Eliminating Sharp Minima with Truncated Heavy-tailed Noise

Xingyu Wang*, Sewoong Oh†, Chang-Han Rhee*

Northwestern University*, University of Washington†

DeepMath 2021

Intro: Generalization Gap and Flat Minima

- **Generalization of DNN**

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Training Set

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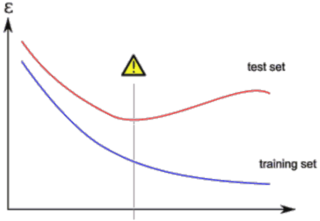

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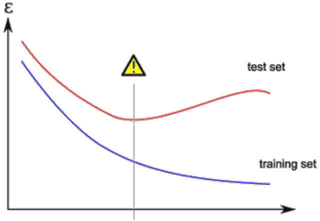

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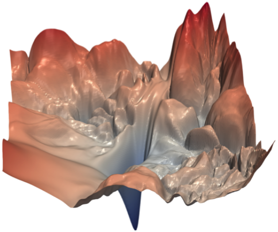

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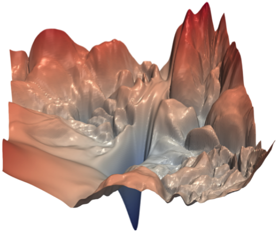

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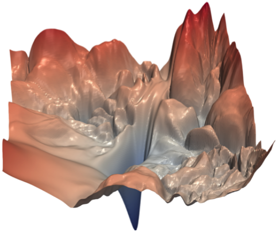

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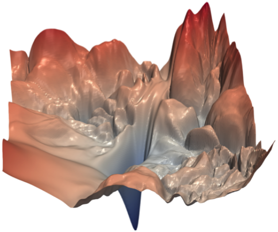

- **Q:** SGD prefers flat minima?

Intro: Heavy-tailed SGD Prefers Flat Minima

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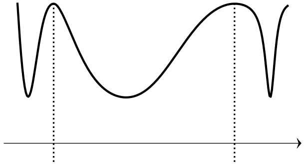

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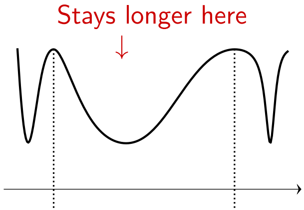

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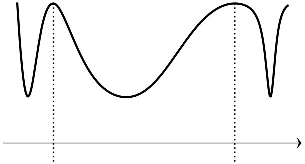
Our Work: Complete Elimination of Sharp Minima



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$$X_j = X_{j-1} - \varphi_b\big(\eta \nabla f(X_{j-1}) + \eta Z_j\big); \quad \varphi_b(x) = \min\{b, \|x\|\} \cdot \frac{x}{\|x\|}$$

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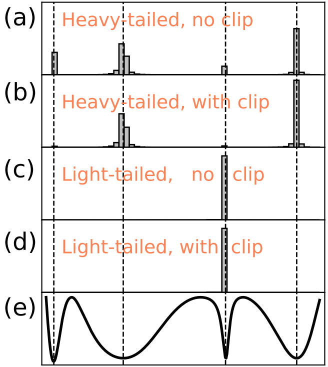

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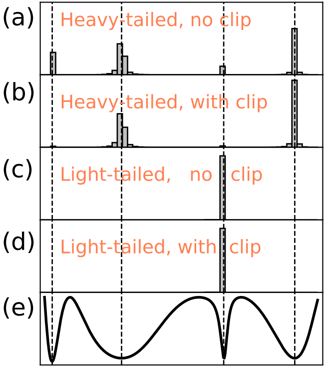

Rare Events depend on "Tail Behaviors"

Light-Tailed Distributions

- Extreme Values are Very Rare
- Normal, Exponential, etc

Heavy-Tailed Distributions

- Extreme Values are Frequent
- Power Law, Weibull, etc

Instagram

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Structural difference in the way systemwide rare events arise.

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(Conspiracy Principle)

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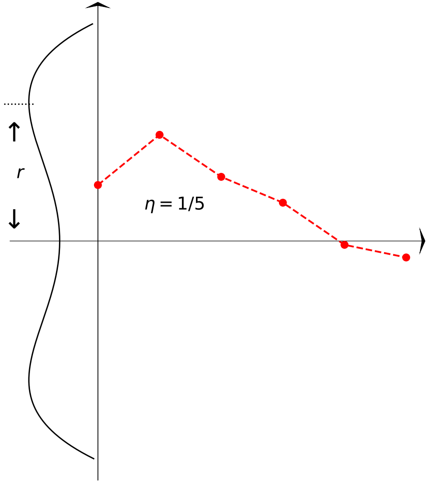
Typical Behavior of SGD



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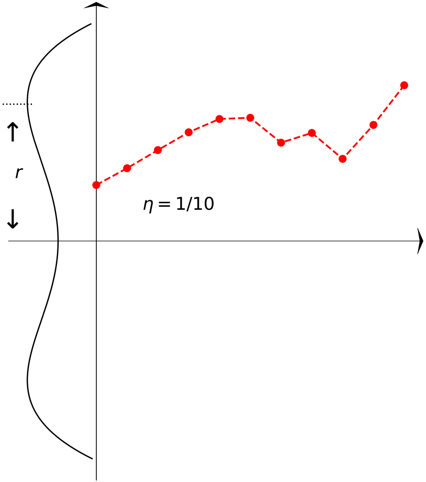

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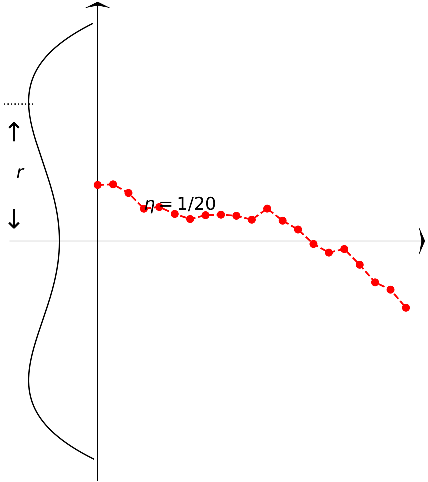

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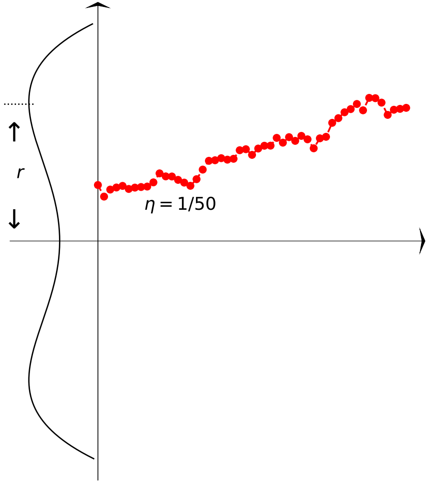

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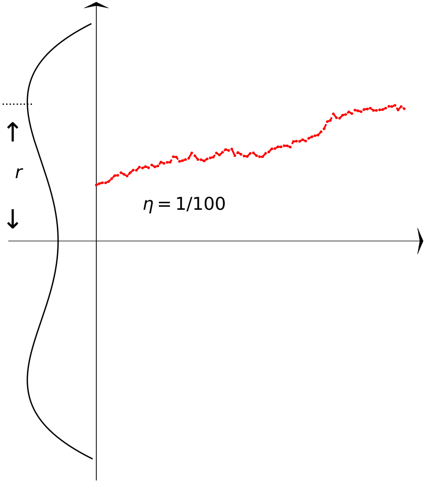

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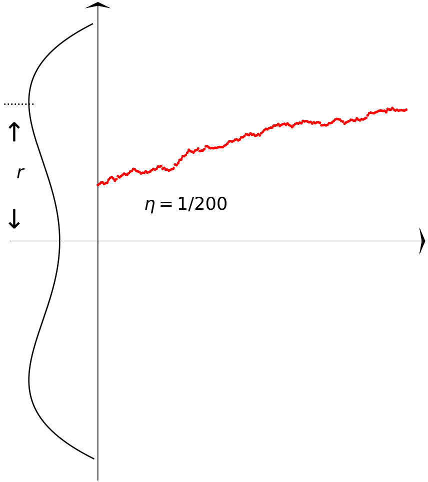

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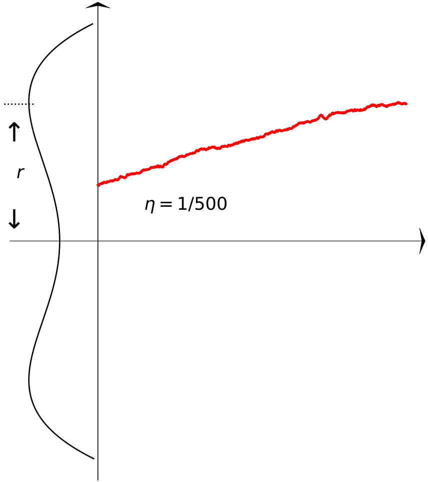

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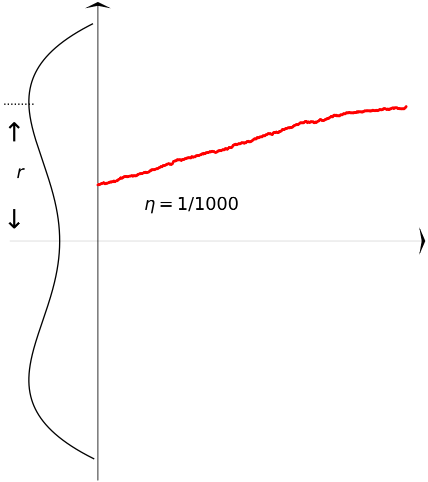

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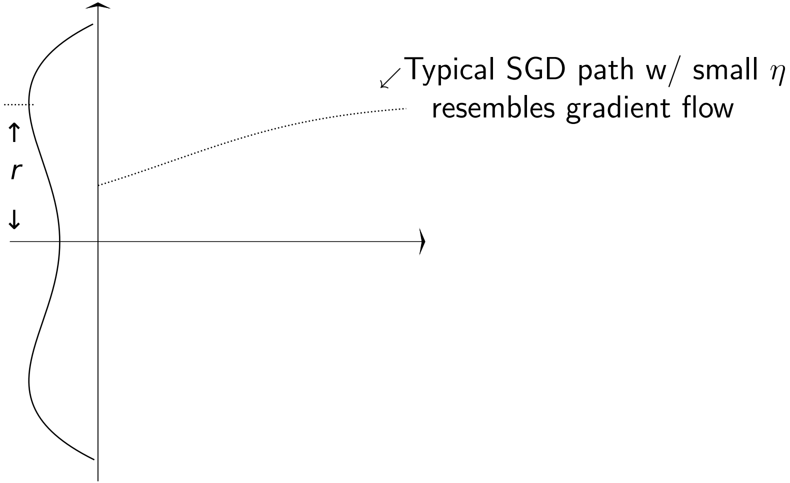

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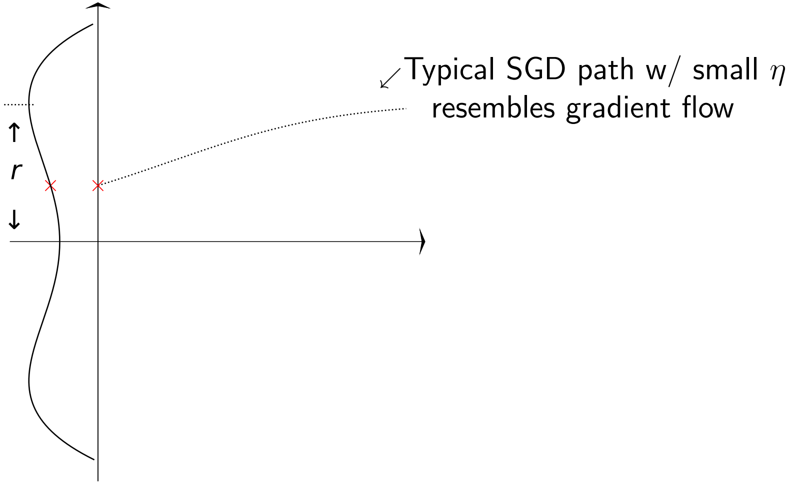

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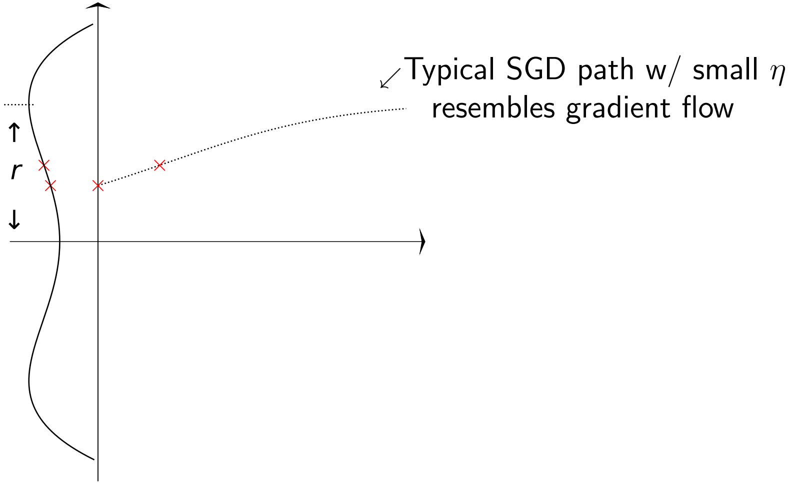

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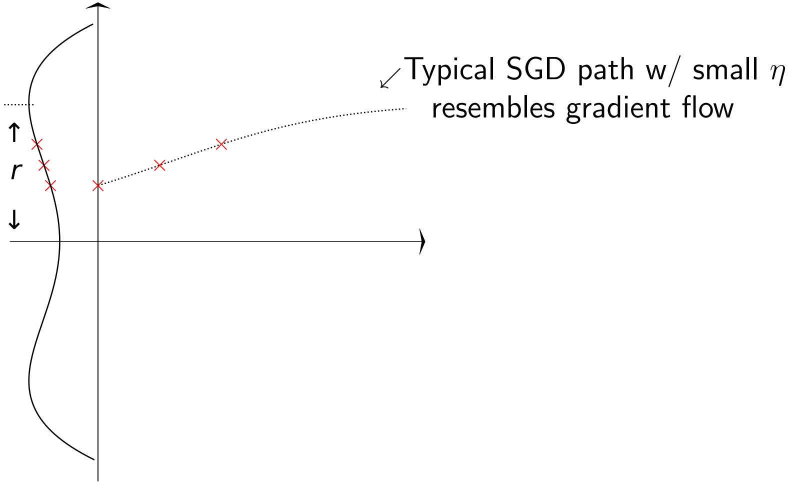

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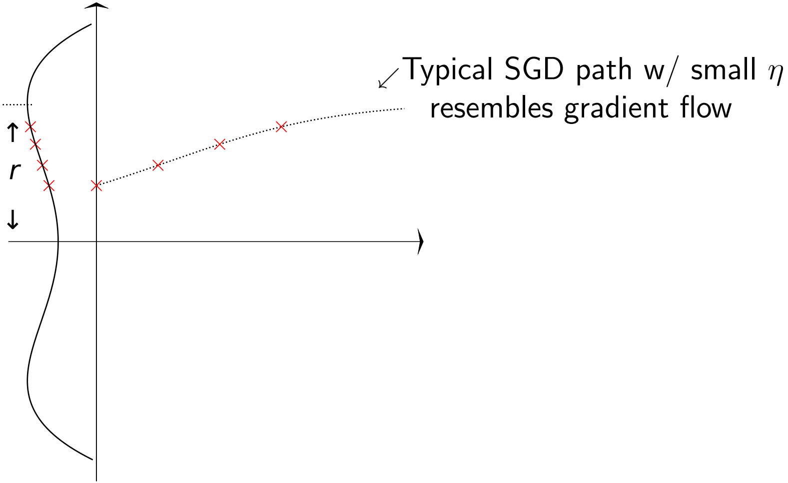

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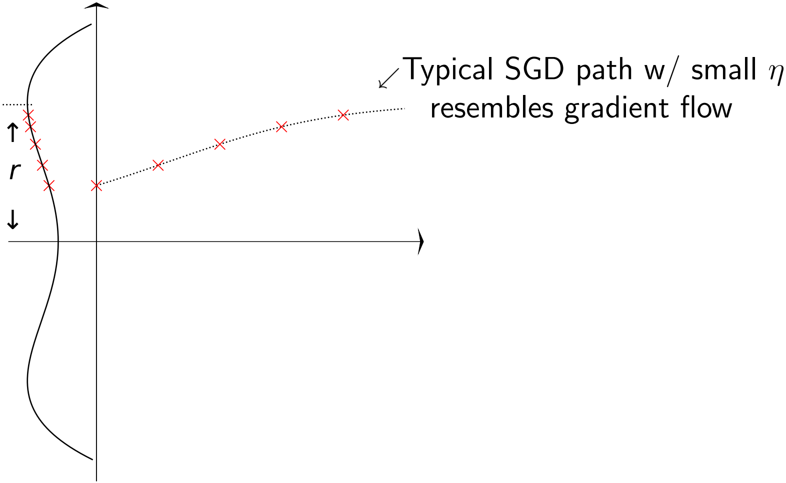

