Differential Privacy Meets Robust Statistics

Sewoong Oh

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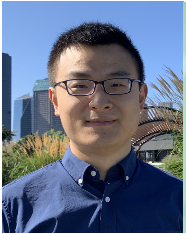
University of Washington

joint work with

Xiyang Liu

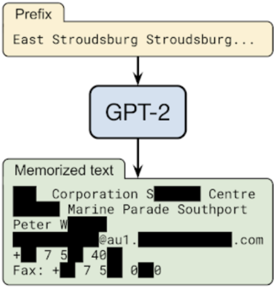


Weihao Kong

Sham Kakade

What can go wrong when training on shared data?

- Increasingly more models are being trained on shared data
- Sensitive information should not be revealed by the trained model
- Membership inference attacks can identify individual's sensitive data used in the training



*

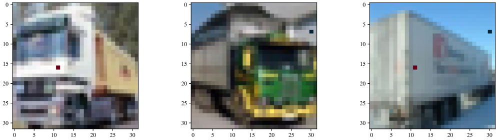
- Potential defense: Differentially Private Stochastic Gradient Descent† when computing the average of the gradients in the mini-batch, use differentially private mean estimation

* [Carlini et al.,2020]

† [Chaudhuri,Monteleoni,Sarwate,2011], [Abadi et al.,2016]

What can go wrong when training on shared data?

- When training on shared data, not all participants are trusted
- Malicious users can easily inject corrupted data
- Data poisoning attacks can create backdoors on the trained model such that any sample with the trigger will be predicts as 'deer'



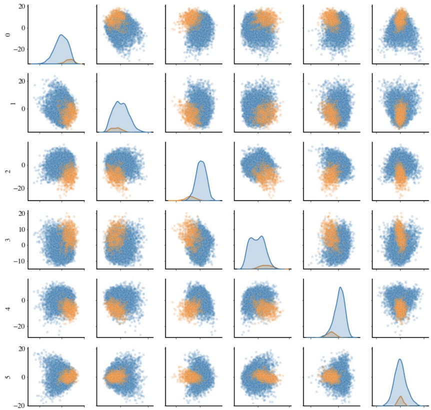
$y_i =$ 'deer'

- Strong defense: Robust estimation*
- Insight: successful backdoor attacks leave a path of activations in the trained model that are triggered only by the corrupted samples

*[Hayase,Kong,Somani,O.,2021] inspired by [Tran,Li,Madry,2018]

Middle layer of a model trained with corrupted data

- All samples with label 'deer': CLEAN and POISONED
- Top-6 PCA projection of node activations at a middle layer
- Can we separate POISONED from CLEAN?



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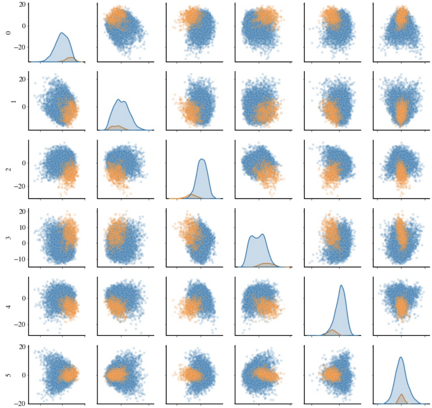
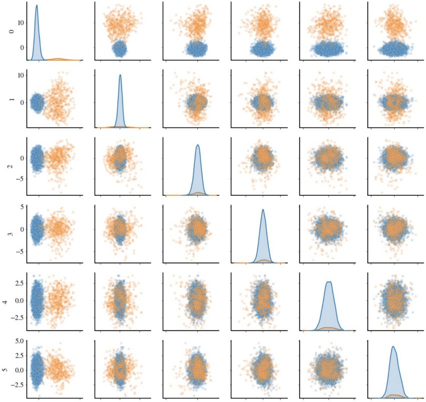


