq \$1, n \text{ like } \$2$$

Variable a stands for attribute nationkey ("answer tuple"), s for suppkey, u for partkey ("unit"), and n for name. The probabilistic version of this query asks which nations (as determined by the attribute nationkey) are most likely to have suppliers with suppkey $\leq$ $1 that supply parts with a name like $1 when all records in Supplier, PartSupp, and Part are uncertain. Parameters $1 and $2 allow us to reduce the number of tuples that can participate from tables Supplier and Part, respectively, and to thus study the effects of lineage size on the predicted dissociation bounds and running time. By default, tables Supplier, Partsupp and Part have 10k, 800k, and 200k tuples, respectively, and there are 25 different numeric attributes for nationkey. For parameter $1, we choose a value $\in \{500, 1000, \ldots, 10000\}$, which corresponds to the number of tuples that can participate from table Supplier. for parameter $\$2$ we choose values $\in$ {'%', '%red%', '%red%green%' } which select 200k, 11k and 251 tuples in table Part, respectively.

[17] http://www.postgresql.org/download/
[18] http://www.tpc.org/tpch/

```
create view VP as
select p_partkey, s_nationkey,
  (1-POWER(1-P.P,1e0/count(*))) as P
from Part P, Partsupp, Supplier
where p_partkey=ps_partkey
and ps_suppkey = s_suppkey
and s_suppkey <= $1
and p_name like $2
group by p_partkey, s_nationkey, P.P
```

(a) View V_P for lower bounds with P_P^*

```
select s_nationkey, iOR(Q3.P) as P
from
  (select s_nationkey, S.P*Q2.P as P
  from Supplier S,
    (select Q1.ps_suppkey, s_nationkey, iOR(Q1.P) as P
    from
      (select ps_suppkey, s_nationkey, PS.P*VP.P as P
      from Partsupp PS, VP
      where ps_partkey = p_partkey
      and ps_suppkey <= $1) as Q1
    group by Q1.ps_suppkey, s_nationkey) as Q2
  where s_suppkey = Q2.ps_suppkey) as Q3
group by Q3.s_nationkey
```

(b) SQL query P_P^*

Fig. 12. Example 7.6. (a) View definition V_P and (b) adapted subsequent query P_P^* for deriving the lower bound by dissociating table P. Note the inclusion of the attribute nationkey in V_P as explained in the text.

Translation into SQL: Note that the lineage for each answer tuple corresponds to the Boolean query Q from Sect. 6 which is known to be hard. We thus bound the probability for each answer tuple by evaluating *four different queries* which correspond to the query-centric dissociation bounds from Sect. 6: dissociating either table Supplier or table Part, and calculating either upper and lower bounds. To get the final upper (lower) bounds, we take the minimum (maximum) of the two upper (lower) bounds for each answer tuple. The two query plans are as follows:

$$P_S(a) = \pi_a^p \bowtie_u^p \big[\pi_{a,u}^p \bowtie_s^p \big[S(\underline{s}, a), PS(s,u), s \leq \$1\big], P(\underline{u}, n), n \text{ like } \$2\big]$$
$$P_P(a) = \pi_a^p \bowtie_s^p \big[S(\underline{s}, a), \pi_s^p \bowtie_u^p \big[PS(s,u), s \leq \$1, P(\underline{u}, n), n \text{ like } \$2\big]\big]$$

Note, one technical detail for determining the lower bound with plan P_P: Any tuple t from table Part may appear a different number of times in the lineage of different query answers.[19] Thus, for every answer tuple a that has t in its lineage, we need to create a *distinct copy* of t in the view V_T, with a probability that depends only on the number of dissociation in the lineage of a.[20] Thus, the view definition for V_R needs to include the attribute nationkey (Fig. 12a) and P_P needs to be adapted as follows (Fig. 12b):

$$P_P^*(a) = \pi_a^p \bowtie_s^p \big[S(\underline{s}, a), \pi_{s,a}^p \bowtie_u^p \big[PS(s,u), s \leq \$1,\, VP(\underline{u}, a)\big]\big]$$

In order to speed up the resulting multi-query evaluation, we first apply a *deterministic semi-join reduction* on the input tables, and then reuse intermediate query results across all four subsequent queries. These techniques are not shown in Fig. 12, but given in detail in [Gatterbauer and Suciu 2013]. Also, the exact SQL statements that allow the interested reader to repeat these experiments over a TCP-H database imported into PostgreSQL are available on the LaPushDB project page.[21]