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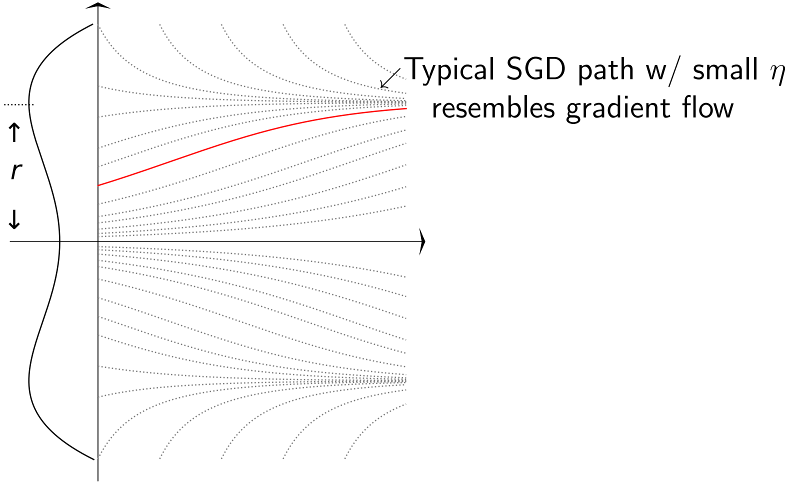

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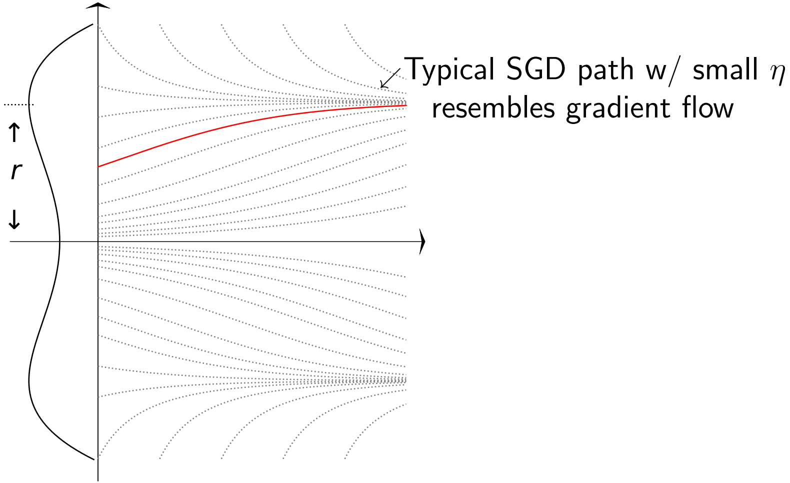

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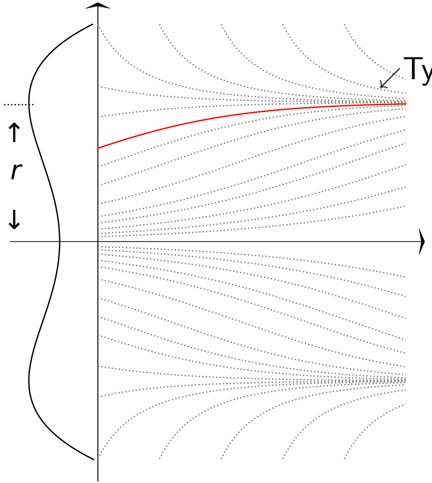

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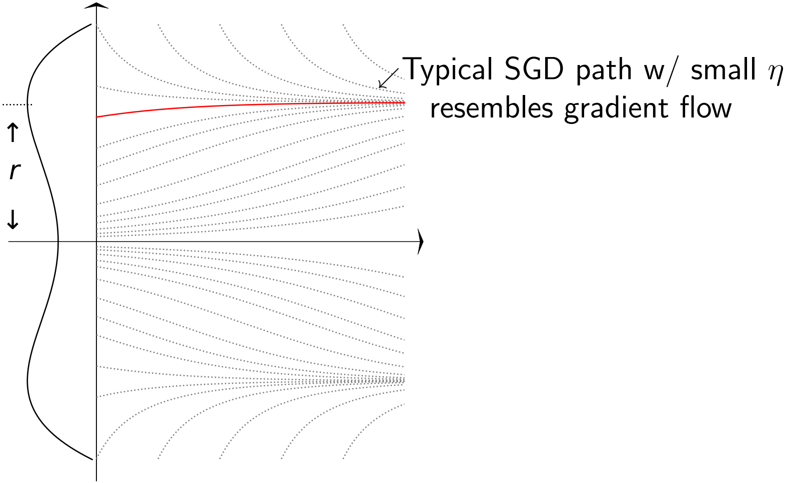

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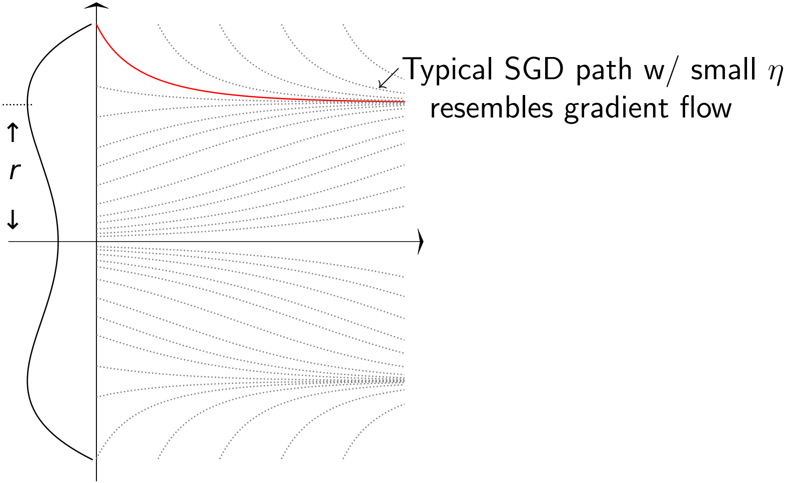

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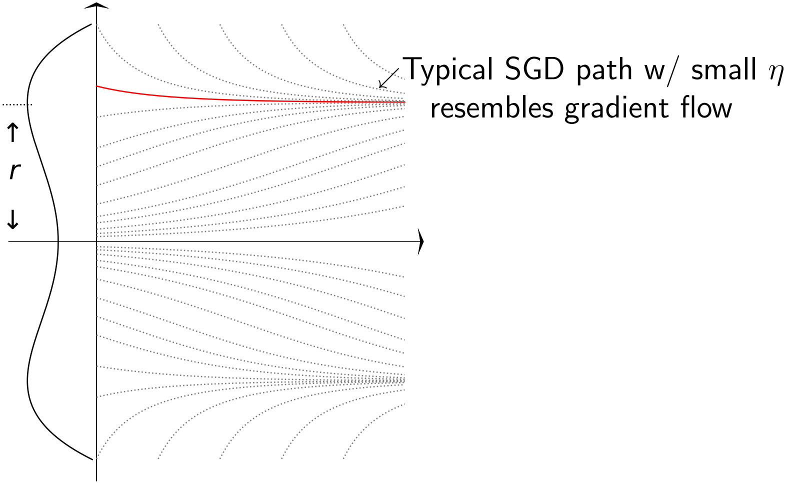

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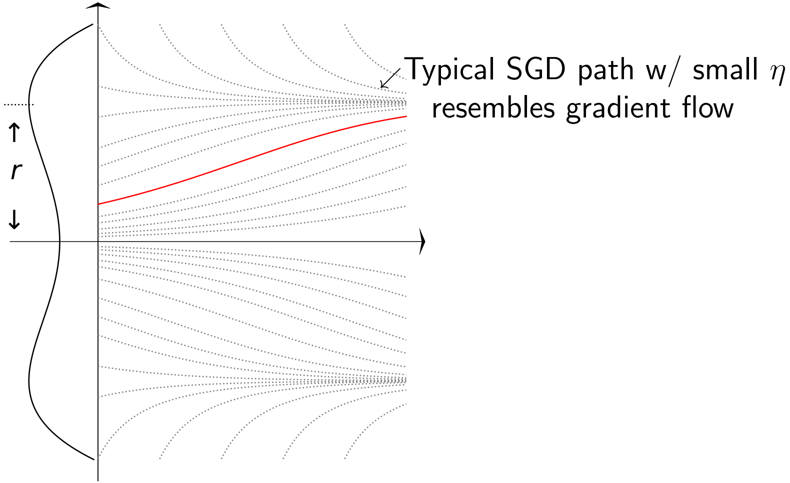

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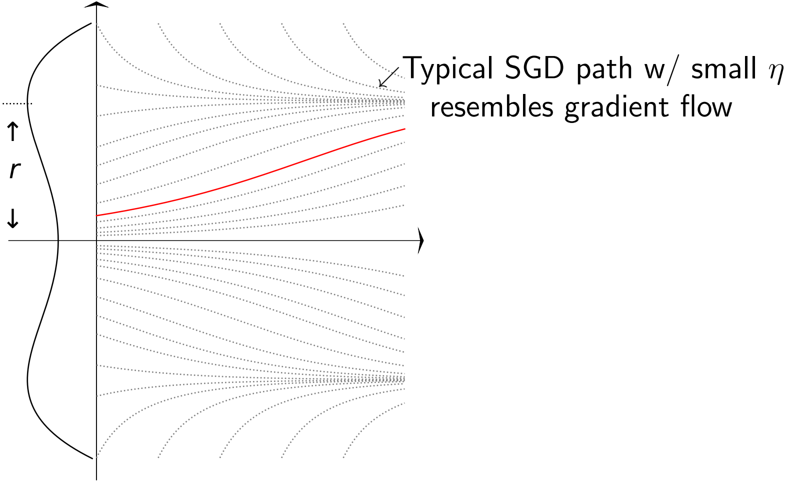

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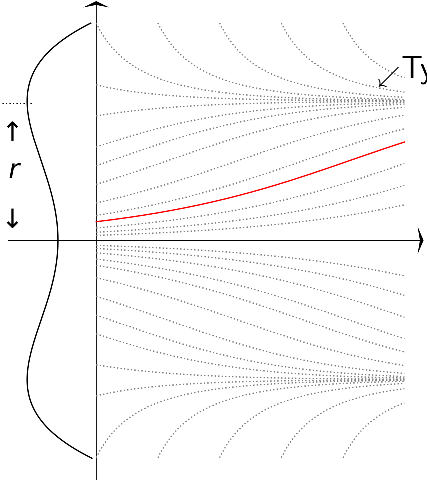

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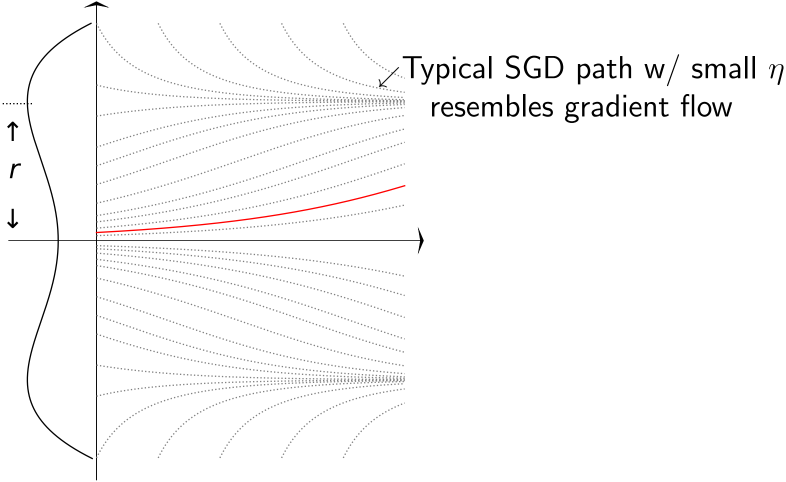

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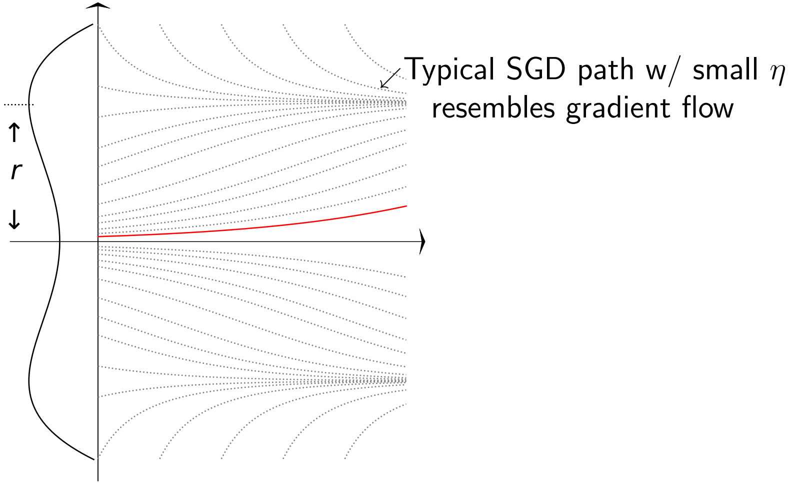

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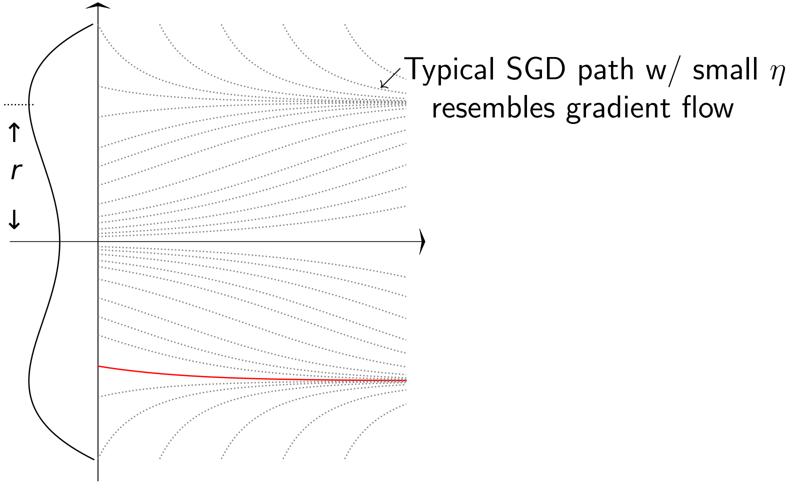

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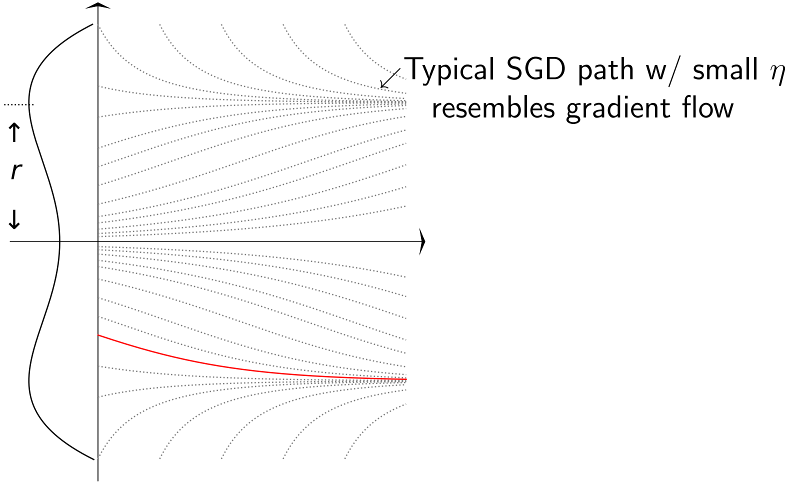

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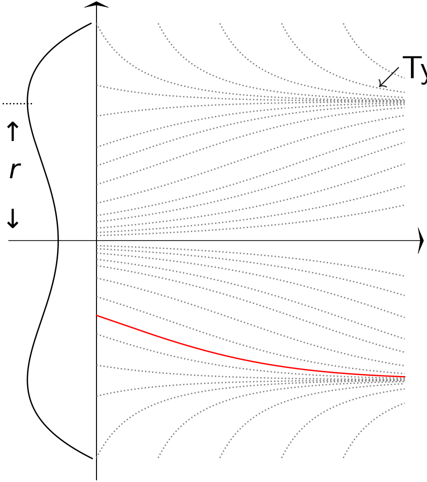

Typical SGD path w/ small η
resembles gradient flow

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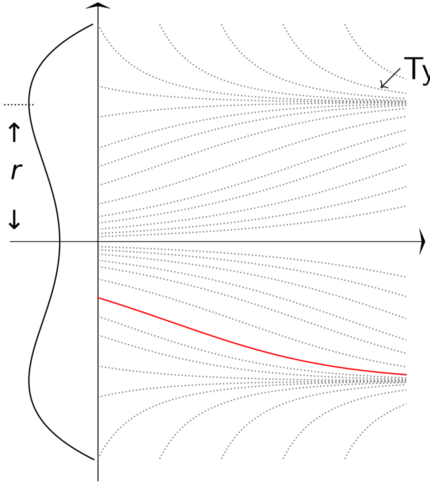

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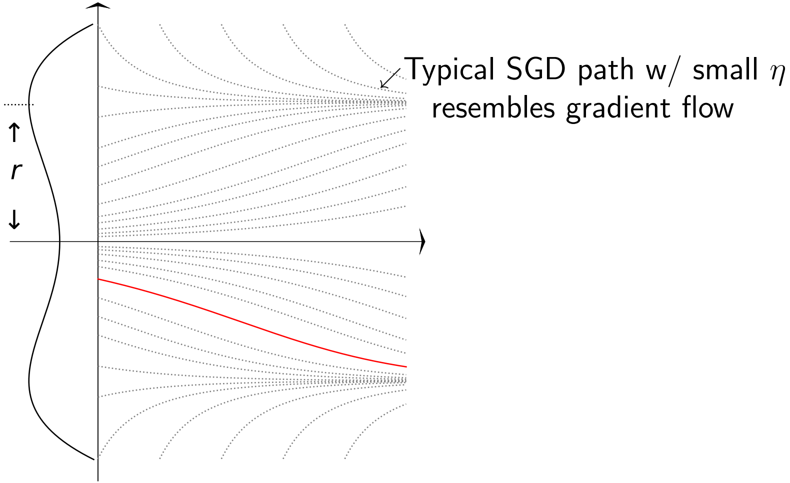

