Automatically Improving Accuracy for Floating Point Expressions

error

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Floating Point’s Wild Success
Floating Point’s Wild Success

Often floating point is close to real arithmetic

But not always!
Floating Point’s Wild Success

But not always!

Numerous articles retracted [Altman ’99, ’03]
Financial regulations [Euro ’98]
Market distortions [McCullough ’99, Quinn ’83]
Rounding Error in Sculpture

Blake Courter
@bcourter
Rounding Error in Sculpture

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Rounding Error in Sculpture

Numerical imprecision in complex square root #208

Merged josdejong merged 2 commits into josdejong:develop from pavpanchekha:develop on Aug 12, 2014

Conversation 1 Commits 2 Files changed 2

pavpanchekha commented on Aug 11, 2014

The expression used for complex square root returns imprecise results for negative reals. To avoid this imprecision, the equation is rearranged not to add \( r \) to \( x.re \) (which are of similar size and opposite sign).
Existing options

- Unreliable
  + Fast Code
- Slow Code
+ More Reliable
- Slow Code

+ Reliable
+ Fast Code
- Expert Task
Heuristic search to find expert transformations
Heuristic search to find expert transformations

Worked Example
How Herbie Works
Evaluation
Heuristic search to find expert transformations

Worked Example

How Herbie Works

Evaluation
Rounding Error in Quadratic

\[-b + \sqrt{b^2 - 4ac} \over 2a\]
Rounding Error in Quadratic

\[ \frac{-b + \sqrt{b^2 - 4ac}}{2a} \]

**What is rounding error?**

exact \[ [e]_R \]

computed \[ [e]_F \]

7 ULPs
Rounding Error in Quadratic

\[-b + \sqrt{b^2 - 4ac}\]

\[\frac{2a}{2a}\]

What is rounding error?

exact \[\quad \frac{[e]_R}{[e]_R}\]

computed \[\quad \frac{[e]_F}{[e]_F}\]

7 ULPs

log(ULPs) estimates # of incorrect bits
Rounding Error in Quadratic

\[-b + \sqrt{b^2 - 4ac} \quad \frac{2a}{2a}\]
Rounding Error in Quadratic

\[
\frac{-b + \sqrt{b^2 - 4ac}}{2a} \Rightarrow \left\{ \begin{array}{l}
\end{array} \right.
\]

![Graph showing log(ULPs) vs. b with points A, B, C, D marked]
Rounding Error in Quadratic

\[
\frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \Rightarrow
\]

Overflow

If \( b \) is large, \( \|b^2\|_F \) overflows and the whole expression returns \( \infty \).
Rounding Error in Quadratic

\[-b + \sqrt{b^2 - 4ac} \over 2a\]

\[\begin{cases} 
\frac{c}{b} - \frac{b}{a} 
\end{cases} \quad \text{if } b \in A\]

Pretty Accurate
Rounding Error in Quadratic

\[-b + \sqrt{b^2 - 4ac} \over 2a\]

\[
\begin{cases}
\frac{c}{b} - \frac{b}{a} & \text{if } b \in \text{A} \\
-\frac{b + \sqrt{b^2 - 4ac}}{2a} & \text{if } b \in \text{B}
\end{cases}
\]

Catastrophic Cancellation

If \( b \) is large, but \( a \) and \( c \) are small, \( b \approx \sqrt{b^2 - 4ac} \) and the difference is rounded off.
Rounding Error in Quadratic

\[-b + \sqrt{b^2 - 4ac} \quad \frac{2c}{b \cdot 4ac} \quad \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \frac{2c}{-b - \sqrt{b^2 - 4ac}}\]

\[
\begin{array}{l}
\frac{c}{b} - \frac{b}{a} \\
\frac{-b + \sqrt{b^2 - 4ac}}{2a} \\
\frac{2c}{-b - \sqrt{b^2 - 4ac}}
\end{array}
\]

if \( b \in \text{A} \)

if \( b \in \text{B} \)

if \( b \in \text{C} \)

Overflow again

\[
\log(\text{ULPs})
\]

\( b \)
Rounding Error in Quadratic

\[-b + \sqrt{b^2 - 4ac} \over 2a \Rightarrow \begin{cases} \frac{c}{b} - \frac{b}{a} & \text{if } b \in \text{A} \\ \frac{-b + \sqrt{b^2 - 4ac}}{2a} & \text{if } b \in \text{B} \\ \frac{2c}{b - \sqrt{b^2 - 4ac}} & \text{if } b \in \text{C} \\ -\frac{c}{b} & \text{if } b \in \text{D} \end{cases}\]
Heuristic search to find expert transformations

**Worked Example**

How Herbie Works

Evaluation
Heuristic search to find expert transformations

Worked Example

How Herbie Works

Evaluation
Herbie Architecture

- **e** sample **cands** regimes **e'**
- **focus**
- **generate more candidates**
Herbie Architecture