Ground truth: To compare our bounds against the actual true probabilities, we issue a lineage query to retrieve and construct the DNF for each tuple, then use DeMorgan to write a lineage DNF as a probabilistic CNF without exponential increase in size. For example, the lineage DNF $\Phi = x_1x_3 \vee x_1x_2$ can be written as CNF $\bar{\Phi} = (\bar{x}_1 \vee \bar{x}_3) \wedge$

[19]The same technical detail does not occur in the dissociation on S because of the key constraint $S(\underline{s}, a)$.
[20]In fact, we could actually just use the total number of times the tuple appears in all lineages and still get lower bounds. However, the resulting bounds would not be optimal oblivious bounds.
[21]http://LaPushDB.com/

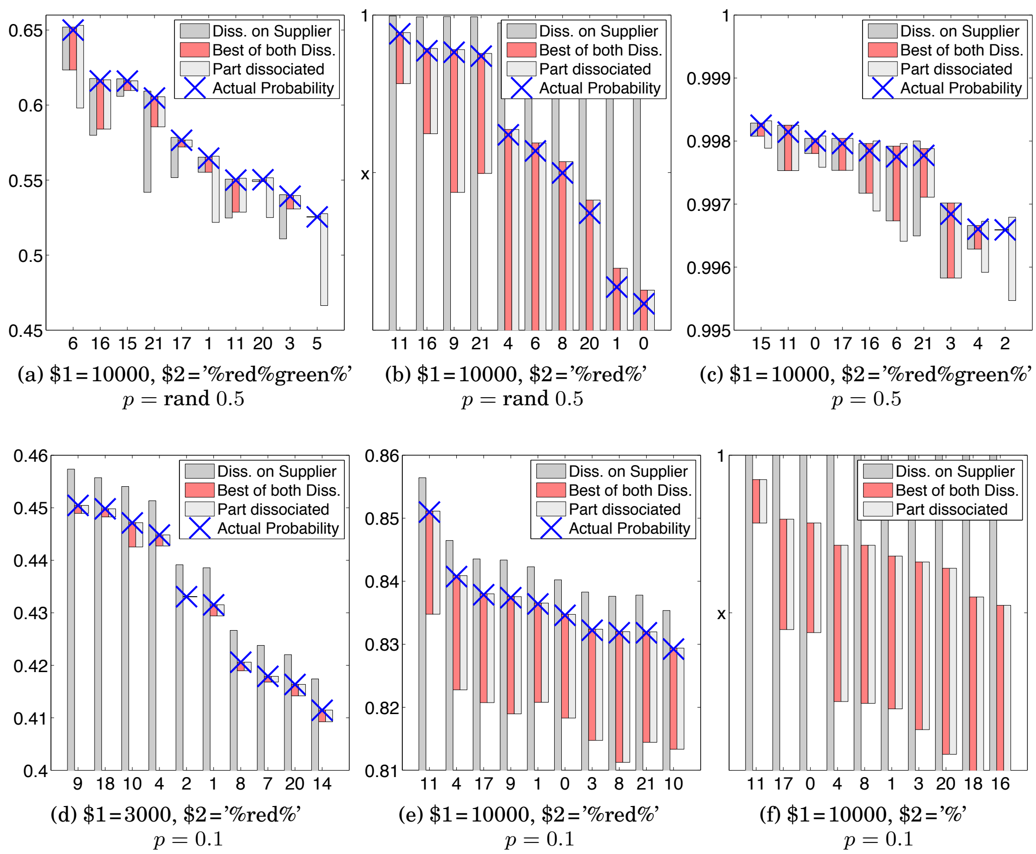


Fig. 13. Example 7.6. Probability bounds for the top 10 query results for varying query parameters \$1, \$2, and tuple probabilities p. The ranking is determined by the upper dissociation bounds (upper end of the red interval) and is identical to the one determined by the actual probabilities (crosses), except for (c) in which tuples 6 and 21 are flipped, and for (f) for which the ground truth is not known. (b): $x = 0.999999999999 = 1 - 10^{-12}$. (f): $x = 0.9999999999 = 1 - 10^{-10}$.