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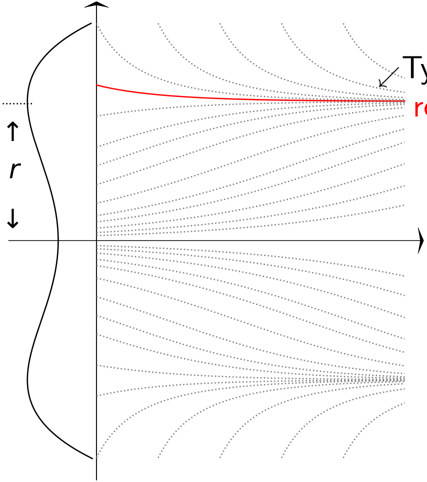

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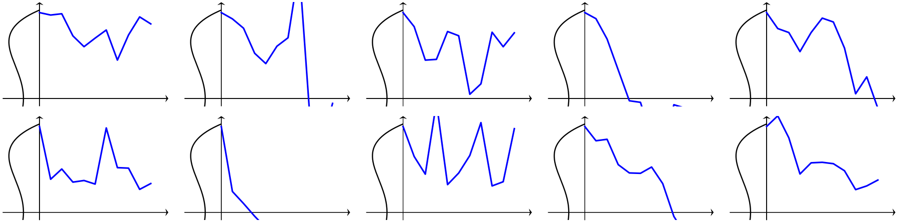

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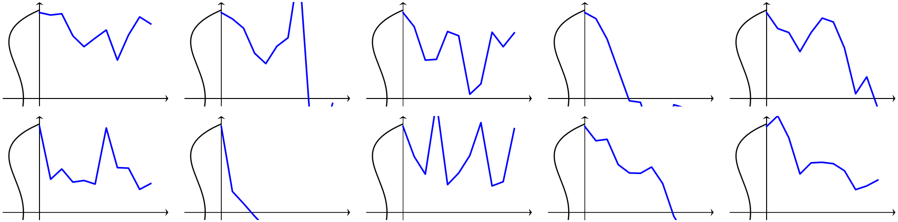

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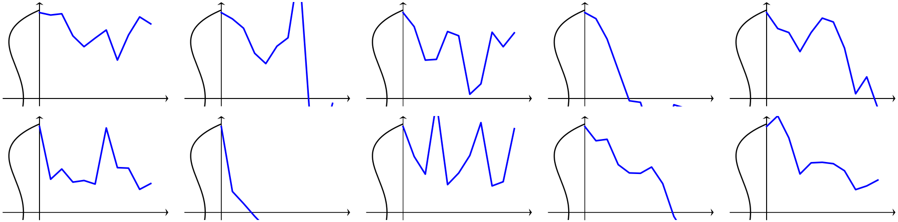

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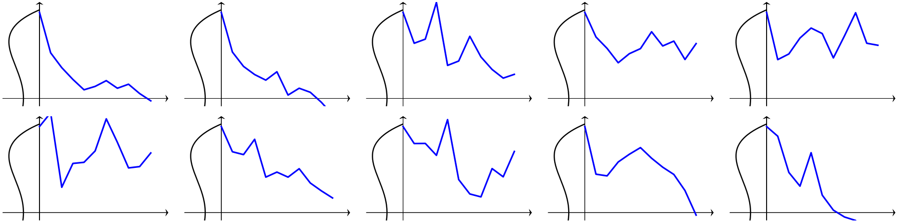

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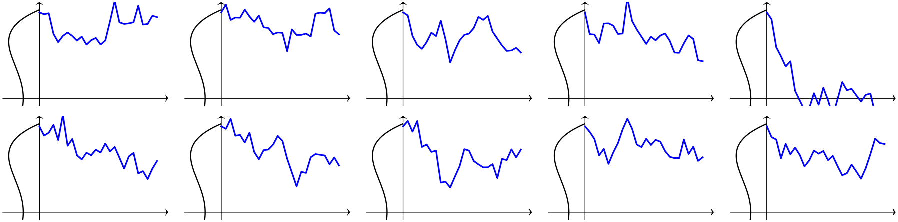

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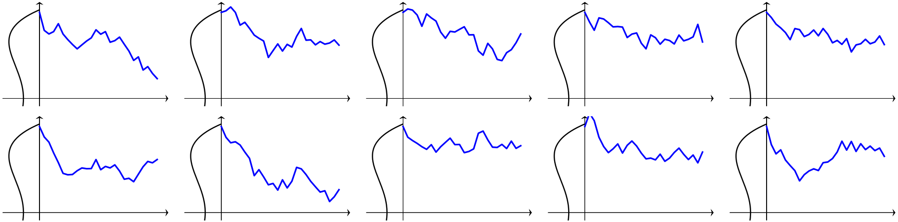

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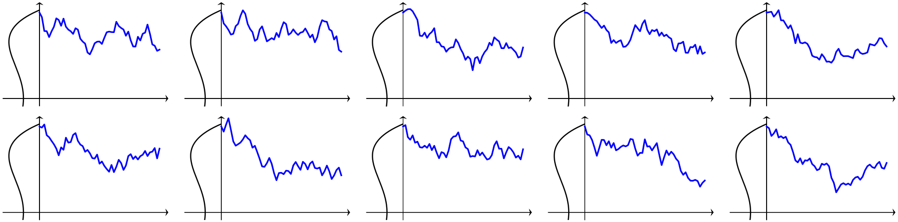

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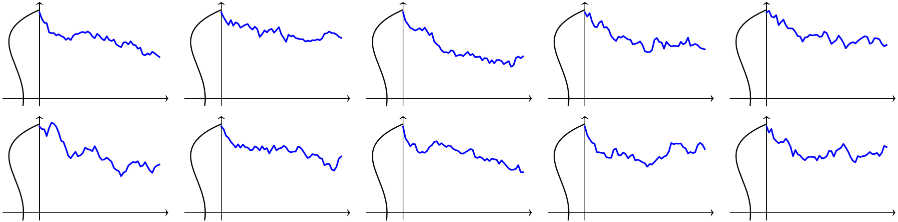

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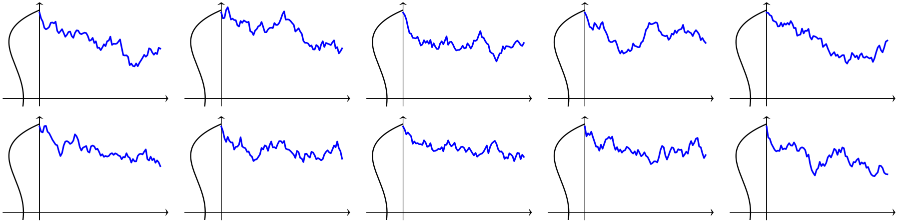

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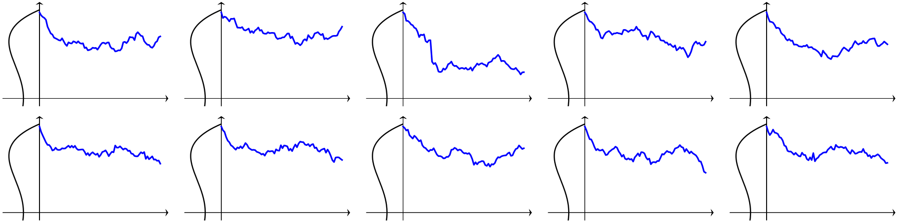

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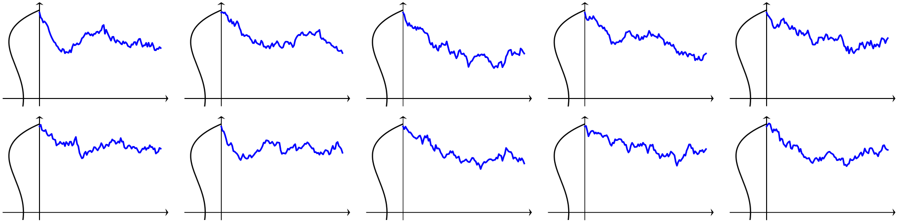

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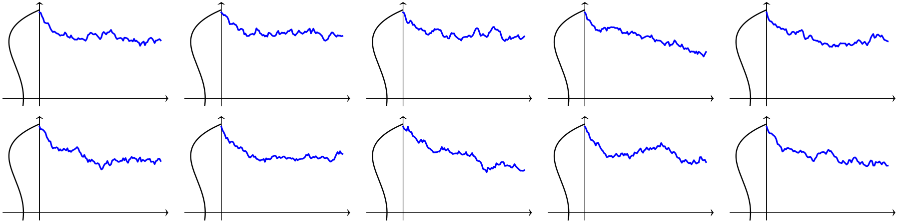

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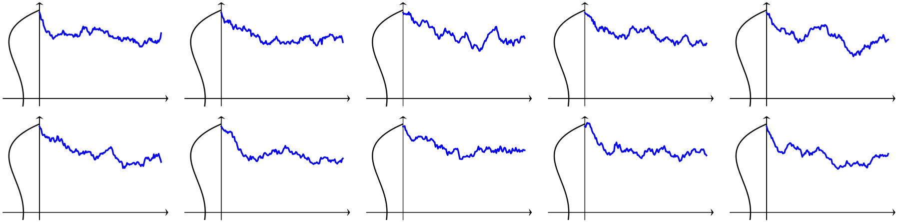

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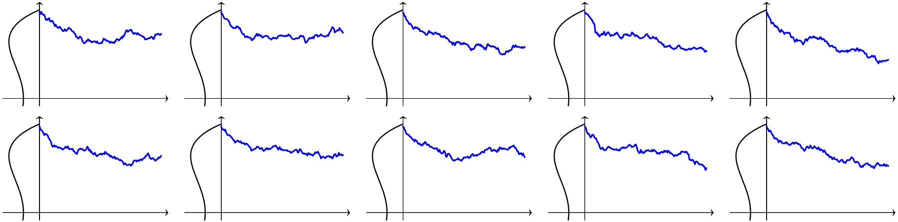

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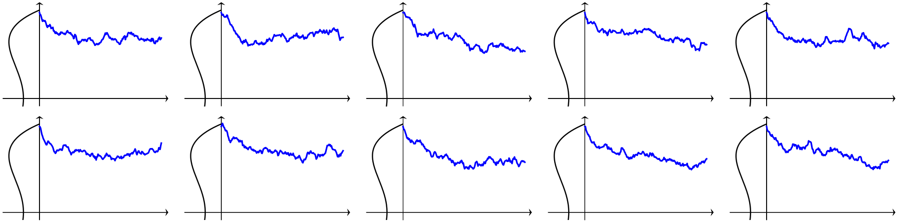

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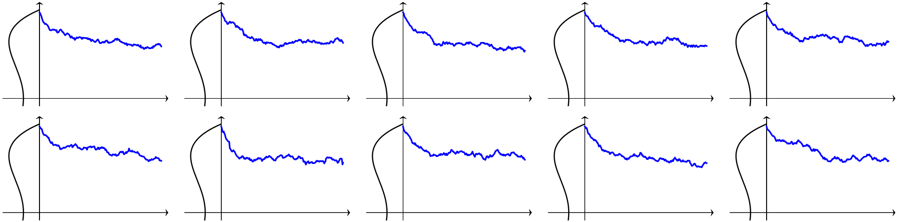

How does SGD escape local minima?

Catastrophe Principle in Heavy-tailed SGD

(Su, Wang, Rhee, 2021+) For "rare event" A,

Catastrophe Principle in Heavy-tailed SGD

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SGD path

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Typical Behavior — Catastrophes

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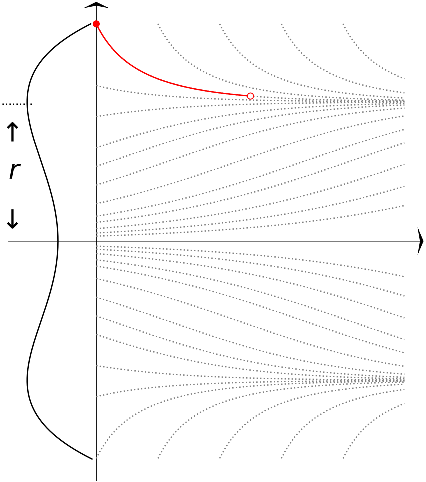
  Typical Behavior ↘ ... ↙ Catastrophes

- Conditioned on $\{X^\eta \in A\}$, X^η resembles piece-wise $\boxed{\text{gradient flow}}$ with $I^*(A)$ $\boxed{\text{jumps}}$
- $I^*(A)$:Min # of jumps (catastrophes) to cause event A

Catastrophe Principle Dictates SGD's Escape Route

This way?



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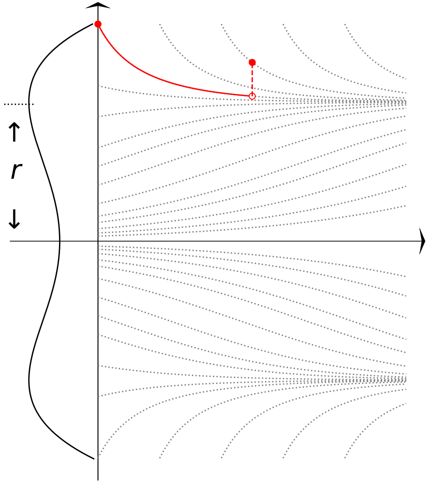

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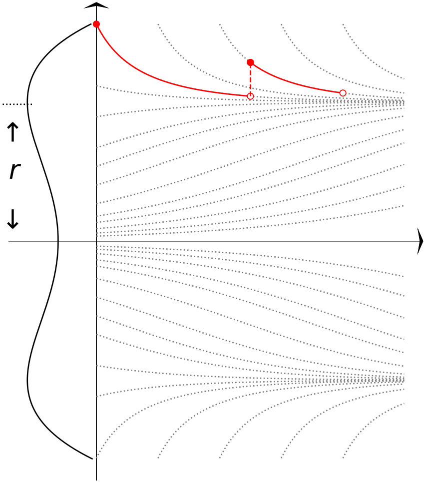

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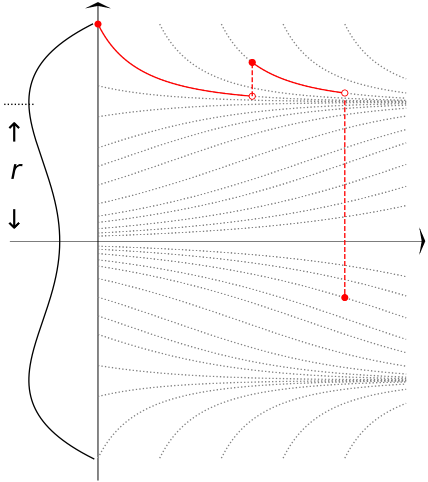

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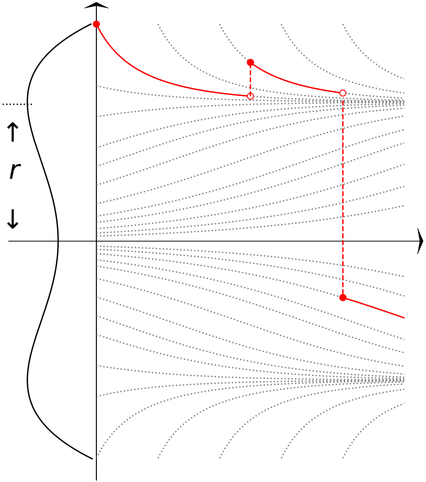

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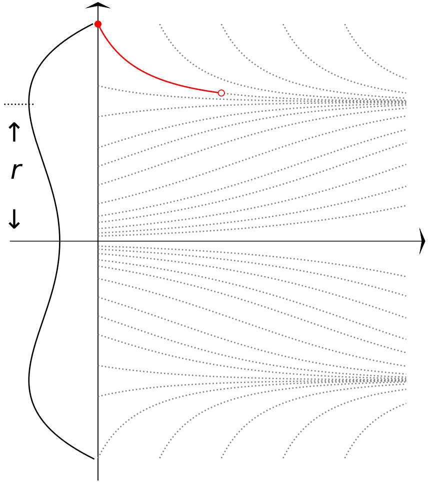

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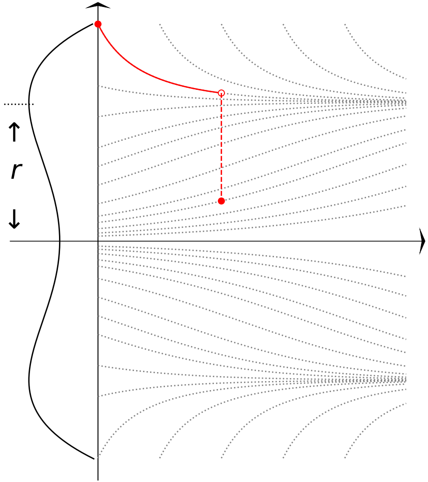