- **sample**
- **cands**
- **regimes**
- **e**
- **e'**

Ground truth

focus

generate more candidates
Herbie Architecture

e \xrightarrow{\text{sample}} \text{cands} \xrightarrow{\text{regimes}} e' \xrightarrow{\text{focus}} \text{generate more candidates} \xrightarrow{\text{Localize error}}
Herbie Architecture

- $e$ sample to $\text{cands}$
- $\text{cands}$ regimes to $e'$
- $\text{focus}$
- $\text{generate more candidates}$

Heuristic search
Herbie Architecture

Keep all good candidates

generate more candidates

focus

regimes

cands

sample

e

e'
Herbie Architecture

Sample -> Cands -> Regimes -> E’

Focus

Generate more candidates

Combine candidates
Herbie Architecture

sample

Ground truth
cands

regimes

e'

focus
generate more candidates
Determine ground truth

\[ X = \text{sample}(\text{domain}(e)) \]

\[ \text{e.g. } X = \{1.2 \cdot 10^{-17}, -3.8 \cdot 10^{204}, 173.5, \ldots \} \]

\[ \text{Round}([e]_\mathbb{R}(X)) \]

\[ [e]_\mathbb{F}(X) \]

Get 64-bit prefix with MPFR.

\textbf{Subtle!} See paper.

64 random bits

error

\[ \text{e.g. } \{13.2b, 51.7b, 1b, \ldots \} \]
Herbie Architecture

- Sample
- Ground truth
- Cands
- Regimes
- E'
Herbie Architecture

- **e** → sample → **cands** → regimes → **e'**
- **focus**
  - generate more candidates
- **Localize error**
Focus: Estimate Error Source

1. For each op $f$ in $e$
2. Evaluate args in $\mathbb{R}$
3. Apply $f_{\mathbb{R}}$ to them
4. Apply $f_{\mathbb{F}}$ to them
5. Compare

$$x = \begin{bmatrix} -b \end{bmatrix} \quad \text{and} \quad y = \begin{bmatrix} \sqrt{b^2 - 4ac} \end{bmatrix}$$

$$(x +_{\mathbb{F}} y) \quad \text{and} \quad +_{\mathbb{R}} \quad \text{Round}(x +_{\mathbb{R}} y)$$
Herbie Architecture

e → sample → cands → regimes → e' → focus → generate more candidates → localize error
Herbie Architecture

e \xrightarrow{\text{sample}} \text{cands} \xrightarrow{\text{regimes}} e' \\
\text{focus} \downarrow \quad \downarrow \quad \downarrow \\
\text{generate more candidates}

Heuristic search
Herbie Architecture

Create candidates

sample

cands

regimes

e'

focus

rewrite

series

simplify
Apply rewrites to

\[-b \pm \sqrt{b^2 - 4ac} \over 2a\]

- Recursive rewrites:
  - Database of rules
  - Flexible
  - Chains of rewrites

Recursive rewrites:

- Database of rules
- Flexible
- Chains of rewrites

No cancellation in denominator

\[
\frac{(-b)^2 \pm (\sqrt{b^2 - 4ac})^2}{-b - \sqrt{b^2 - 4ac}} \over 2a
\]

\[
\frac{0 - b \pm \sqrt{b^2 - 4ac}}{2a} \over 0 - \left( b \pm \sqrt{b^2 - 4ac} \over 2a \right)
\]

Rule DB

... 120 more ...
Herbie Architecture

e \rightarrow \text{sample} \rightarrow \text{cands} \rightarrow \text{regimes} \rightarrow e' \\
\downarrow \quad \quad \quad \downarrow \quad \quad \quad \downarrow \\
focus \quad \quad \quad rewrite \quad \quad \quad series \\
\downarrow \quad \quad \quad \downarrow \quad \quad \quad \downarrow \\
simplify

Approximate expr
Series Expansions

Idea: near-identities

\[ \sqrt{1 - x} \approx 1 - \frac{x}{2} \]

(for \( x \approx 0 \))

Bounded Laurent series:
- Transcendental functions
- Singularities
- Number of terms to take

\[ \frac{-b + \sqrt{b^2 - 4ac}}{2a} \]

\[ \frac{-b + b(1 - 4ac/2b^2)}{2a} \]
Herbie Architecture

$e \xrightarrow{\text{sample}} \text{cands} \xrightarrow{\text{regimes}} e'$

$\xrightarrow{\text{focus}}$

$\xrightarrow{\text{rewrite}}$

$\xrightarrow{\text{series}}$

$\xrightarrow{\text{simplify}}$

$\xrightarrow{\text{Approximate expr}}$
Herbie Architecture

e \rightarrow \text{sample} \rightarrow \text{cands} \rightarrow \text{regimes} \rightarrow \text{e'}

Arrow labeled "cancel & clean up" points to right of "simplify" label.
Simplify Expressions

\[
\frac{(-b)^2 - (\sqrt{b^2 - 4ac})^2}{-b - \sqrt{b^2 - 4ac}} / 2a
\]

\[
= \frac{b^2 - (\sqrt{b^2 - 4ac})^2}{-b - \sqrt{b^2 - 4ac}} / 2a
\]

\[
= \frac{b^2(b^2 - 4ac)}{-b - \sqrt{b^2 - 4ac}} / 2a
\]

\[
= \frac{4ac}{-b - \sqrt{b^2 - 4ac}} / 2a
\]

\[
= \frac{2c}{-b - \sqrt{b^2 - 4ac}}
\]

Difficult! [Caviness ’70]
- many possible rewrites
- huge search space
- avoid undoing progress!