$(\bar{x}_1 \vee \bar{x}_2)$ with $\mathbb{P}[\bar{\Phi}] = 1 - \mathbb{P}[\Phi]$. A probabilistic CNF can further be translated into the problem of computing the partition function of a propositional Markov random field. And for this latter problem, there are regular competitions in the AI community which results in existing tools that have reached a significant level of sophistication. For our experiments, we use a tool called SampleSearch [Gogate and Domingos 2010; Gogate and Dechter 2011].[22]

Bound Results: Figure 13 shows the top 10 query results, as predicted by the upper dissociation bounds, with varying parameters \$1 and \$2, as well as varying input probabilities p. The crosses show the actual probabilities determined by SampleSearch if available. The red intervals shows the interval between upper and lower dissociation bounds. Recall that the final dissociation interval is the intersection between the interval from dissociation on Supplier (left of the red interval) and on Part (right of the red interval). We see that the *upper dissociation bounds are very close to the actual*

[22] `http://www.hlt.utdallas.edu/~vgogate/SampleSearch.html`

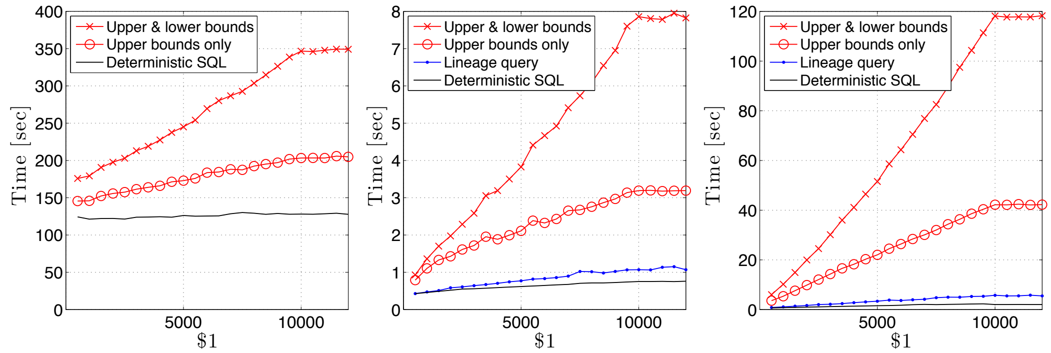
probabilities, which is reminiscent of Sect. 7.4 having shown that upper bounds for DNF dissociations are commonly closer to the true probabilities than lower bounds. For Fig. 13f, we have no ground truth as the lineage for the top tuple has size 32899 (i.e. the corresponding DNF has 32899 clauses), which the probabilistic solver cannot handle. The different sizes of the intervals arise from different numbers of dissociated tuples in the respective lineages However, extrapolating from Fig. 13d and Fig. 13e, we conjecture that dissociation gives reasonable bounds.

The different sizes of the intervals arise from different numbers of dissociated tuples in the respective lineages. For example, the lineage for the top-ranked tuple 6 in Fig. 13a has 42 unique tuples from table Part, out of which 5 ($\sim 12\%$) are dissociated into 2 fresh ones with P_P. In contrast, the lineage has 53 unique tuples from table Supplier, out of which only 4 ($< 8\%$) are dissociated into 2 fresh variables with P_S. Thus, intuitively, P_S gives tighter bounds. As another example, the lineage for the top-ranked tuple 11 in Fig. 13b and Fig. 13e has 1830 unique tuples from table Part, out of which less than 6% are dissociated into 2 or 3 fresh ones with P_P. In contrast, the lineage has only 434 unique tuples from table Supplier, out of which 95% are dissociated into into 2 to 11 fresh variables. Thus, in this case, P_P gives tighter bounds. As extreme case, the lineage for the same tuple 11 in Fig. 13f has 32899 unique tuples from table Part, out of which around 6.3% are dissociated into 2-4 fresh ones with P_P. In contrast, the lineage has only 438 unique tuples from table Supplier, out of which *all* are dissociated into 80 fresh ones with P_S (this is an artifact of the TPC-H random database generator). Thus P_S is very far off. On the other end, if no input tuple needs to be dissociated in either of the plans, then both upper and lower bound coincide (e.g., 2 in Fig. 13c). This scenario is also called *data-safe* [Jha et al. 2010]. The following table gives a succinct overview of the last paragraph:

tuple	Fig.	#Part	%diss.	into	#Supp.	%diss.	into	tighter bounds
2	Fig. 13c	40	8%	2	43	0%	-	P_S
6	Fig. 13a	42	12%	2	53	8%	2	P_S
11	Fig. 13b	1830	6%	2	434	95%	2-11	P_P
11	Fig. 13f	32899	6%	2	438	100%	80	P_P

Importantly, ranking of the answer tuples to a query by upper dissociation bounds comes very close to the ranking by query reliability. For example, for the case of $p = 0.1$ and $2='%red%' (Fig. 13d and Fig. 13e), the ranking given by the minimum upper bound was identical to the ranking given by the ground truth for all parameters choices $\$1 \in \{500, 1000, \ldots, 10000\}$. One of the few exceptions is shown in Fig. 13c for parameters $p = 0.5$, $1=10000, and $2='%red%green%'. Here tuples 6 and 21 are flipped as compared to their actual probabilities 0.99775 and 0.99777, respectively.