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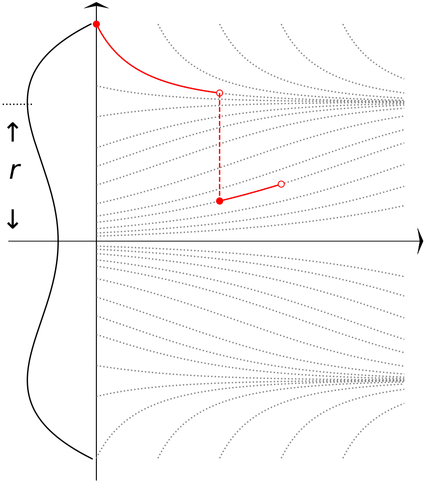

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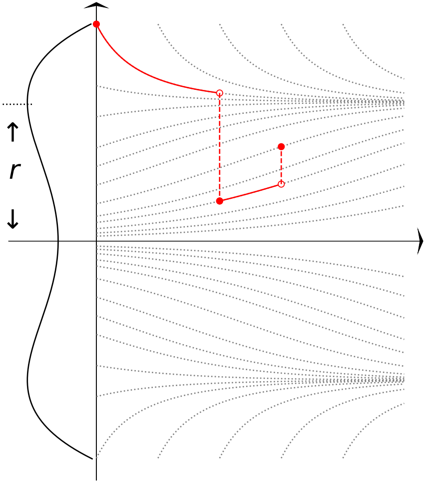

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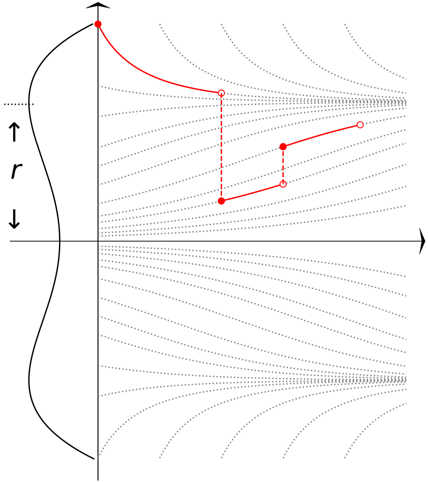

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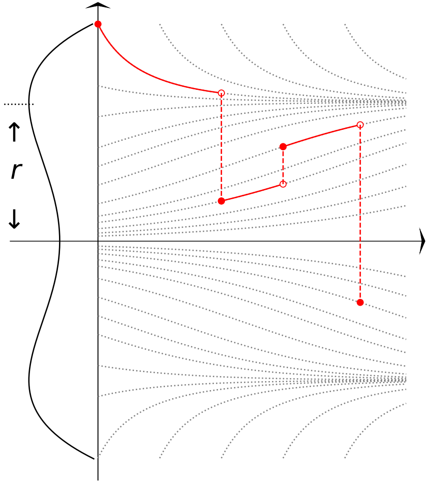

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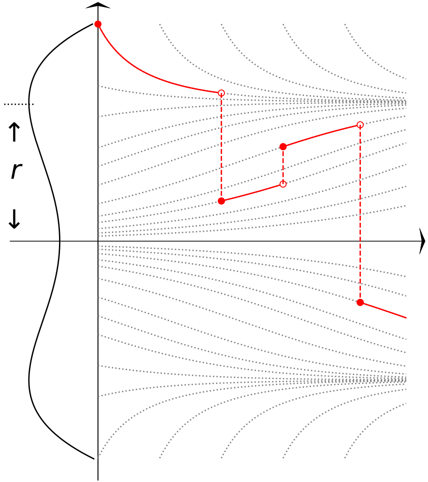

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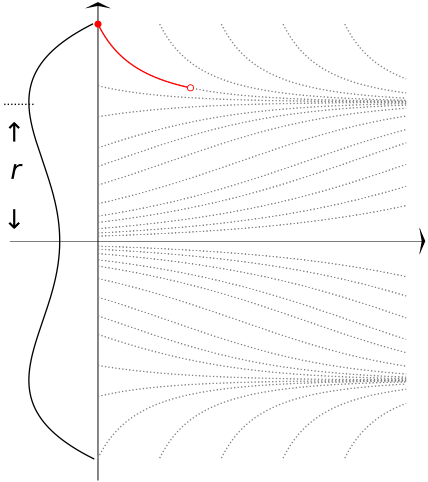

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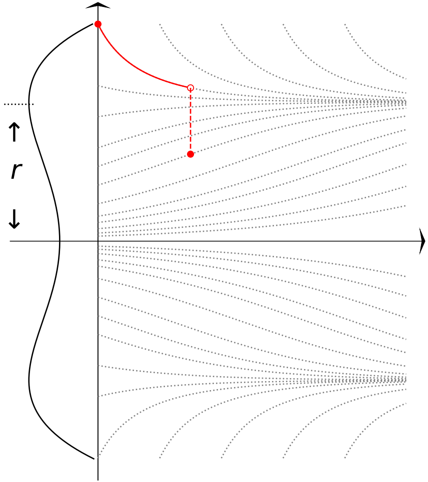

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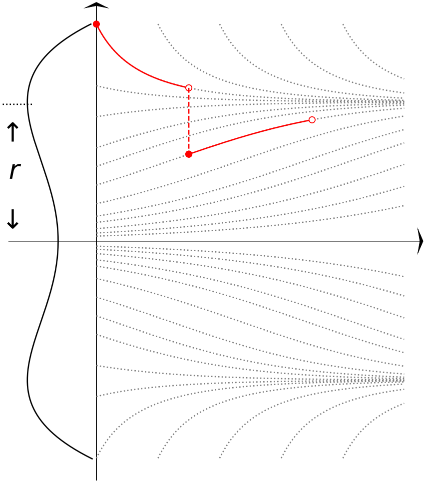

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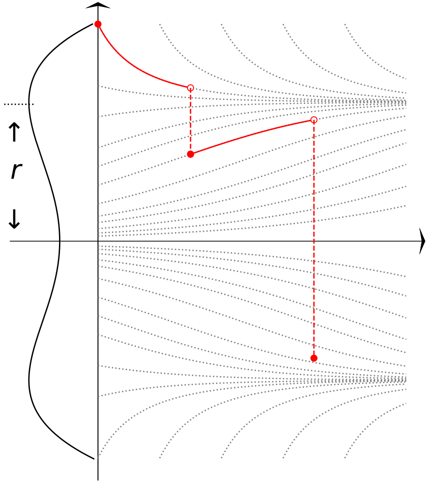

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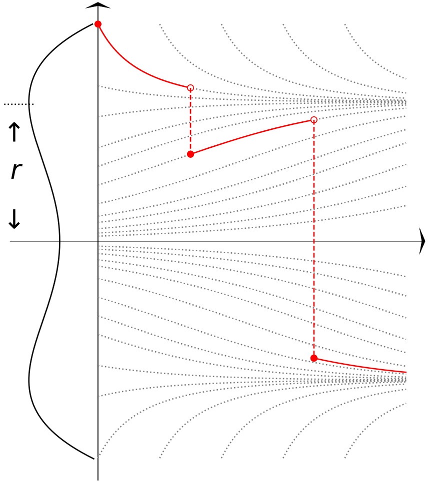
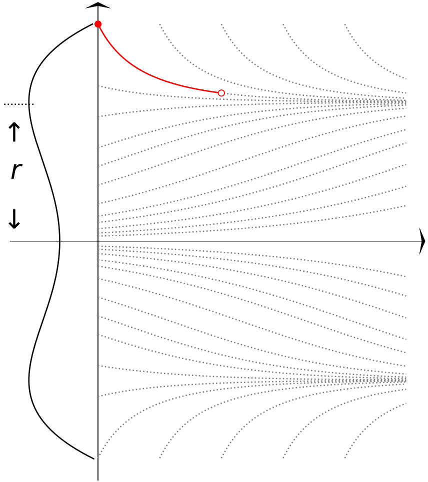


Catastrophe Principle Dictates SGD's Escape Route

Most likely path under heavy-tailed noises: with $l^* = 1$ jump



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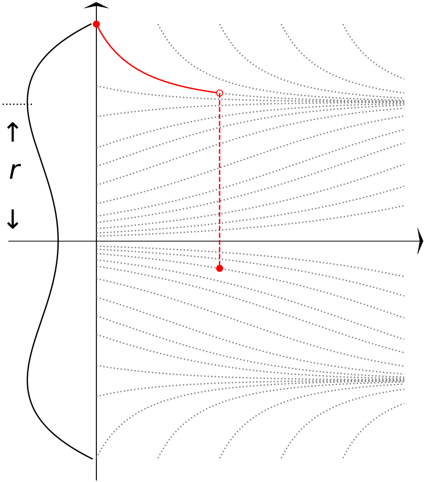

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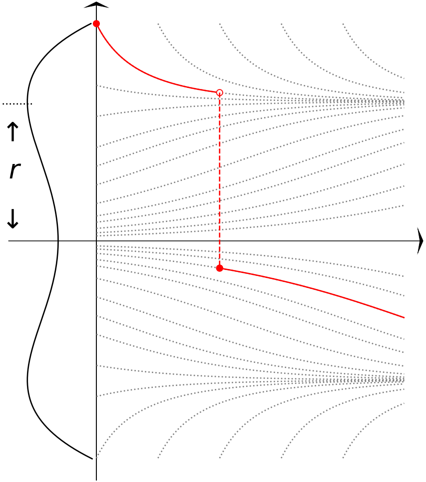

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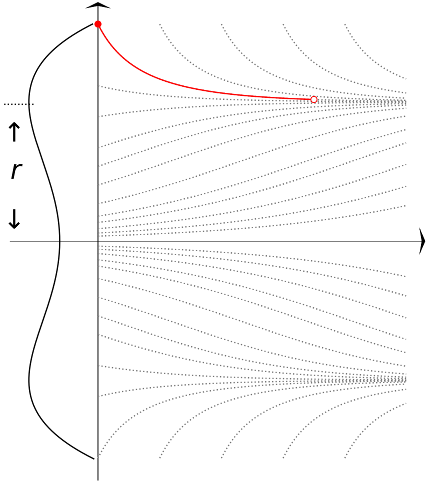

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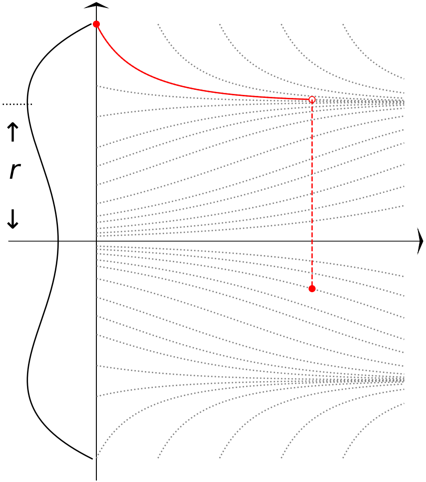

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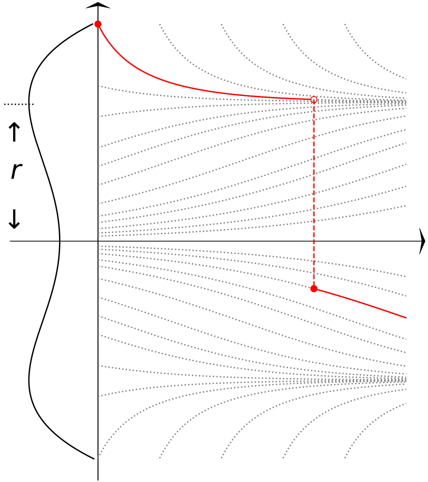

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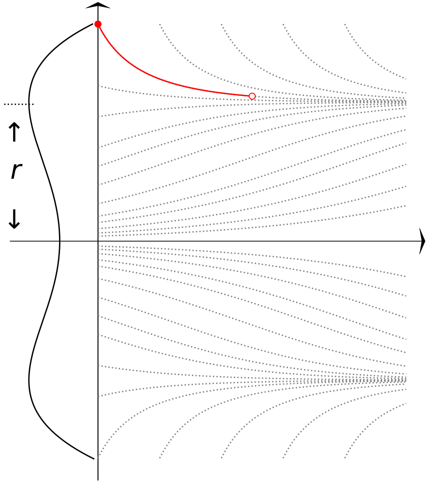

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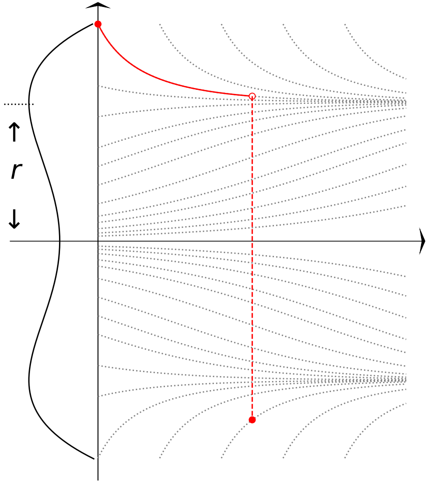

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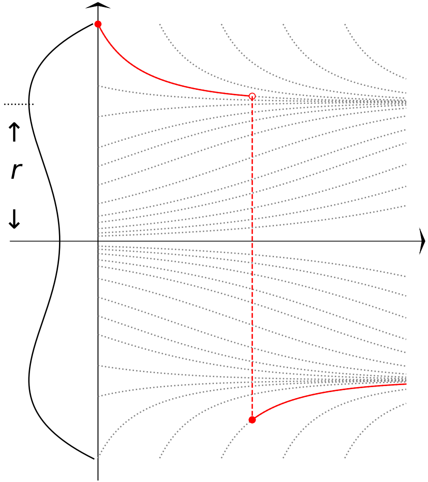

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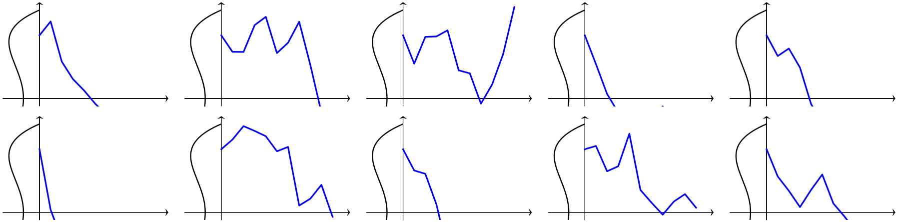
Trajectory of SGD X^{η} conditional on exit: **light-tailed** noises with $\eta = 1/10$



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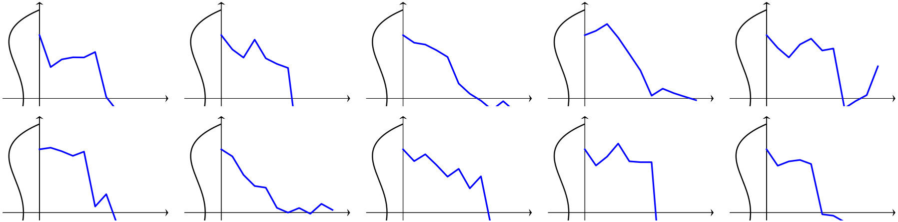

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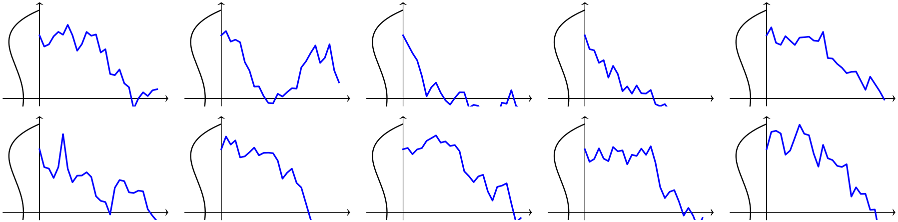

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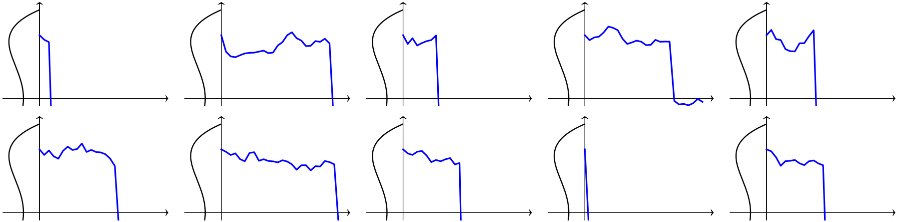

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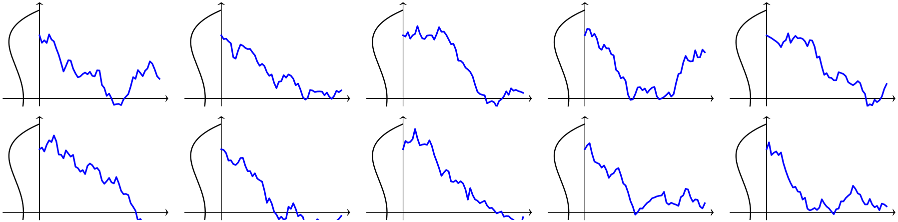

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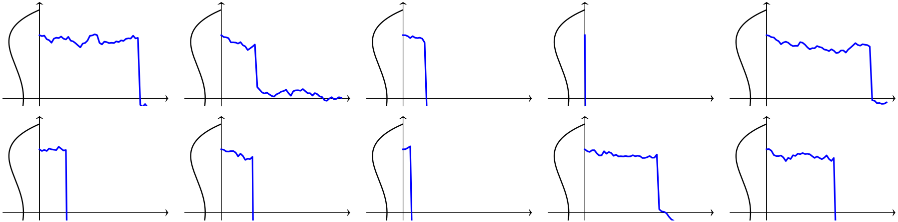