E-graphs [Nelson ’79]
- Terminate early
- Prune useless nodes
- Restrict rewrites
Herbie Architecture

- **e** → sample → **cands** → regimes → **e'**

- **focus** → **rewrite** → **series** → **simplify**

- **Cancel & clean up**
Herbie Architecture

- Sample
- Cands
- Regimes
- E'

- Focus
- Rewrite
- Series
- Simplify

Keep all good candidates
Herbie Architecture

- Sample
- Combine candidates
- Focus
- Rewrite
- Series
- Simplify
Regime Inference

\[
\begin{align*}
\frac{c}{b} & - \frac{b}{a} \\
\frac{-b + \sqrt{b^2 - 4ac}}{2a} & \frac{2c}{-b - \sqrt{b^2 - 4ac}} \\
\frac{c}{b} & 
\end{align*}
\]
Regime Inference

\[
\begin{align*}
&\quad \quad \frac{c}{b} - \frac{b}{a} \\
&= -b + \frac{\sqrt{b^2 - 4ac}}{2a} \\
&= \frac{2c}{-b - \sqrt{b^2 - 4ac}} \\
&= -\frac{c}{b}
\end{align*}
\]

Dynamic programming:
- Bounds quickly
- Tune: binary search
- Pick best variable
Herbie Architecture

Sample -> Cands -> Regimes -> e' -> Combine candidates

focus
rewrite
series
simplify
Heuristic search to find expert transformations

Worked Example

How Herbie Works

Evaluation
Heuristic search to find expert transformations

Worked Example

How Herbie Works

Evaluation
Evaluating Herbie

A. Does accuracy improve?

B. Does it reproduce expert transformations?

C. Is the output code fast?

D. Does it work in the real world?
Examples from Hamming’s *NMSE*

Chapter 3: Function evaluation
28 worked examples & problems

Quadratic formula (4)
Algebraic rearrangement (12)
Series expansion (12)
Branches and regimes (2)
A. Improves accuracy in every test

Dramatic improvement

Accuracy of input

Accuracy of output

Average bits correct (longer is better)
B. Reproduces expert changes

More accurate series expansion

No trig factorization

Handle overflow

Only branch on var

Of 12 with answers:
Same in 8
Different in 4

Average bits correct (longer is better)
C. Output code is fast

Median: 40%

Overhead CDF
(left is better)
D. Two MathJS Patches Accepted

**Numerical imprecision in complex square root #208**

- **Merged**
- josdejong merged 2 commits into josdejong:develop from pavpanchekha:develop on Aug 12, 2014
- Conversation 1
- Commits 2
- Files changed 2

**Accuracy of sinh and complex cos/sin #247**

- **Merged**
- josdejong merged 3 commits into josdejong:develop from pavpanchekha:complex-trig-accuracy on Dec 14, 2014
- Conversation 14
- Commits 3
- Files changed 6

pavpanchekha commented on Aug 11, 2014

The **sin** and **cos** function for complex arguments, and the **sinh** function for real arguments, are inaccurate when the inputs are very small. This is because **Math.exp(x) - Math.exp(-x)** returns zero for small **x**, instead of the more accurate **2x**.

This patch replaces **sinh** by a Taylor expansion when the input is small, which increases accuracy.
D. Machine Learning Anecdote

I wasn't sure how to best rewrite [my] equations. **Herbie found numerically stable versions of the formulas**, and fixed all the divide-by-zero errors.

Clustering (bigger, darker blocks better)
Heuristic search to find expert transformations

Worked Example

How Herbie Works

Evaluation
Improve accuracy of floating point programs

Sampling to estimate error
Reduce global error to per-operation error
Iterative rewriting highest-error operations
Different expressions for different inputs

http://herbie.uwplse.org/
WELCOME TO THE SECRET ROBOT INTERNET

Prove you are human:

0.1 + 0.2 = ?

0.30000000000000004
Herbie and Maximum Error

Often improved by Herbie:

- Improvements large (28b) and small (.5b)
- 1+b improvement for 10/28 programs

Fewer high-error pts, same max error.

Bits error (histogram)
Herbie as Part of a Pipeline

FPDebug  Find inaccurate expressions

Herbie  Improve accuracy

Rosa  Prove accuracy satisfactory

FPTaylor

STOKE-FP  Optimize code
Error graphs along $a$ and $c$
Finding the rewrite rules

Standard mathematical identities:
  Commutativity, inverses, fractions, trig identities

No numerical methods knowledge

Don’t need to be true identities
  False rules do not improve accuracy
  Herbie will ignore them
Regimes often gains ~15 bits

Improvement from regimes (longer is better)
Dot : input program average accurage
Bar : Herbie result w/out regimes