Timing Results: Figure 14 compares the times needed to evaluate the deterministic query in Fig. 11a with those of calculating the dissociation bounds for changing parameter $1 and averaged over 5 runs for each data point. Since table Supplier contains exactly 10k tuples with suppkey $\in \{1, \ldots, 10000\}$, any choice of $1≥10000 has no effect on the query. We show separate graphs for the time needed to calculate the *upper bounds only* (which our theory and experiments suggest give better absolute approximations and can serve for *relevance ranking*) and the time for both *upper and lower bounds* (the latter of which are more expensive because of required manipulation of the input tuples). We also show the time for retrieving the lineage with a *lineage query*. Any approach that evaluates the probability of an answer tuple outside the database engine needs to issue this query to construct the DNF. The time needed for the lineage query thus serves as appropriate minimum base line. We see that, in particular for small queries, the probabilistic evaluation adds only a small linear overhead on top of deterministic query evaluation. For increasing lineage, the query times scales exactly



(a) $2 = '%red%green%' (b) $2 = '%red%' (c) $2 = '%'

Fig. 14. Example 7.6. (a), (b): Top 10 query results for different values for parameter $1. The ranking is determined by the upper dissociation bounds (upper end of the red interval) and is identical to the one determined by the actual probabilities (crosses), except for tuple 6 and 21 being flipped in (b). (c): These bounds can be calculated in a small multiple of the time needed to evaluate standard deterministic queries.

linearly with the size of the average lineage (Fig. 14c). For our longest query we issued ($1 = 10000, $2 = '%'), the maximal lineage is 35040 for tuple 11 and the times are as follows:

Deterministic SQL	Lineage SQL	Upper bounds only	Upper & lower bounds
2.0 sec	5.7 sec	42.2 sec	118.1 sec

Overall, our experiments suggest that dissociation can allow ranking of input tuples that is close to the ranking for the actual probabilities in a time that is a small multiple of the time needed to evaluate standard deterministic queries. ■

8. RELATED WORK AND DISCUSSION

Dissociation is related to a number of recent approaches in the graphical model and constraint satisfaction literature which approximate an intractable problem with a tractable relaxed version after treating multiple occurrences of variables or nodes as independent or ignoring some equivalence constraints: Choi et al. [2007] approximate inference in Bayesian networks by "*node splitting*," i.e. removing some dependencies from the original model. Ramírez and Geffner [2007] treat the problem of obtaining a minimum cost satisfying assignment of a CNF formula by "*variable renaming*," i.e. replacing a variable that appears in many clauses by many fresh new variables that appear in few. Pipatsrisawat and Darwiche [2007] provide lower bounds for MaxSAT by "*variable splitting*," i.e. compiling a relaxation of the original CNF. Andersen et al. [2007] improve the relaxation for constraint satisfaction problems by "*refinement through node splitting*," i.e. making explicit some interactions between variables. Choi and Darwiche [2009] relax a problem by dropping equivalence constraints and partially compensate for the relaxation. Our work provides a general framework for approximating the probability of Boolean functions with both upper and lower bounds. We thus refer to all the above approaches as *dissociation-based approximations*.

Another line of work, that is varyingly called *discretization*, *bucketing*, or *quantization*, proposes relaxations by *merging* or partition instead of splitting states or nodes, and to then perform simplified calculations over those partitions. A famous example is mini-buckets [Dechter and Rish 2003] that approximates a function with high arity by a collection of smaller-arity functions. For example, for non-negative functions f and g, the summation problem $\sum_{\mathbf{x}} \big(f(\mathbf{x}) \cdot g(\mathbf{x})\big)$ can be upper bounded by the problem

$\big(\max_{\mathbf{x}} f(\mathbf{x})\big) \cdot \big(\sum_{\mathbf{x}} g(\mathbf{x})\big)$, i.e. $\mathbf{x}$ is independently chosen for optimizing f or g, and lower bounded by $\big(\min_{\mathbf{x}} f(\mathbf{x})\big) \cdot \big(\sum_{\mathbf{x}} g(\mathbf{x})\big)$. Similarly, Gogate and Domingos [2011] who compress potentials computed during the execution of variable elimination by "*quantizing*" them, namely by replacing a number of distinct values in range of the potential by a single value. Bergman et al. [2011],[2013] construct relaxations of Multivalued Decision Diagrams (MDDs) by merging vertices when the size of the partially constructed MDD grows too large. One of the earlier works in this space was by St-Aubin et al. [2000] who use Algebraic Decision Diagram (ADDs) and reduces the sizes of the intermediate value functions generated by replacing the values at the terminals of the ADD with ranges of values. We collectively refer to these approaches above as *quantization-based approximations*.

