CSE 599Q: Lecture 6

Bell inequalities and the EPR paradox

[Experimental validation $\Longrightarrow$ 2022 Nobel prize in Physics]

Partial measurements:

U 2×2 unitary $\quad (U \otimes I)|\psi\rangle$

Bob

$$|\psi\rangle = \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle \qquad \left\{ \begin{array}{l} |\alpha_{00}|^2 + |\alpha_{01}|^2 + \\ |\alpha_{10}|^2 + |\alpha_{11}|^2 = 1 \end{array} \right.$$

Alice

Alice measures her qubit in the $\{|0\rangle, |1\rangle\}$ basis

$$p_0 := \mathbb{P}[\text{Alice measures } |0\rangle] = |\alpha_{00}|^2 + |\alpha_{01}|^2 \rightarrow \frac{\alpha_{00}|00\rangle + \alpha_{01}|01\rangle}{\sqrt{|\alpha_{00}|^2 + |\alpha_{01}|^2}} = \frac{\alpha_{00}|00\rangle + \alpha_{01}|01\rangle}{\sqrt{p_0}}$$

$$p_1 := \mathbb{P}[\text{Alice measures } |1\rangle] = |\alpha_{10}|^2 + |\alpha_{11}|^2$$

Suppose we measure $|\psi\rangle$ in the $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ basis.

Then $\mathbb{P}[\text{measure "}|00\rangle\text{"}] = |\alpha_{00}|^2 \rightarrow$ state collapses to $|00\rangle$

$\mathbb{P}[\text{measure "}|01\rangle\text{"}] = |\alpha_{01}|^2 \rightarrow$ state collapses to $|01\rangle$

$\mathbb{P}[\text{measure "}|0\rangle\text{" in the 1st bit}] =$

$\mathbb{P}[\text{measure } 00] + \mathbb{P}[\text{measure } 01]$

Two random coins $A, B \in \{0,1\}$

$$\mathbb{P}[B=0 \mid A=0] = \frac{\mathbb{P}[(A,B)=(0,0)]}{\mathbb{P}[A=0]}$$

$$\mathbb{P}[B=1 \mid A=0] = \mathbb{P}[(A,B)=(0,1)]$$

$+ \;=\; \mathbb{P}[A=0]$

$$\mathbb{P}[(A,B)=(0,1)] = \mathbb{P}[A=0]\cdot \mathbb{P}[B=1 \mid A=0]$$

$$\frac{\alpha_{00}|00\rangle + \alpha_{01}|01\rangle}{\sqrt{|\alpha_{00}|^2 + |\alpha_{01}|^2}} \qquad \frac{\alpha_{00}|00\rangle + \alpha_{01}|01\rangle}{|\alpha_{00}|^2 + |\alpha_{01}|^2}$$

$$\downarrow \qquad\qquad \downarrow$$

$$\frac{|\alpha_{00}|^2 + |\alpha_{01}|^2}{|\alpha_{00}|^2 + |\alpha_{01}|^2} = 1 \qquad \frac{|\alpha_{00}|^2 + |\alpha_{01}|^2}{(|\alpha_{00}|^2 + |\alpha_{01}|^2)^2}$$

Born rule for measurements

$$|\psi\rangle = \alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \cancel{\alpha_{10}|10\rangle} + \cancel{\alpha_{11}|11\rangle}$$

$$p_0 := \mathbb{P}[\text{Alice measures } |0\rangle] = |\alpha_{00}|^2 + |\alpha_{01}|^2 \longrightarrow \frac{\alpha_{00}|00\rangle + \alpha_{01}|01\rangle}{\sqrt{|\alpha_{00}|^2 + |\alpha_{01}|^2}}$$

$$\frac{\alpha_{00}|00\rangle + \alpha_{01}|01\rangle}{\sqrt{p_0}} = |0\rangle \otimes \left(\frac{\alpha_{00}|0\rangle + \alpha_{01}|1\rangle}{\sqrt{p_0}}\right)$$

$$p_1 := \mathbb{P}[\text{Alice measures } |1\rangle] = |\alpha_{10}|^2 + |\alpha_{11}|^2$$