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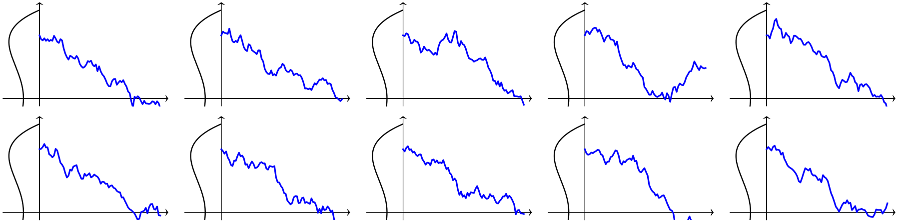

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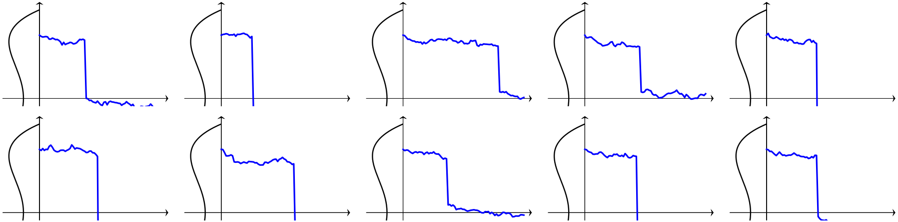

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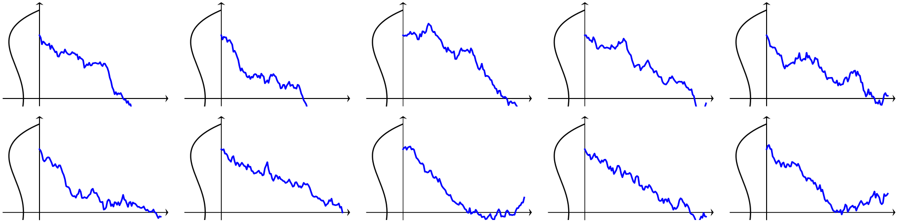

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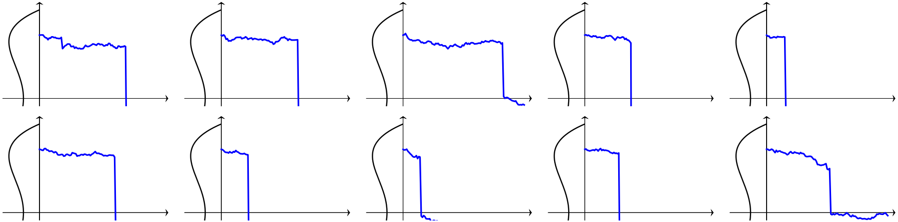

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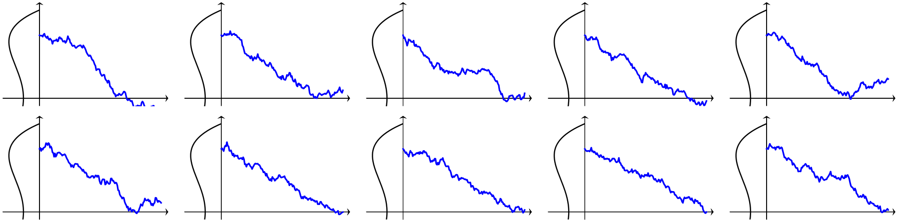

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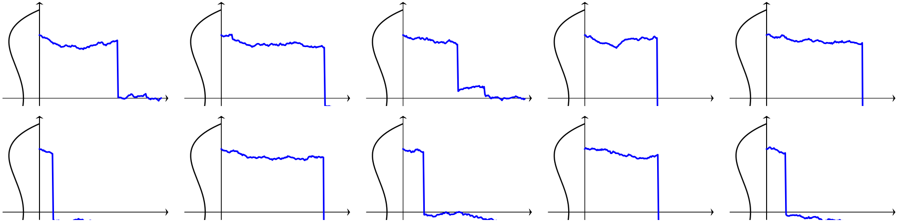
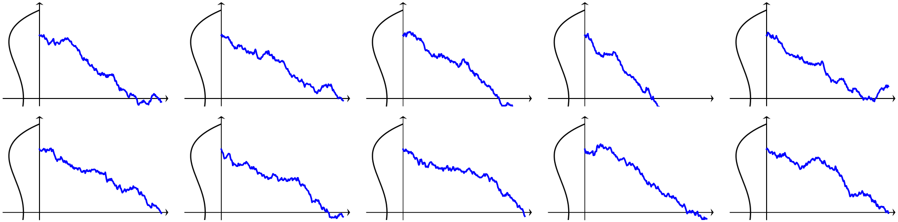
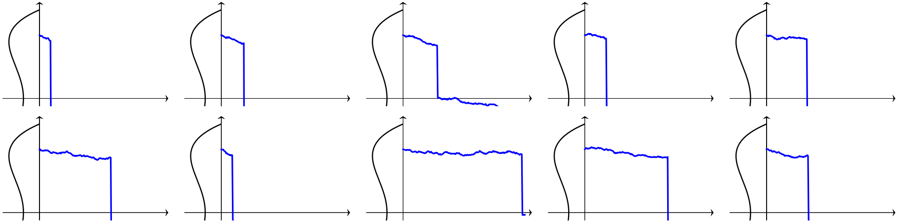


Catastrophe Principle Dictates SGD's Escape Route

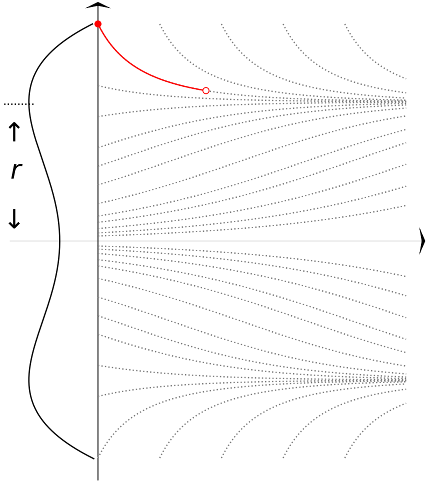
Trajectory of SGD X^η conditional on exit: **light-tailed** noises with $\eta = 1/200$



Trajectory of SGD X^η conditional on exit: **heavy-tailed** noises with $\eta = 1/200$



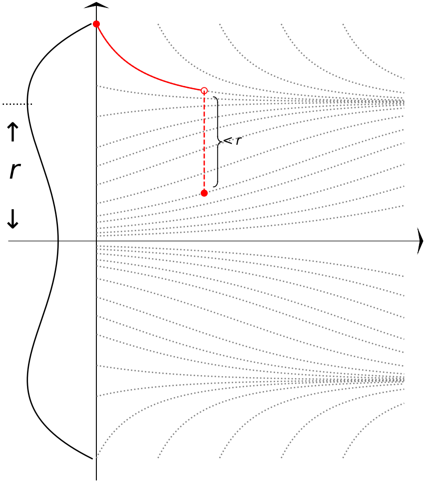
SGD's Escaping Route under Gradient Clipping



Clipping threshold

$$X_j^\eta = X_{j-1}^\eta + \varphi_b\big(-\eta\nabla f(X_{j-1}^\eta) + \eta Z_j\big),\ b \in (r/2, r)$$

SGD's Escaping Route under Gradient Clipping



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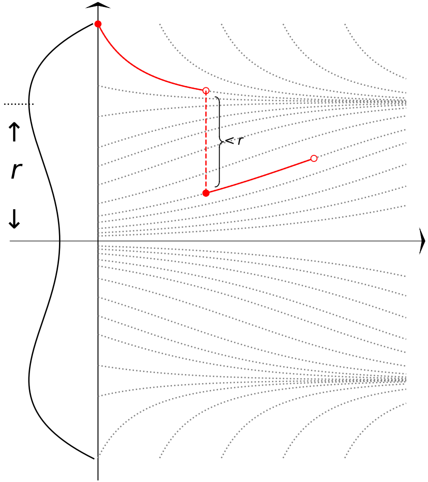

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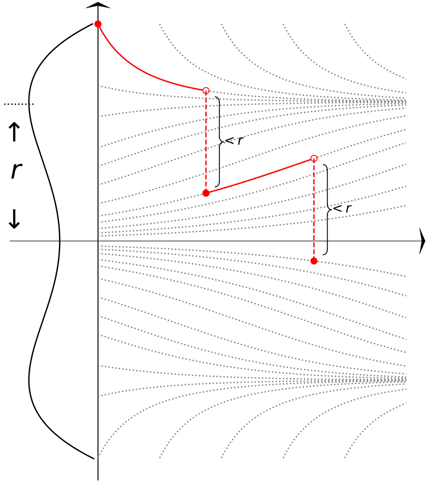
SGD's Escaping Route under Gradient Clipping



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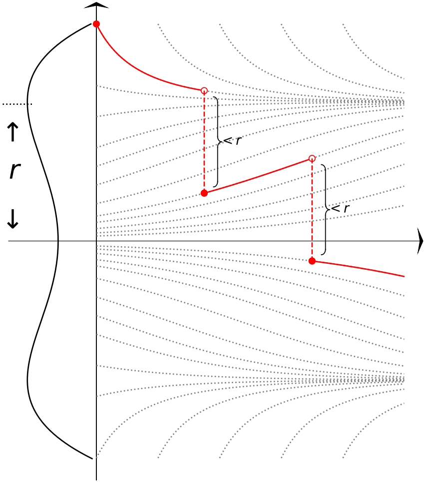

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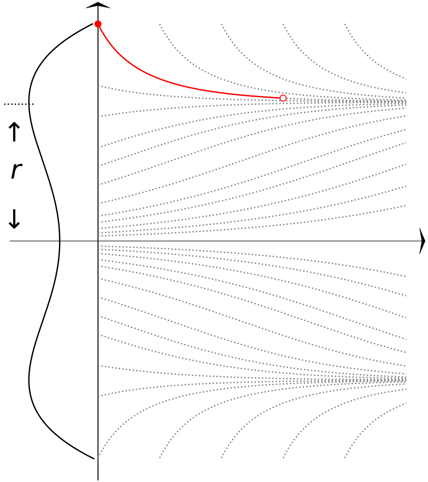

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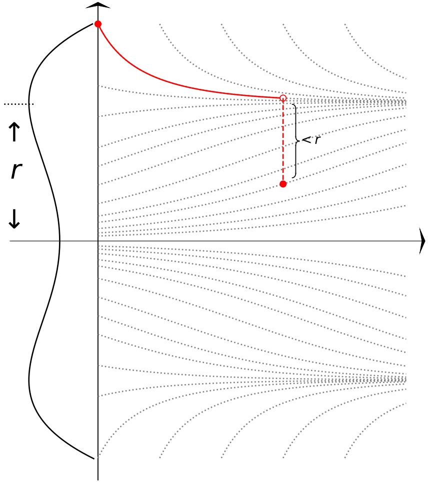

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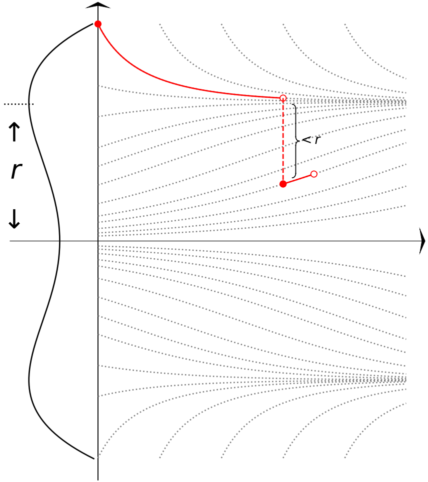

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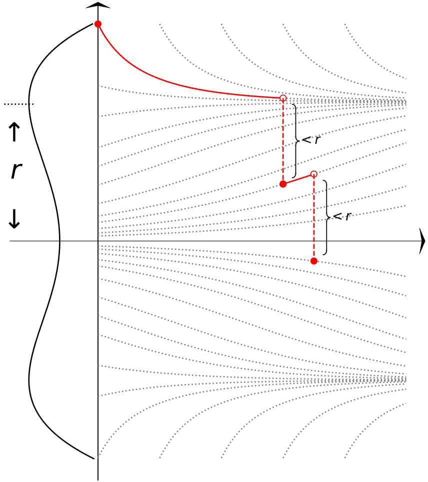

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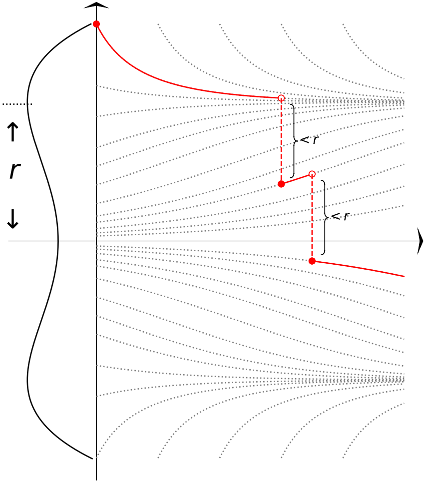

Clipping threshold

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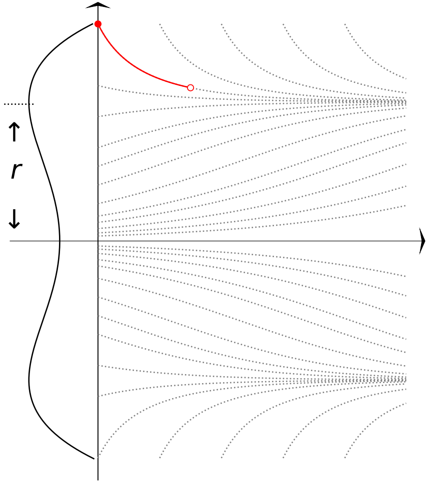

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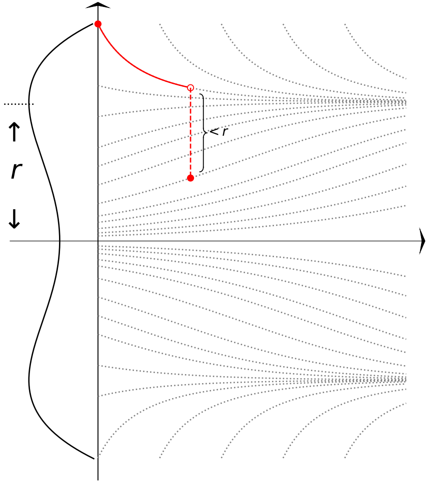

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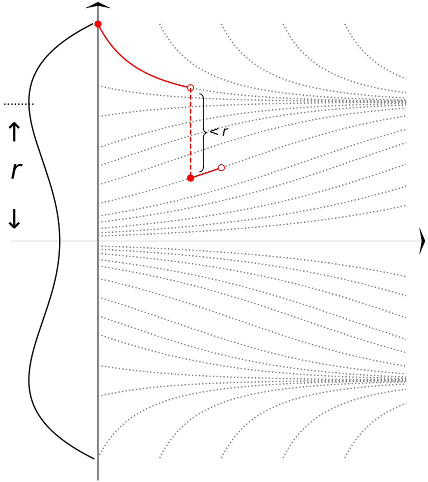

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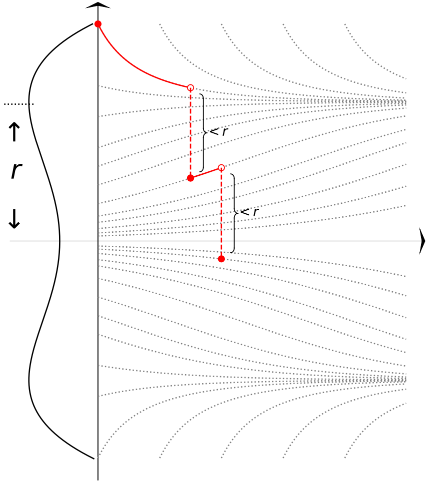

Clipping threshold

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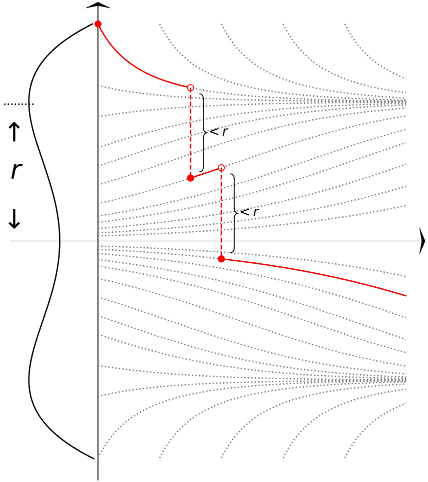
SGD's Escaping Route under Gradient Clipping



Clipping threshold

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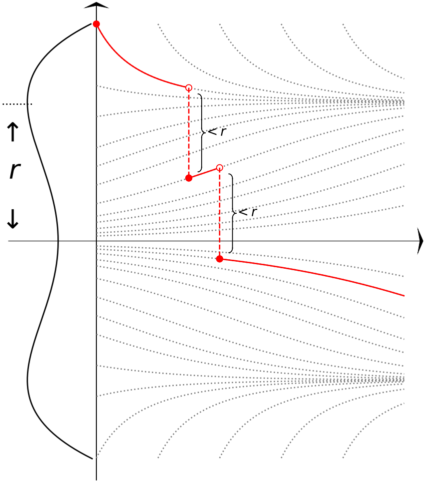
SGD's Escaping Route under Gradient Clipping