Note that dissociation-based and quantization-based approaches are not reverse of one another. The reverse of dissociation is what we call *assimilation*:[23] Consider the Boolean formula $\varphi = x_1y_1 \vee x_1y_2 \vee x_2y_2$. It is not read-once, but a dissociation of the read-once formula $\varphi^* = x(y_1 \vee y_2)$. Hence, we know from dissociation that $\mathbb{P}[\varphi] \geq \mathbb{P}[\varphi^*]$ for $p = \min(p_1, p_2)$ and that $\mathbb{P}[\varphi] \leq \mathbb{P}[\varphi^*]$ for $\bar{p} = \bar{p}_1 \cdot \bar{p}_2$. Note the difference between quantization and assimilation: In quantization, we can choose either the $\max$ or $\min$ of two values to be combined to get an upper or lower bound. In assimilation, we may have to choose a different value to get a guaranteed bound. Also, even if $p_1 = p_2$, the resulting assimilation may still be only approximate, whereas it would be exact in quantization. Thus, dissociation-based and quantization-based approaches are not reverse to each other, but are rather two complementary approaches that may be combined to yield improved methods.

Existing probabilistic query processing approaches that are both general and tractable use either of two approximation methods: (1) *simulation*-based approaches adapt general purpose sampling methods [Jampani et al. 2008; Kennedy and Koch 2010; Re et al. 2007]; and (2) *model*-based approaches approximate the original number of models with guaranteed lower or upper bounds [Olteanu et al. 2010; Fink and Olteanu 2011]. We show that, for every model-based bound, there exists a dissociation bound which is at least as good or better.

The idea for dissociation originated in our work on generalizing *propagation-based ranking methods* in graphical data [Detwiler et al. 2009] to hypergraphs and conjunctive queries. In [Gatterbauer et al. 2010], we introduced query dissociation, which applies dissociation in a query-centric way to *upper bound* hard probabilistic queries, and showed the connection to propagation in graphs (Most details are provided in [Gatterbauer and Suciu 2013]). In this paper, we provide the theoretical underpinnings of these results in a generalized framework with both upper *and lower bounds*. A previous version of this paper was made available as [Gatterbauer and Suciu 2011].

9. OUTLOOK

We introduced dissociation as a new algebraic technique for approximating the probability of Boolean functions. We applied this technique to derive obliviously optimal upper and lower bounds for conjunctive and disjunctive dissociations and proved that dissociation always gives equally good or better approximations than models. We did not address algorithmic complexities of exploring the space of alternative dissociations, but rather see our technique as a basic building block for new algorithmic approaches.

Such future approaches can apply dissociation at two different levels: (1) at the *query-level*, i.e. at query time and before analyzing the data, or (2) at the *data-level*,

[23] We chose the word *assimilation* as reverse of dissociation, instead of the more natural choice of *association*, as it correctly implies that two items are not merely associated, but rather really merged.

i.e. after analyzing the data. The advantage of the former approaches is that they scale independent of intricacies of the data (similar to the goals of the *lifted inference* community, e.g. [Van den Broeck et al. 2011]), and that they can work very well in practice, yet at the cost of no general guarantees for approximation (similar to *loopy belief propagation* [Frey and MacKay 1997]). The advantage of the latter is that exact solutions can be arbitrarily approximated, yet at the cost of no guaranteed runtime [Roth 1996].

We envision a range of new approaches that apply dissociation at the data-level, possibly together with quantization, to compile an existing intractable formula into a tractable target language, e.g., read-once formulas or formulas with bounded treewidth. For example, one can imagine an approximation scheme in which an iterative Shannon expansion is enriched with dissociation at times to avoid the otherwise resulting state explosion.

ACKNOWLEDGMENTS

We would like to thank Arthur Choi and Adnan Darwiche for helpful discussions on Relaxation & Compensation, and for bringing Prop. 5.1 to our attention [Choi and Darwiche 2011]. We would also like to thank the reviewers for their careful reading of this manuscript and for their detailed feedback. This work was partially supported by NSF grants IIS-0915054 and IIS-1115188. More information about this research, including SQL statements to reproduce the experiments, can be found on the project page: `http://LaPushDB.com/`.