$|\psi\rangle$ collapses to

$$\frac{\alpha_{10}|10\rangle + \alpha_{11}|11\rangle}{\sqrt{p_1}} = |1\rangle \otimes \left(\frac{\alpha_{10}|0\rangle + \alpha_{11}|1\rangle}{\sqrt{p_1}}\right)$$

$$|\psi\rangle = \underbrace{\frac{1}{\sqrt{3}}|000\rangle}_{} + \underbrace{\frac{1}{\sqrt{3}}|111\rangle + \frac{1}{\sqrt{3}}|101\rangle}_{} \qquad \boxed{|11\rangle_A \otimes |0\rangle_B} \qquad \boxed{U_A \otimes I}$$

Measure qubits 1 and 3

$A \in \mathbb{C}^{4\times 4}$ unitary

$I_2 \otimes A \leftarrow 8\times 8$ matrix

$$\mathbb{P}[\text{measure } 1^{st} \text{ qubit } 1 \text{ and } 3^{rd} \text{ qubit } 1] = \frac{2}{3}$$

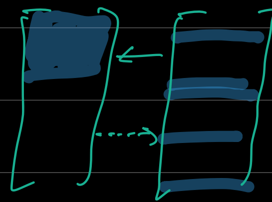
$$|\psi\rangle \longrightarrow \frac{\frac{1}{\sqrt{3}}|101\rangle + \frac{1}{\sqrt{3}}|111\rangle}{\sqrt{2/3}} = \frac{1}{\sqrt{2}}|101\rangle + \frac{1}{\sqrt{2}}|111\rangle$$

$$= \underbrace{|1\_1\rangle}_{\mathbb{C}^4} \otimes \underbrace{\frac{1}{\sqrt{2}}\left(|\_0\_\rangle + |\_1\_\rangle\right)}_{\mathbb{C}^2}$$

$$e_1 \otimes e_1 \otimes e_1 = e_1 \otimes (e_1 \otimes e_1) = (e_1 \otimes e_1) \otimes e_1$$

$$\left[\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right] \left[\begin{array}{c} u_A \\ \hdashline u_B \end{array}\right] = \left[\begin{array}{c} A u_A \\ \hdashline B u_B \end{array}\right]$$

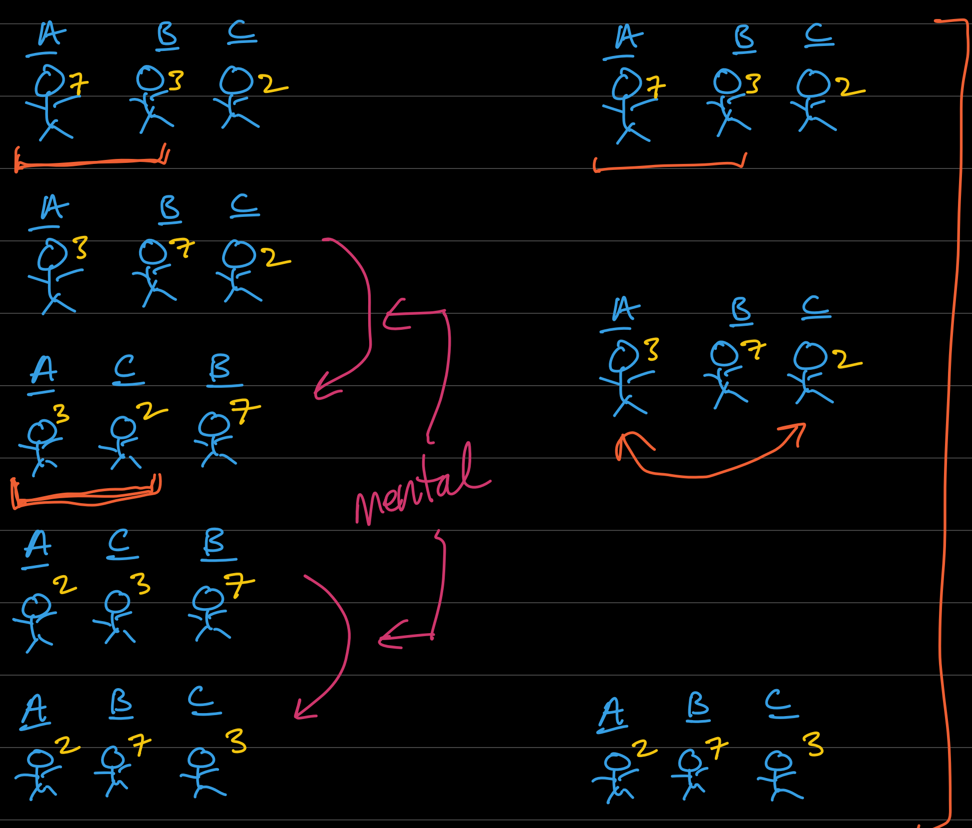
$$\underline{e_1^A} \otimes e_1^B \otimes \underline{e_1^C}$$