Most likely path under heavy-tailed noises: with $l^* = 2$ jumps



Clipping threshold

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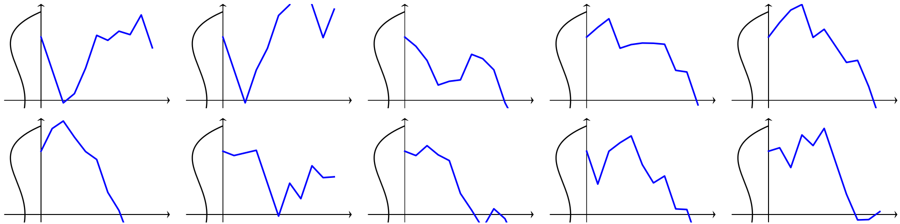
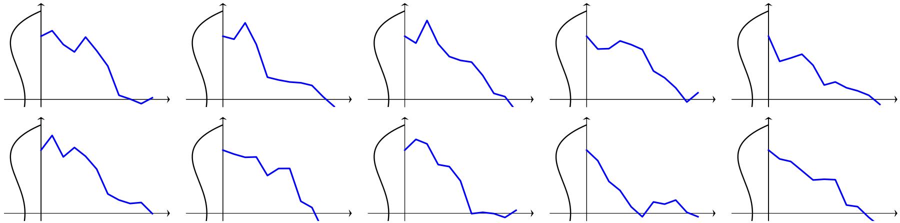
SGD's Escaping Route under Gradient Clipping

Trajectory of SGD X^η conditional on exit: **light-tailed** noises with $\eta = 1/10$

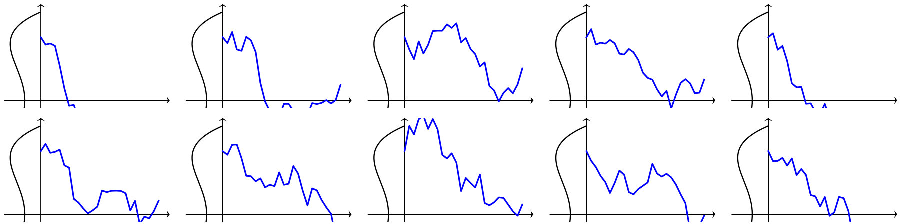


Trajectory of SGD X^η conditional on exit: **heavy-tailed** noises with $\eta = 1/10$

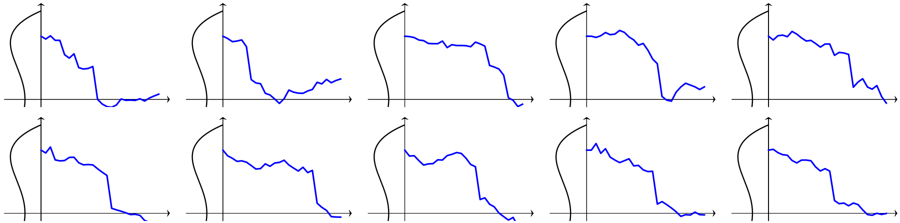


SGD's Escaping Route under Gradient Clipping

Trajectory of SGD X^η conditional on exit: **light-tailed** noises with $\eta = 1/25$

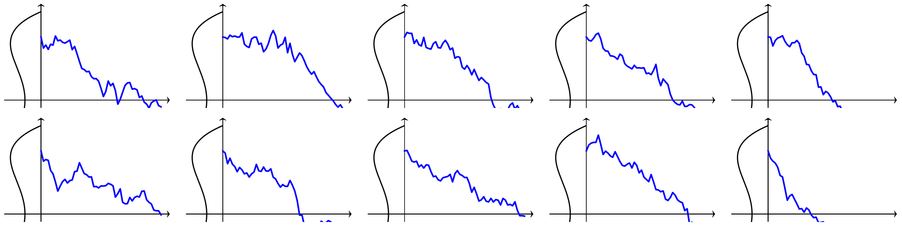


Trajectory of SGD X^η conditional on exit: **heavy-tailed** noises with $\eta = 1/25$

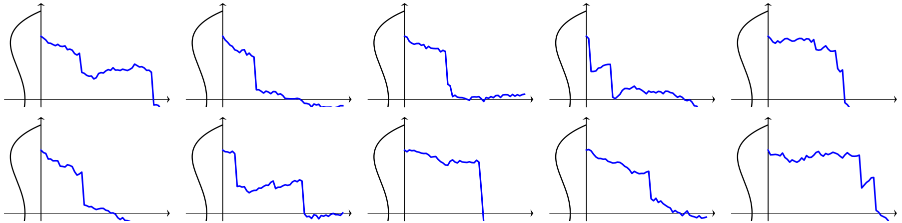


SGD's Escaping Route under Gradient Clipping

Trajectory of SGD X^η conditional on exit: **light-tailed** noises with $\eta = 1/50$

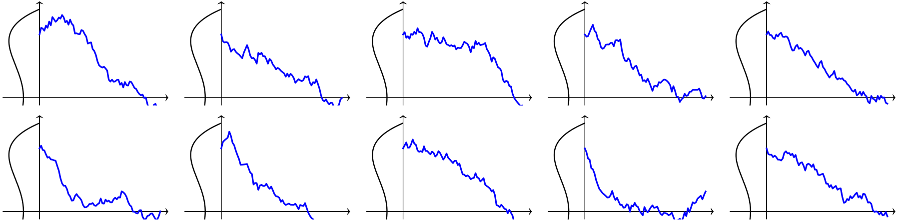


Trajectory of SGD X^η conditional on exit: **heavy-tailed** noises with $\eta = 1/10$

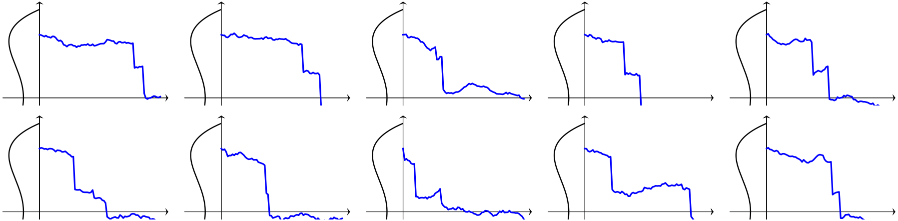


SGD's Escaping Route under Gradient Clipping

Trajectory of SGD X^η conditional on exit: **light-tailed** noises with $\eta = 1/75$

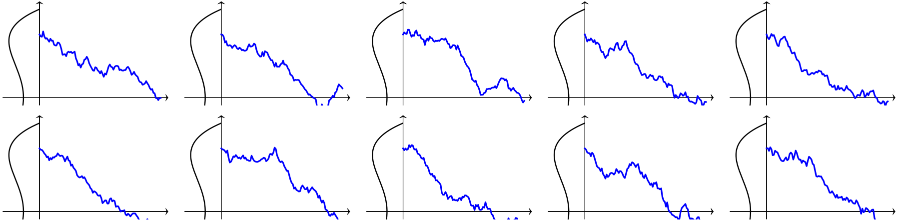


Trajectory of SGD X^η conditional on exit: **heavy-tailed** noises with $\eta = 1/75$

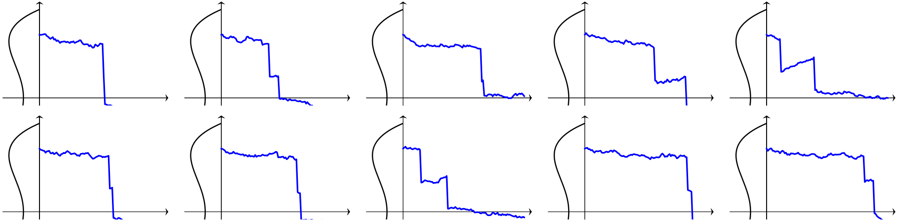


SGD's Escaping Route under Gradient Clipping

Trajectory of SGD X^η conditional on exit: **light-tailed** noises with $\eta = 1/100$

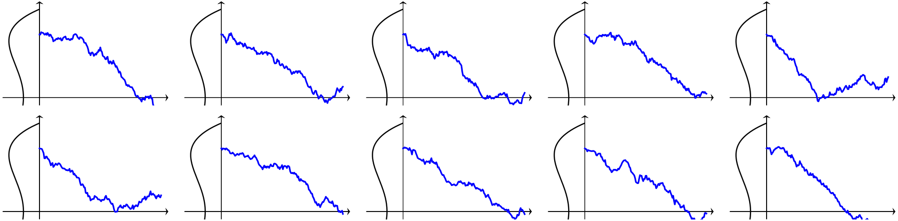


Trajectory of SGD X^η conditional on exit: **heavy-tailed** noises with $\eta = 1/100$

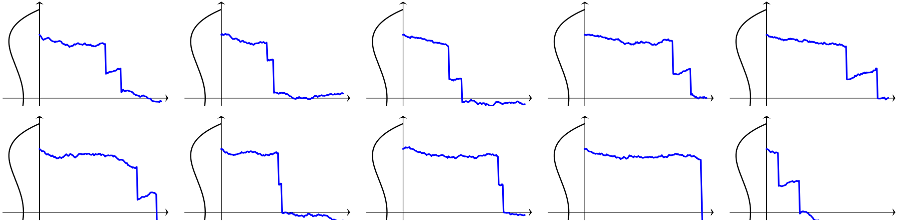


SGD's Escaping Route under Gradient Clipping

Trajectory of SGD X^η conditional on exit: **light-tailed** noises with $\eta = 1/150$

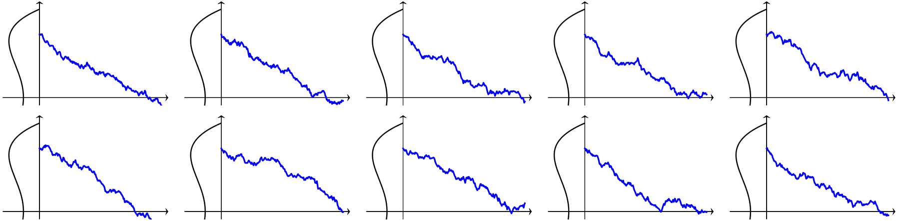


Trajectory of SGD X^η conditional on exit: **heavy-tailed** noises with $\eta = 1/150$

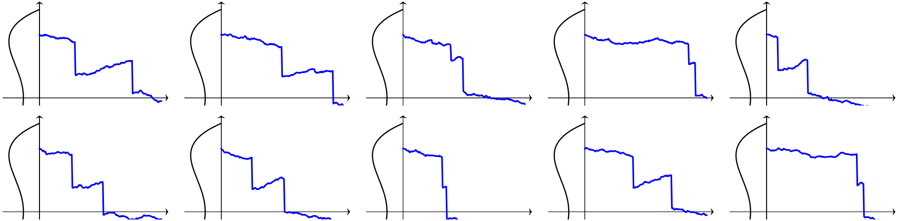


SGD's Escaping Route under Gradient Clipping

Trajectory of SGD X^{η} conditional on exit: **light-tailed** noises with $\eta = 1/200$

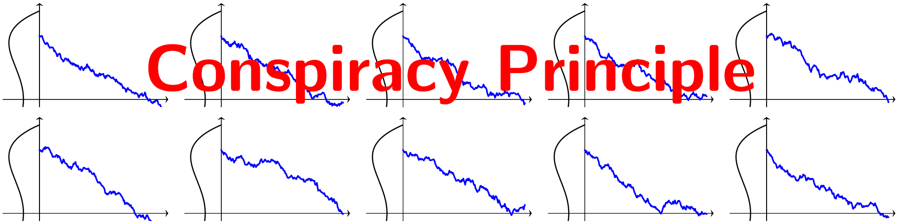


Trajectory of SGD X^{η} conditional on exit: **heavy-tailed** noises with $\eta = 1/200$



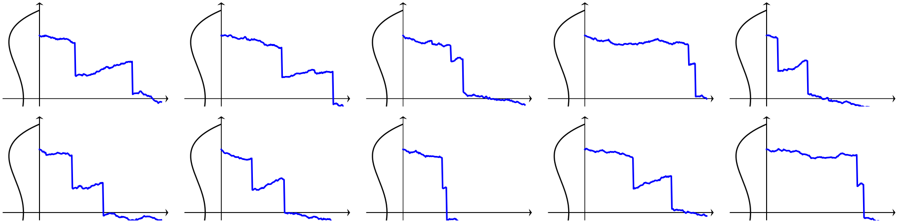
SGD's Escaping Route under Gradient Clipping

Trajectory of SGD X^η conditional on exit: **light-tailed** noises with $\eta = 1/200$



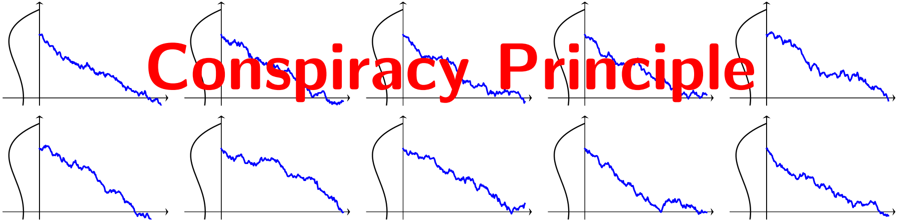
Conspiracy Principle

Trajectory of SGD X^η conditional on exit: **heavy-tailed** noises with $\eta = 1/200$



SGD's Escaping Route under Gradient Clipping

Trajectory of SGD X^η conditional on exit: **light-tailed** noises with $\eta = 1/200$



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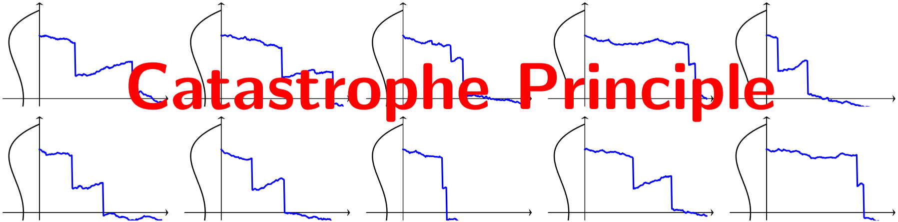
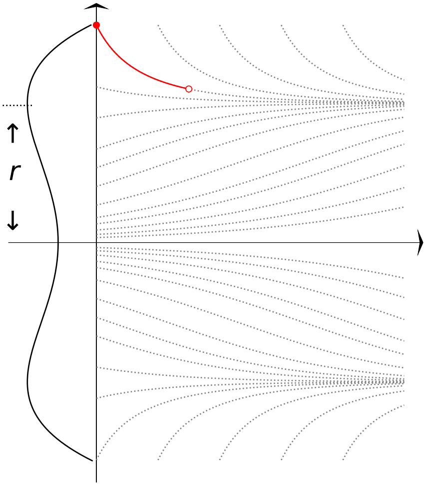


SGD's Escaping Route under Gradient Clipping



Clipping threshold

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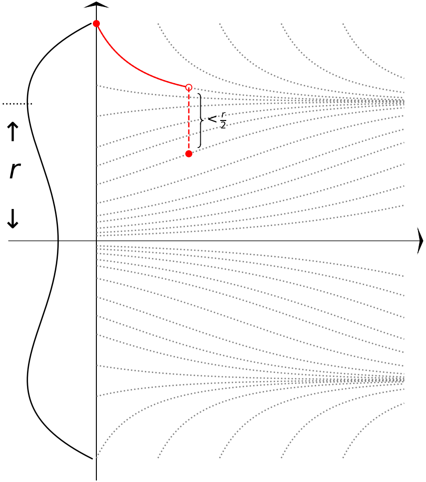
SGD's Escaping Route under Gradient Clipping



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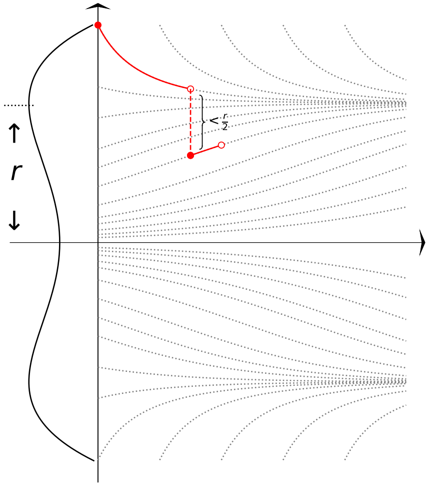

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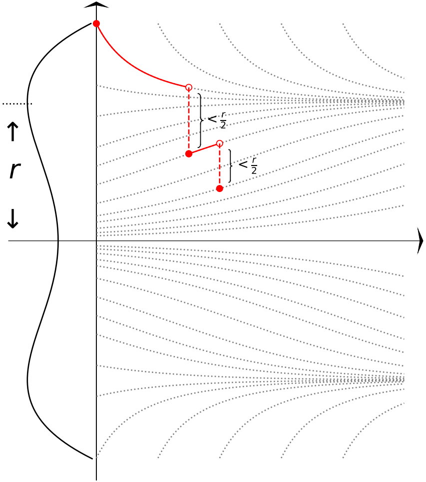

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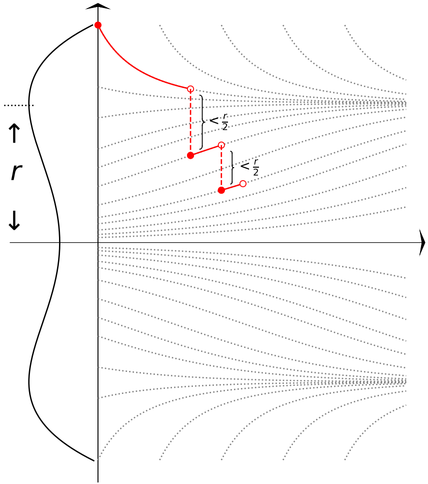
SGD's Escaping Route under Gradient Clipping