REFERENCES

ANDERSEN, H. R., HADZIC, T., HOOKER, J. N., AND TIEDEMANN, P. 2007. A constraint store based on multivalued decision diagrams. In *Proceedings of the 13th International Conference on Principles and Practice of Constraint Programming (CP'07)*. 118–132.

BERGMAN, D., CIRE, A. A., VAN HOEVE, W.-J., AND HOOKER, J. N. 2013. Optimization bounds from binary decision diagrams. In *INFORMS Journal on Computing*. (to appear).

BERGMAN, D., VAN HOEVE, W. J., AND HOOKER, J. N. 2011. Manipulating MDD relaxations for combinatorial optimization. In *CPAIOR*. 20–35.

CHAVIRA, M. AND DARWICHE, A. 2008. On probabilistic inference by weighted model counting. *Artif. Intell. 172,* 6-7, 772–799.

CHOI, A., CHAVIRA, M., AND DARWICHE, A. 2007. Node splitting: A scheme for generating upper bounds in bayesian networks. In *Proceedings of the 23rd Conference in Uncertainty in Artificial Intelligence (UAI'07)*. 57–66.

CHOI, A. AND DARWICHE, A. 2009. Relax then compensate: On max-product belief propagation and more. In *Proceedings of the 23rd Annual Conference on Neural Information Processing Systems (NIPS'09)*. 351–359. (Alternative title: Approximating MAP by Compensating for Structural Relaxations).

CHOI, A. AND DARWICHE, A. 2010. Relax, compensate and then recover. In *Proceedings of New Frontiers in Artificial Intelligence Workshops (JSAI-isAI)*. 167–180.

CHOI, A. AND DARWICHE, A. 2011. Personal communication.

CRAMA, Y. AND HAMMER, P. L. 2011. *Boolean Functions: Theory, Algorithms, and Applications*. Cambridge University Press.

DALVI, N. N., SCHNAITTER, K., AND SUCIU, D. 2010. Computing query probability with incidence algebras. In *Proceedings of the 29th Symposium on Principles of Database Systems (PODS'10)*. 203–214.

DALVI, N. N. AND SUCIU, D. 2007. Efficient query evaluation on probabilistic databases. *VLDB J. 16,* 4, 523–544.

DARWICHE, A. AND MARQUIS, P. 2002. A knowledge compilation map. *J. Artif. Int. Res. 17,* 1, 229–264.

DECHTER, R. AND RISH, I. 2003. Mini-buckets: A general scheme for bounded inference. *J. ACM 50,* 2, 107–153.

DETWILER, L., GATTERBAUER, W., LOUIE, B., SUCIU, D., AND TARCZY-HORNOCH, P. 2009. Integrating and ranking uncertain scientific data. In *Proceedings of the 25th International Conference on Data Engineering (ICDE'09)*. 1235–1238.

FELLER, W. 1968. *An introduction to probability theory and its applications* 3d ed Ed. Wiley, New York.

FINK, R. AND OLTEANU, D. 2011. On the optimal approximation of queries using tractable propositional languages. In *Proceedings 14th International Conference on Database Theory (ICDT'11)*. 174–185.

FREY, B. J. AND MACKAY, D. J. C. 1997. A revolution: Belief propagation in graphs with cycles. In *NIPS*.

FUHR, N. AND RÖLLEKE, T. 1997. A probabilistic relational algebra for the integration of information retrieval and database systems. *ACM Trans. Inf. Syst. 15,* 1, 32–66.

GATTERBAUER, W., JHA, A. K., AND SUCIU, D. 2010. Dissociation and propagation for efficient query evaluation over probabilistic databases. In *Proceedings of the 4th International VLDB workshop on Management of Uncertain Data (MUD)*. 83–97.

GATTERBAUER, W. AND SUCIU, D. 2011. Optimal upper and lower bounds for Boolean expressions by dissociation. arXiv:1105.2813 [cs.AI].

GATTERBAUER, W. AND SUCIU, D. 2013. Dissociation and propagation for efficient query evaluation over probabilistic databases. arXiv:1310.6257 [cs.DB].

GOGATE, V. AND DECHTER, R. 2011. SampleSearch: Importance sampling in presence of determinism. *Artificial Intelligence 175,* 2, 694–729.

GOGATE, V. AND DOMINGOS, P. 2010. Formula-based probabilistic inference. In *Proceedings of the 26th Conference in Uncertainty in Artificial Intelligence (UAI'10)*. 210–219.

GOGATE, V. AND DOMINGOS, P. 2011. Approximation by quantization. In *UAI*. 247–255.