$$\mathbb{C}^8 = \mathbb{C}^4 \otimes \mathbb{C}^2$$

$$= \underline{\mathbb{C}^2} \otimes \mathbb{C}^2 \otimes \underline{\mathbb{C}^2}$$

$\mathbb{C}^4$



(EPR paradox) CHSH game:

Alice — Referee — Bob

$[a_1 a_2 \ldots a_m]$ $\quad$ $[b_1 b_2 \ldots b_m]$

$x, y \in \{0,1\}$
unif. at random

$\xleftarrow{x}$ $\qquad$ $\xrightarrow{y}$

$A(x) \in \{0,1\}$ $\qquad$ $B(y) \in \{0,1\}$

Ref checks if $A(x) \oplus B(y) = x \wedge y$ $\quad$ 25%
win if true!

If Alice and Bob are classical, then
$\mathbb{P}[\text{win}] \leq 75\%$

A: $0 \to 1$, $1 \to 0$
B: $0 \to 0$, $1 \to 1$

How many strategies

0	0
0	0
0	1
0	1

75%

xy	A(x) B(y)	$x \wedge y$	$A(x) \oplus B(y)$
00	1 0	0	1
01	1 1	0	0
10	0 0	0	0
11	0 1	1	1

75% win rate!

xy	$x \wedge y$	$A(x) \oplus B(y)$
00	0	0
01	0	0
10	0	0
11	1	0

$\Longrightarrow$ Randomized strategies can win $\leq 75\%$ of the time.

Next time: If Alice and Bob share a bell state:

$$|\text{bell}\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

Bob

Alice

then $\exists$ strategy wins w/ prob $\geq \boxed{0.875} > 3/4$

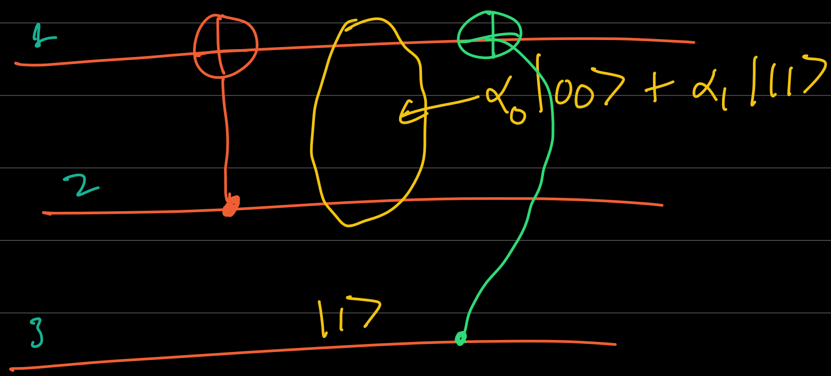
$|0\rangle$

$\otimes$

$\alpha_0|0\rangle + \alpha_1|1\rangle$

$\otimes$

$|1\rangle$



$$|0\rangle \otimes (\alpha_0|0\rangle + \alpha_1|1\rangle) \otimes |1\rangle$$

$$= \alpha_0(|0\rangle \otimes |0\rangle \otimes |1\rangle) + \alpha_1(|0\rangle \otimes |1\rangle \otimes |1\rangle)$$

$$= \underline{\alpha_0|001\rangle + \alpha_1|011\rangle}$$

$$\vec{V} = (V_1, V_2, \ldots, V_n)$$

$$\vec{V} = (V_2, V_1, V_3, \ldots, V_n)$$