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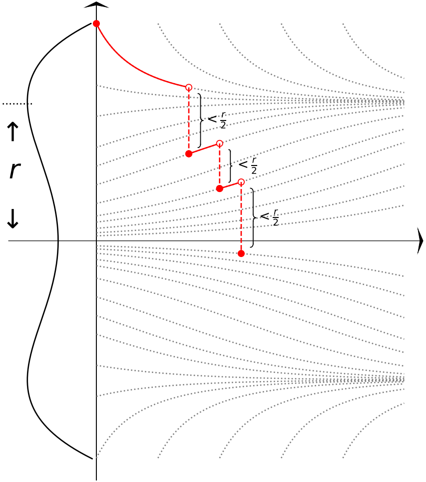
SGD's Escaping Route under Gradient Clipping



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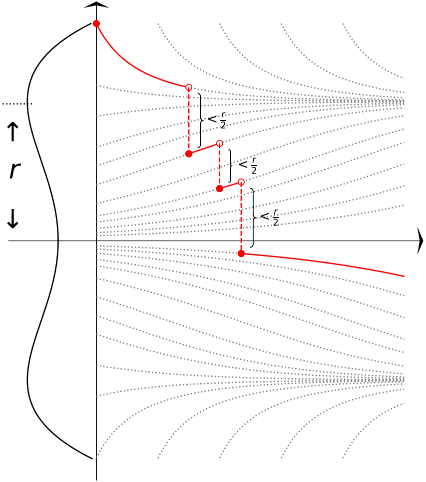

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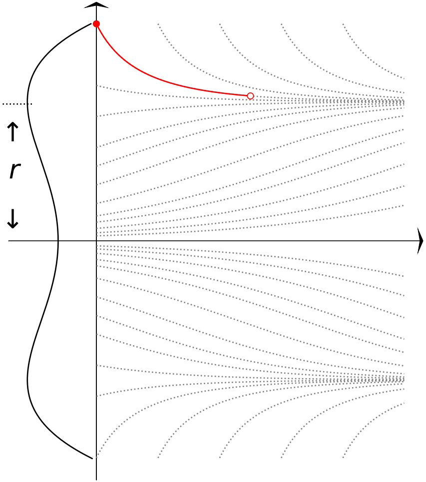

Clipping threshold

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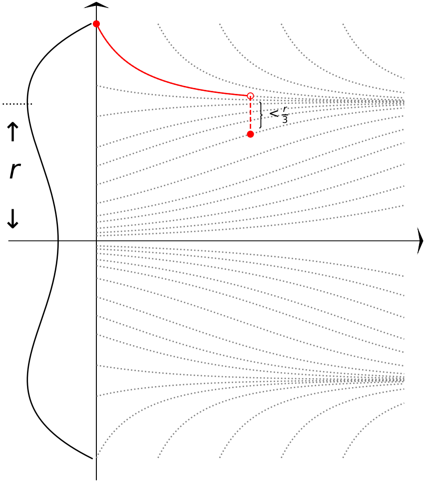
SGD's Escaping Route under Gradient Clipping



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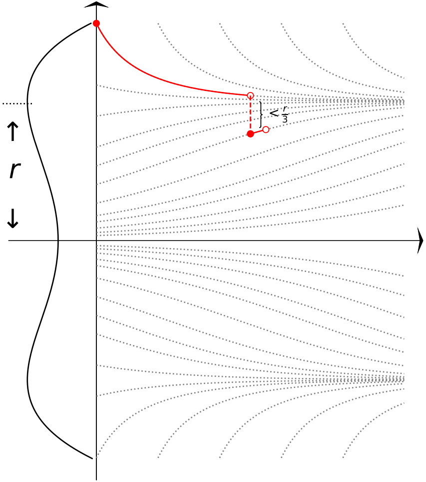
SGD's Escaping Route under Gradient Clipping



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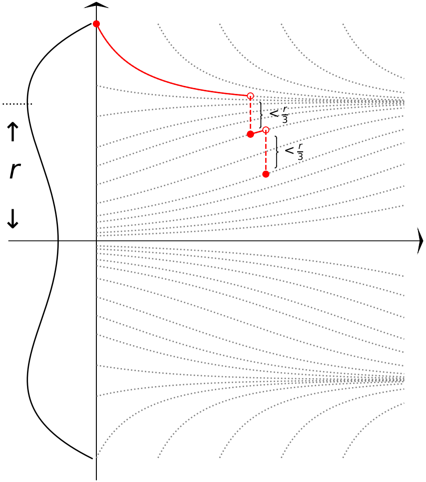
SGD's Escaping Route under Gradient Clipping



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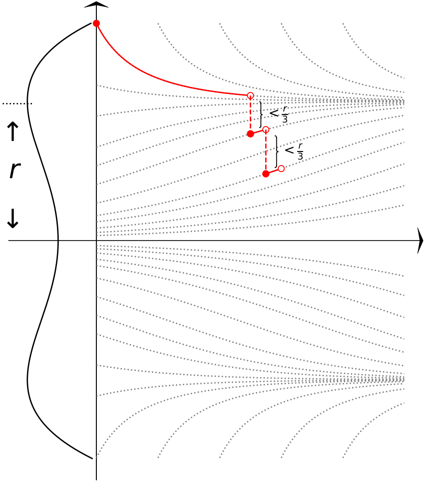

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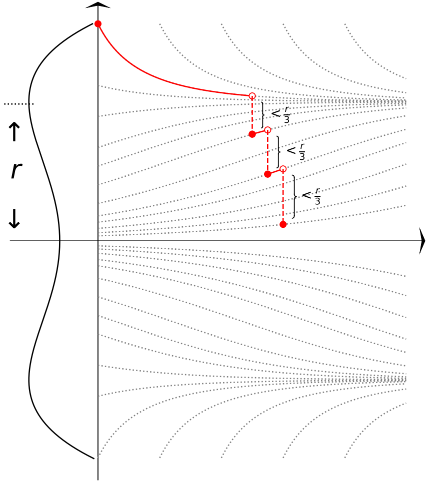
SGD's Escaping Route under Gradient Clipping



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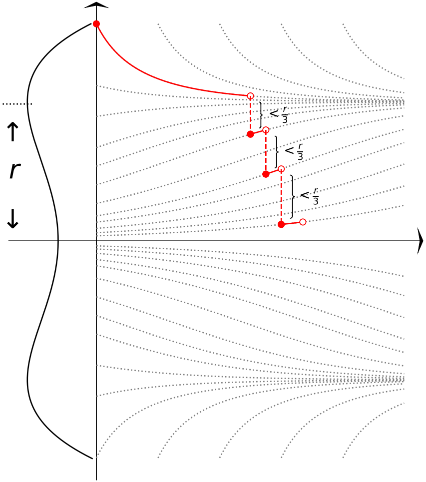
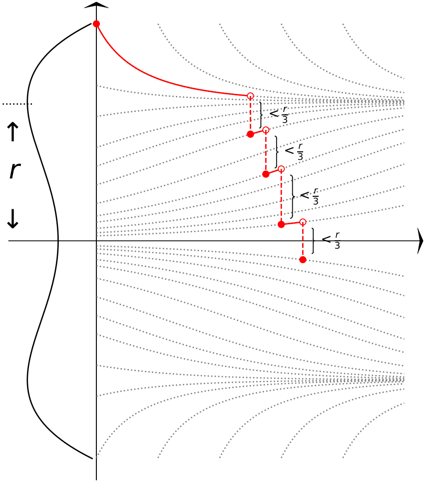
SGD's Escaping Route under Gradient Clipping



Clipping threshold

$$X_j^\eta = X_{j-1}^\eta + \varphi_b\big(-\eta\nabla f(X_{j-1}^\eta) + \eta Z_j\big),\; b \in (r/4, r/3)$$

SGD's Escaping Route under Gradient Clipping



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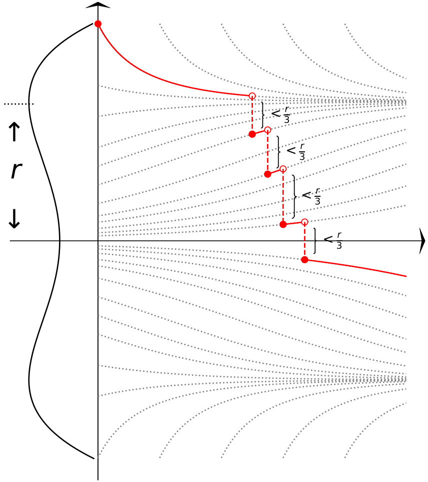

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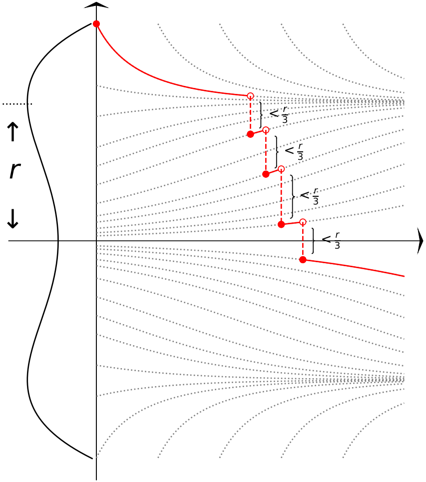
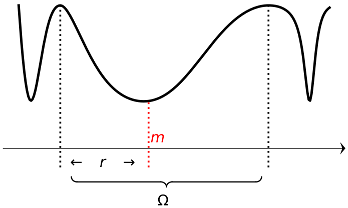


Clipping threshold

(Min # of jumps for escape) $l^* = \lceil r/b \rceil$

First Exit Time Analysis



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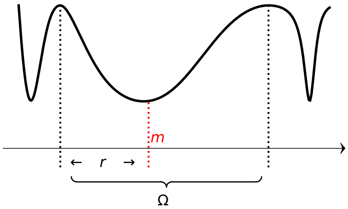

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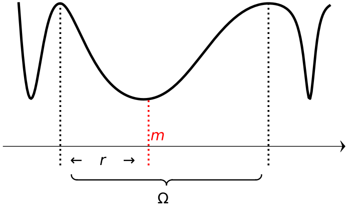

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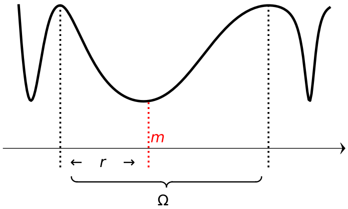

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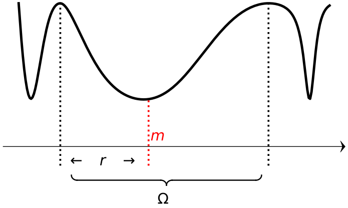

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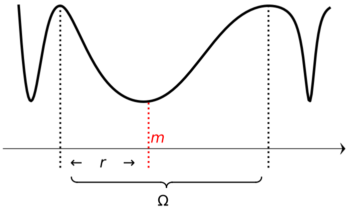

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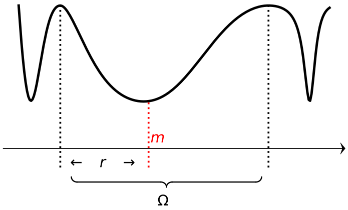

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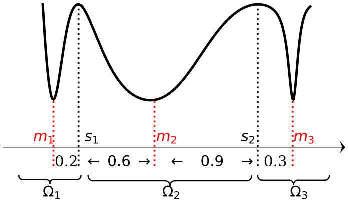
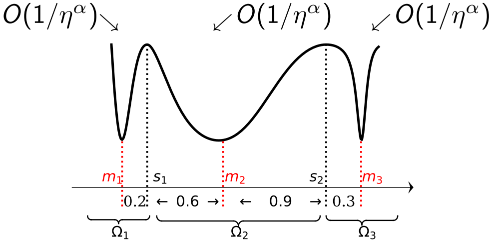


Without Clipping

Elimination of Narrow Minima



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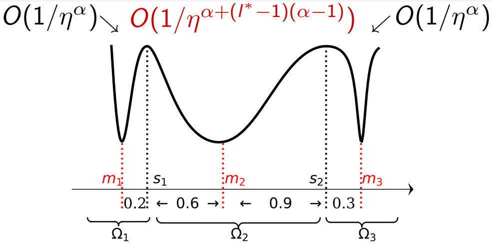

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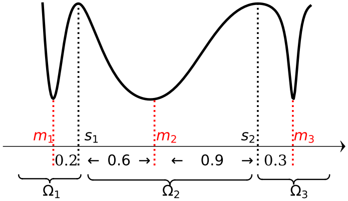

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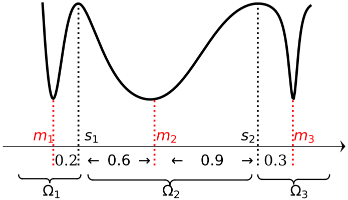

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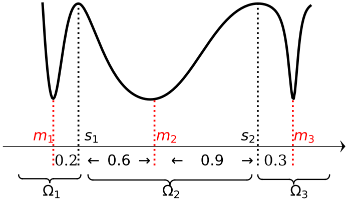

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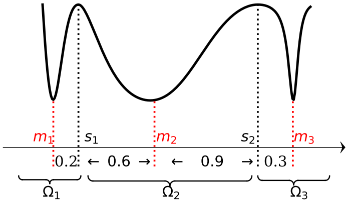

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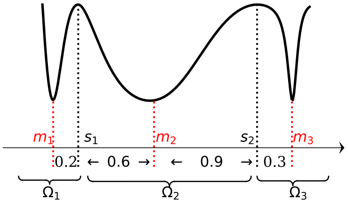

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Under structural conditions on loss landscape, for any $t > 0$ and $\beta > 1 + (\alpha - 1)\max_i l_i^*$,

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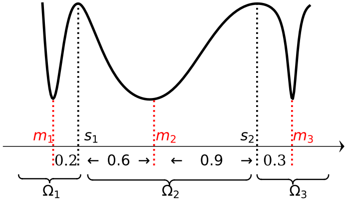

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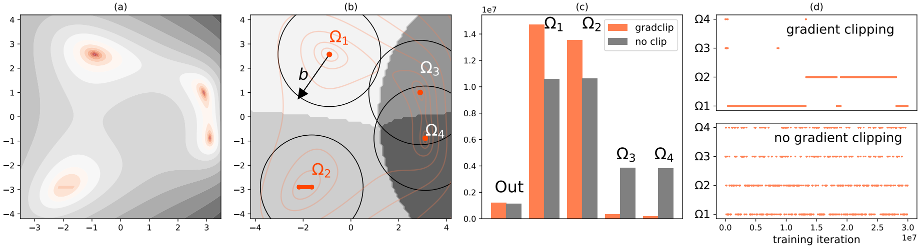
↖Proportion of time at narrow minima

$\mathbb{R}^d$ Case

- Same Elimination Effect in $\mathbb{R}^d$



New Training Algorithm

Truncated Heavy-tailed SGD in Deep Learning

- **Our Method**: $X \leftarrow X - \varphi_b\big(\eta \cdot g_{\text{heavy}}(X)\big)$ where

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$\nwarrow$ $\nearrow$

Same or independent batches? $\Rightarrow$two versions

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Experiments

Test accuracy	LB	SB	SB + Clip	SB + Noise	Our 1	Our 2
CorrputedFMNIST, LeNet	68.66%	69.20%	68.77%	64.43%	69.47%	**70.06%**
SVHN, VGG11	82.87%	85.92%	85.95%	38.85%	**88.42%**	88.37%
CIFAR10, VGG11	69.39%	74.42%	74.38%	40.50%	75.69%	**75.87%**
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- **Flatter geometry & Improved generalization performance**
- Requires both **heavy-tailed** noise and **truncation**

Experiments

CIFAR10-VGG11	SB + Clip	Our 1	Our 2
Test Accuracy	89.54%	**90.76%**	90.45%
Expected Sharpness	0.167	**0.085**	0.096
PAC-Bayes Sharpness	1.31×10^4	$\mathbf{9 \times 10^3}$	10^4
Maximal Sharpness	1.66×10^4	1.29×10^4	$\mathbf{1.22 \times 10^4}$
CIFAR100-VGG16	**SB + Clip**	**Our 1**	**Our 2**
Test Accuracy	56.32%	**65.44%**	62.99%
Expected Sharpness	0.857	**0.441**	0.479
PAC-Bayes Sharpness	2.49×10^4	$\mathbf{1.9 \times 10^4}$	1.98×10^4
Maximal Sharpness	2.75×10^4	$\mathbf{2.12 \times 10^4}$	2.16×10^4

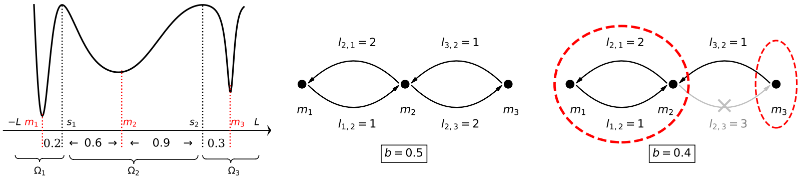
- **More training techniques:** Data augmentation, learning rate scheduler.