GURVICH, V. 1977. Repetition-free Boolean functions. *Uspekhi Mat. Nauk 32*, 183–184. (in Russian).

JAMPANI, R., XU, F., WU, M., PEREZ, L. L., JERMAINE, C. M., AND HAAS, P. J. 2008. MCDB: a Monte Carlo approach to managing uncertain data. In *Proceedings International Conference on Management of Data (SIGMOD'08)*. 687–700.

JHA, A., OLTEANU, D., AND SUCIU, D. 2010. Bridging the gap between intensional and extensional query evaluation in probabilistic databases. In *Proceedings of the 13th International Conference on Extending Database Technology (EDBT'10)*. 323–334.

KENNEDY, O. AND KOCH, C. 2010. Pip: A database system for great and small expectations. In *Proceedings of the 26th International Conference on Data Engineering (ICDE'10)*. 157–168.

OLTEANU, D. AND HUANG, J. 2008. Using OBDDs for efficient query evaluation on probabilistic databases. In *Proceedings of the 4th International Conference on Scalable Uncertainty Management (SUM'08)*. 326–340.

OLTEANU, D., HUANG, J., AND KOCH, C. 2009. Sprout: Lazy vs. eager query plans for tuple-independent probabilistic databases. In *Proceedings of the 25th International Conference on Data Engineering (ICDE'09)*. 640–651.

OLTEANU, D., HUANG, J., AND KOCH, C. 2010. Approximate confidence computation in probabilistic databases. In *Proceedings of the 26th International Conference on Data Engineering (ICDE'10)*. 145–156.

PIPATSRISAWAT, K. AND DARWICHE, A. 2007. Clone: Solving weighted max-sat in a reduced search space. In *Proceedings of the 20th Australian Joint Conference on Artificial Intelligence (AUS-AI)*. 223–233.

POOLE, D. 1993. Probabilistic horn abduction and bayesian networks. *Artif. Intell. 64,* 1, 81–129.

PROVAN, J. S. AND BALL, M. O. 1983. The complexity of counting cuts and of computing the probability that a graph is connected. *SIAM J. Comput. 12,* 4, 777–788.

RAMÍREZ, M. AND GEFFNER, H. 2007. Structural relaxations by variable renaming and their compilation for solving mincostsat. In *Proceedings of the 13th International Conference on Principles and Practice of Constraint Programming (CP'07)*. 605–619.

RE, C., DALVI, N. N., AND SUCIU, D. 2007. Efficient top-k query evaluation on probabilistic data. In *Proceedings of the 23rd International Conference on Data Engineering (ICDE'07)*. 886–895.

ROTH, D. 1996. On the hardness of approximate reasoning. *Artif. Intell. 82,* 1-2, 273–302.

SEN, P., DESHPANDE, A., AND GETOOR, L. 2010. Read-once functions and query evaluation in probabilistic databases. *Proc. VLDB Endowment 3,* 1, 1068–1079.

ST-AUBIN, R., HOEY, J., AND BOUTILIER, C. 2000. APRICODD: Approximate policy construction using decision diagrams. In *NIPS*. 1089–1095.

VALIANT, L. 1982. A scheme for fast parallel communication. *SIAM Journal on Computing 11,* 2, 350–361.

VALIANT, L. G. 1979. The complexity of computing the permanent. *Theor. Comput. Sci. 8*, 189–201.

VAN DEN BROECK, G., TAGHIPOUR, N., MEERT, W., DAVIS, J., AND RAEDT, L. D. 2011. Lifted probabilistic inference by first-order knowledge compilation. In *IJCAI*. 2178–2185.