Conclusion

- **Theoretical Contribution**
  - Rigorously established that truncated heavy-tailed noises can eliminate sharp minima
  - Catastrophe principle, first exit time analysis, and metastability for heavy-tailed SGD
- **Algorithmic Contribution**
  - Proposed a tail-inflation strategy to find flatter solution with better generalization

Remarks on Technical Results

- **"Regularity conditions"**



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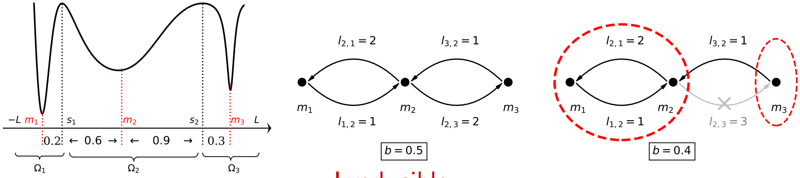

Irreducible

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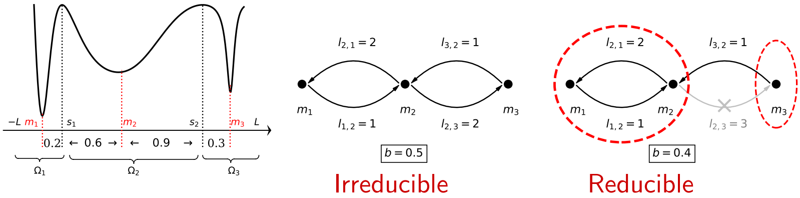

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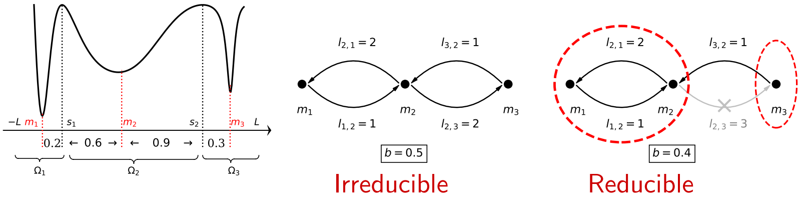

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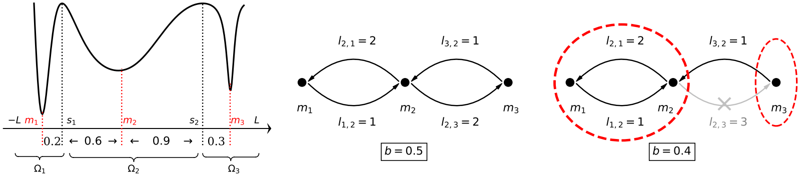

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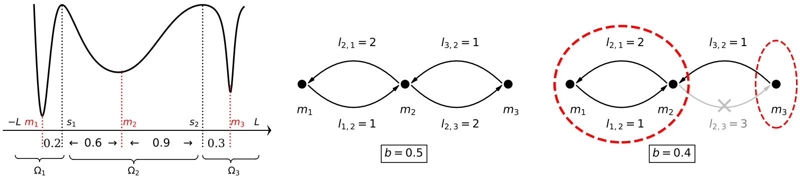

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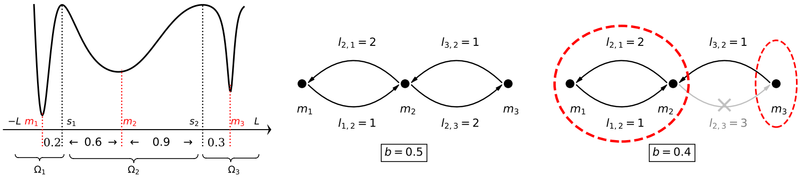
- $\mathbb{R}^d$ **Extension**
  - First exit time results in $\mathbb{R}^d$

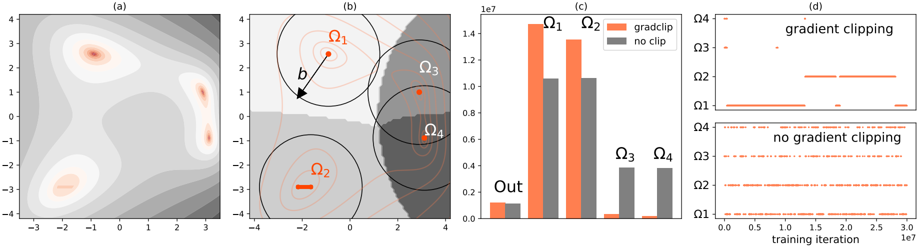
Remarks on Technical Results

- **“Regularity conditions”**: Irreducibility



  - We established similar results for the reducible case.

- $\mathbb{R}^d$ **Extension**
  - First exit time results in $\mathbb{R}^d$
  - $\mathbb{R}^d$ simulation experiments



First Exit Time Analysis

- **First Exit Time:** $\sigma(\eta) \triangleq \min\{j \geq 0 : X_j^\eta \notin \Omega\}$
- $l^* \triangleq \lceil r/b \rceil$

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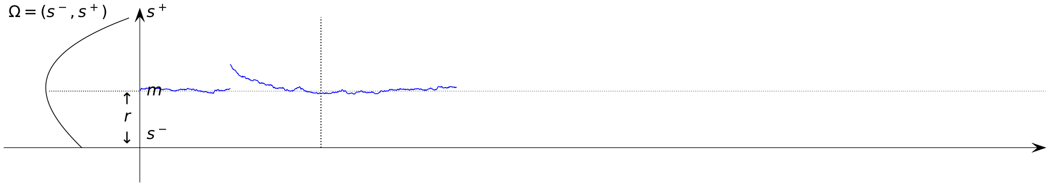

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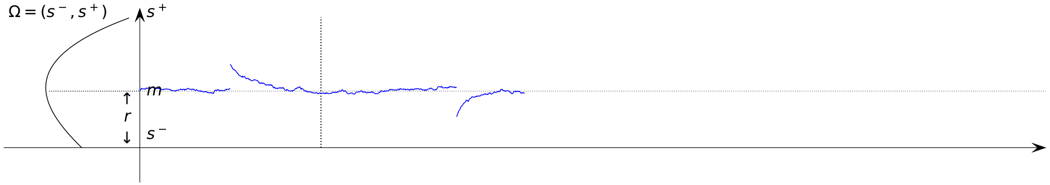

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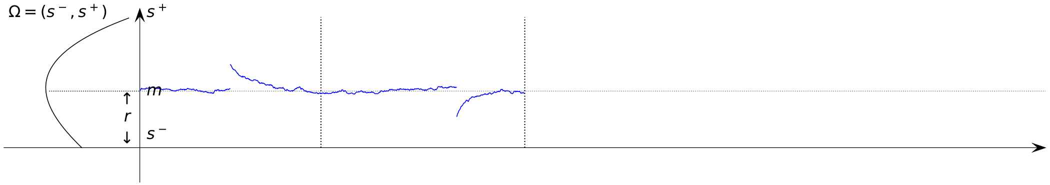

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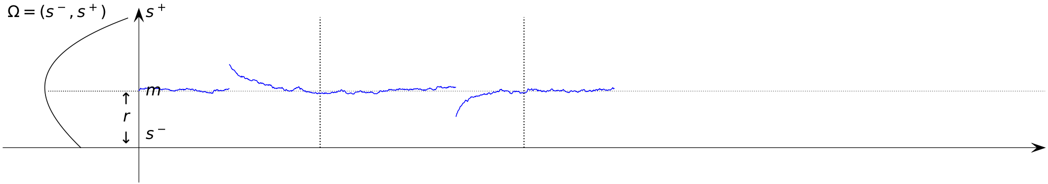

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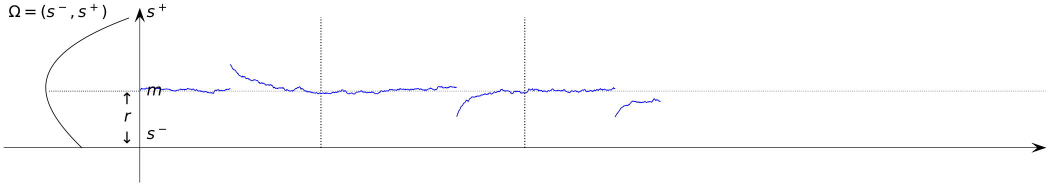

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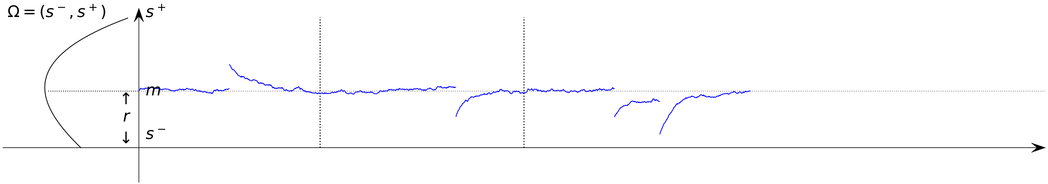

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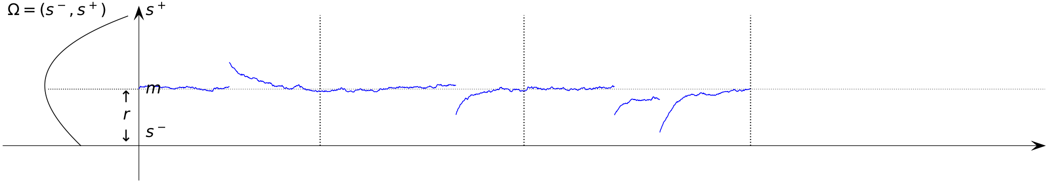

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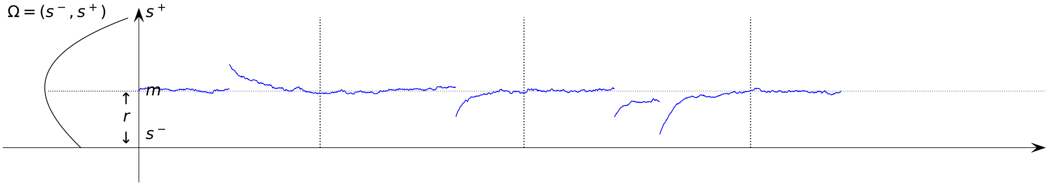

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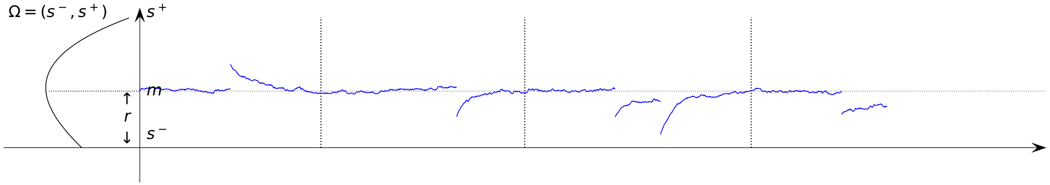

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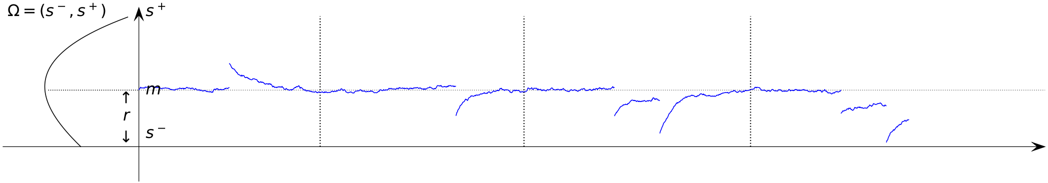

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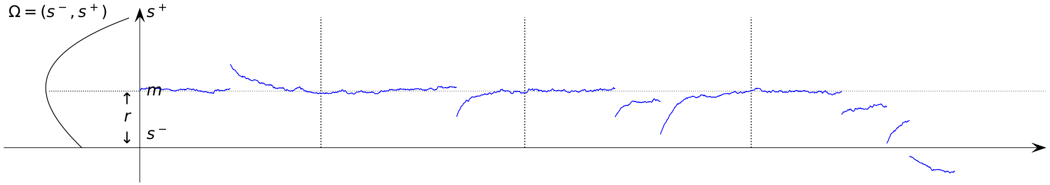

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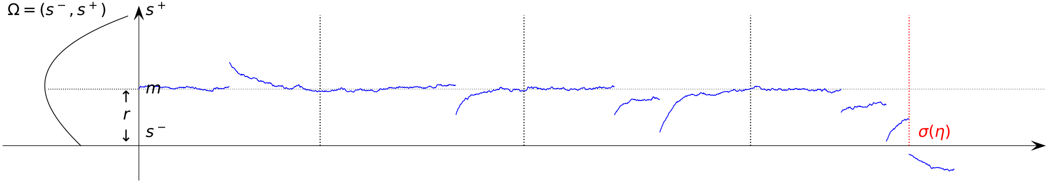

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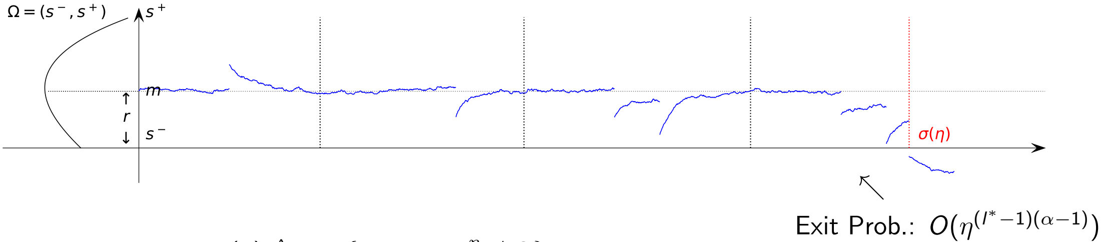

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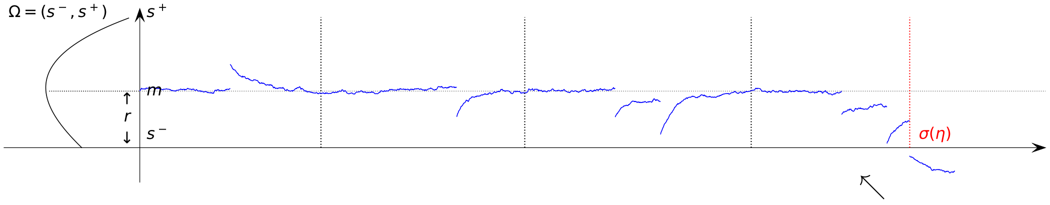

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Duration: $O(1/\eta^{\alpha})$

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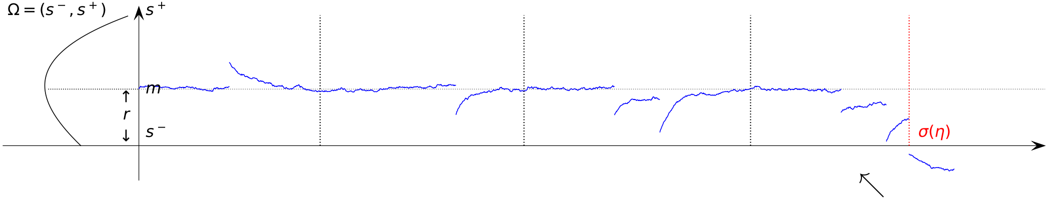

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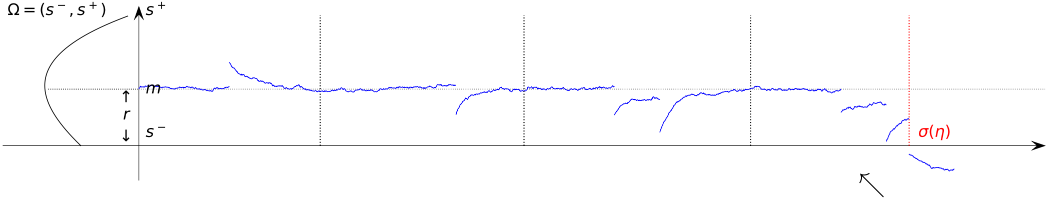

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Theorem (Wang, Oh, Rhee, 2021)

For (Lebesgue) almost every $b > 0$, there exist some $q > 0$ and $\lambda(\eta) \in RV_{\alpha+(l^*-1)(\alpha-1)}(\eta)$ such that

$$\sigma(\eta)\lambda(\eta) \Rightarrow Exp(q) \text{ as } \eta \downarrow 0.$$

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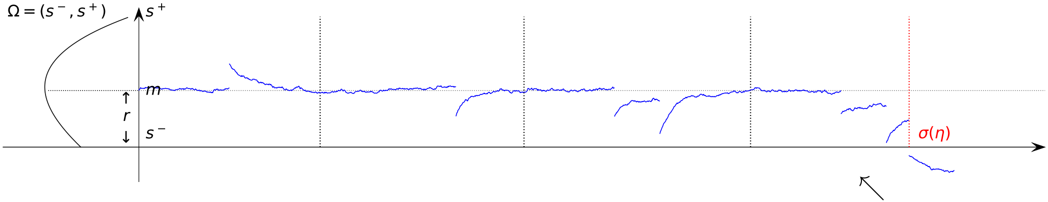

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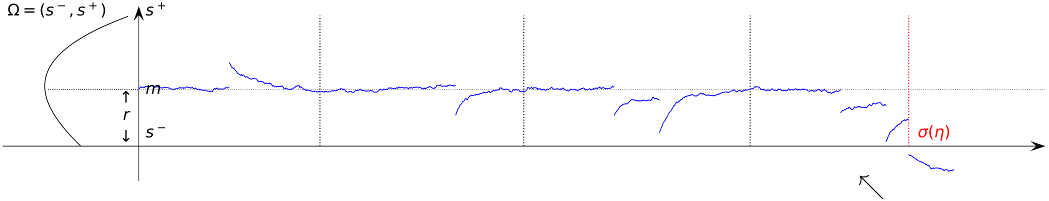

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