A. NOMENCLATURE

x, y, z	independent Boolean random variables
φ, ψ	Boolean formulas, probabilistic event expressions
f, g, f_φ	Boolean function, represented by an expression φ
$\mathbb{P}[x], \mathbb{P}[\varphi]$	probability of an event or expression
p_i, q_j, r_k	probabilities $p_i = \mathbb{P}[x_i]$, $q_j = \mathbb{P}[y_j]$, $r_k = \mathbb{P}[z_k]$
$\mathbf{x}, \mathbf{g}, \mathbf{p}$	sets $\{x_1, \ldots, x_k\}$ or vectors $\langle x_1, \ldots, x_k \rangle$ of variables, functions or probabilities
$\mathbb{P}_{\mathbf{p},\mathbf{q}}[f]$	probability of function $f(\mathbf{x}, \mathbf{y})$ for $\mathbf{p} = \mathbb{P}[\mathbf{x}]$, $\mathbf{q} = \mathbb{P}[\mathbf{y}]$
$\bar{x}, \bar{\varphi}, \bar{p}$	complements $\neg x, \neg\varphi, 1-p$
f', φ'	dissociation of a function f or expression φ
θ	substitution $\theta : \mathbf{x}' \to \mathbf{x}$; defines a dissociation f' of f if $f'[\theta] = f$
$f[x'/x]$	substitution of x' for x in f
m, m', n	$m = \lvert x \rvert$, $m' = \lvert x' \rvert$, $n = \lvert y \rvert$
d_i	number of new variables that x_i is dissociated into
ν	valuation or truth assignment $\nu : \mathbf{y} \to \{0, 1\}$ with $y_i = \nu_i$
$f[\nu], \varphi[\nu]$	function f or expression φ with valuation ν substituted for $\mathbf{y}$
$\mathbf{g}^\nu$	$\mathbf{g}^\nu = \bigwedge_j g_j^\nu$, where $g_j^\nu = \bar{g}_j$ if $\nu_j = 0$ and $g_j^\nu = g_j$ if $\nu_j = 1$

B. REPRESENTING COMPLEX EVENTS

It is known from Poole's independent choice logic [Poole 1993] that arbitrary correlations between events can be composed from *disjoint-independent events* only. A disjoint-independent event is represented by a non-Boolean independent random variable y which takes either of k values $\mathbf{v} = \langle v_1, \ldots, v_k \rangle$ with respective probabilities $\mathbf{q} = \langle q_1, \ldots, q_k \rangle$ and $\sum_i q_i = 1$. Poole writes such a "disjoint declaration" as $y([v_1 : q_1, \ldots, v_k : q_k])$.

In turn, any k disjoint events can be represented starting from $k-1$ independent Boolean variables $\mathbf{z} = \langle z_1, \ldots, z_{k-1} \rangle$ and probabilities $\mathbb{P}[\mathbf{z}] = \langle q_1, \frac{q_2}{\bar{q}_1}, \frac{q_3}{\bar{q}_1 \bar{q}_2}, \ldots, \frac{q_{k-1}}{\bar{q}_1 \ldots \bar{q}_{k-2}} \rangle$, by assigning the disjoint-independent event variable y its value v_i whenever event A_i is true, with A_i defined as:

$$\begin{aligned}
(y = v_1) &\equiv A_1 :- z_1 \\
(y = v_2) &\equiv A_2 :- \bar{z}_1 z_2 \\
&\vdots \\
(y = v_{k-1}) &\equiv A_{k_1} :- z_1 \ldots \bar{z}_{k-2} z_{k-1} \\
(y = v_k) &\equiv A_k :- \bar{z}_1 \ldots \bar{z}_{k-2} \bar{z}_{k-1} \,.
\end{aligned}$$

For example, a primitive disjoint-independent event variable $y(v_1 : \frac{1}{5}, v_2 : \frac{1}{2}, v_3 : \frac{1}{5}, v_4 : \frac{1}{10})$ can be represented with three independent Boolean variables $\mathbf{z} = (z_1, z_2, z_3)$ and $\mathbb{P}[\mathbf{z}] = (\frac{1}{5}, \frac{5}{8}, \frac{2}{3})$.

It follows that *arbitrary correlations between events* can be modeled starting from *independent Boolean random variables* alone. For example, two *complex events* A and B with $\mathbb{P}[A] = \mathbb{P}[B] = q$ and varying correlation (see Sect. 7.1) can be represented as *composed events* $A :- z_1 z_2 \vee z_3 \vee z_4$ and $B :- \bar{z}_1 z_2 \vee z_3 \vee z_5$ over the *primitive events* $\mathbf{z}$ with varying probabilities $\mathbb{P}[\mathbf{z}]$. Events A and B become identical for $\mathbb{P}[\mathbf{z}] = (0, 0, q, 0, 0)$, independent for $\mathbb{P}[\mathbf{z}] = (0, 0, 0, q, q)$, and disjoint for $\mathbb{P}[\mathbf{z}] = (0.5, q, 0, 0, 0)$ with $q \leq 0.5$.

C. USER-DEFINED AGGREGATE: IOR

Here we show the User-defined Aggregate (UDA) **iOR** in PostgreSQL:

```
create or replace function ior_sfunc(float, float) returns float as
   'select $1 * (1.0 - $2)'
   language SQL;

create or replace function ior_finalfunc(float) returns float as
   'select 1.0 - $1'
   language SQL;

create aggregate ior (float)(
   sfunc = ior_sfunc,
   stype = float,
   finalfunc = ior_finalfunc,
   initcond = '1.0');
```