After whitening with the covariance of CLEAN



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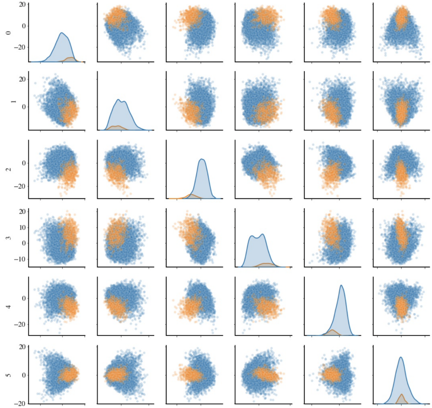
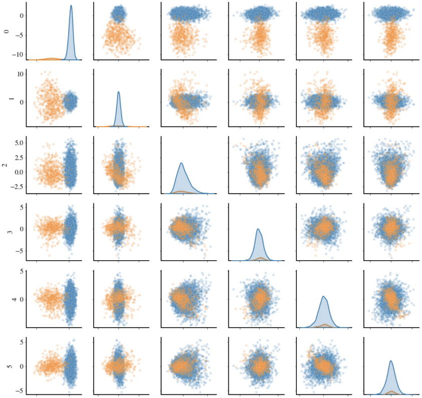
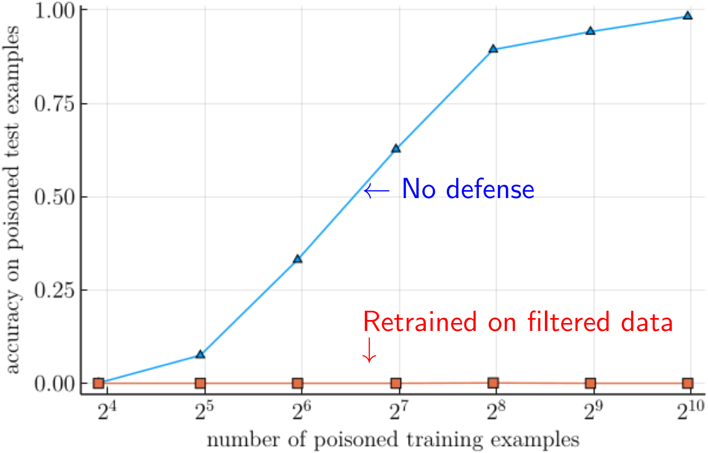


After whitening with
estimated **robust** mean and covariance



SPECTRE: Defense against backdoor attacks

[Hayase,Somani,Kong,O.2021][‡]



‡ `https://github.com/SewoongLab/backdoor-suite`

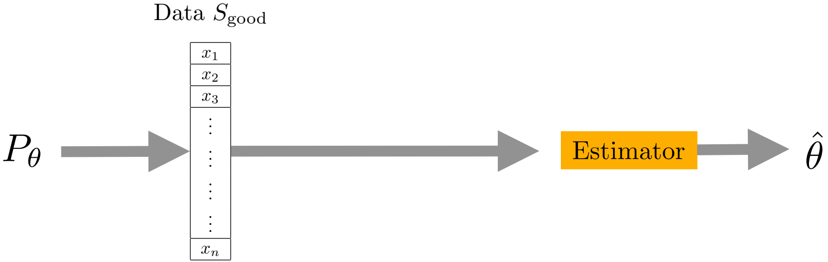
We need privacy and robustness, simultaneously

- When learning from shared data
  - Differential privacy is crucial in defending against inference attacks
  - Robust estimation is crucial in defending against data poisoning attacks

- We provide the first efficient estimators that are provably robust against data corruption and differentially private

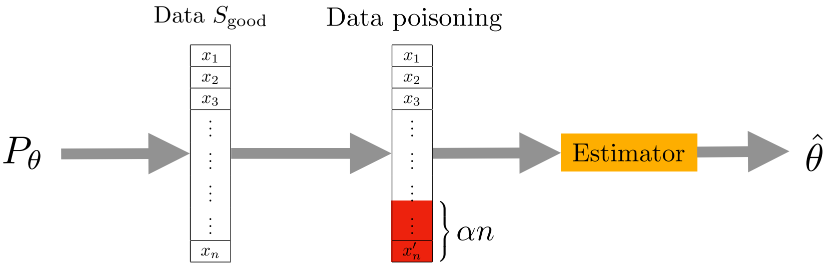
Statistical estimation, robustly and privately

- Statistics



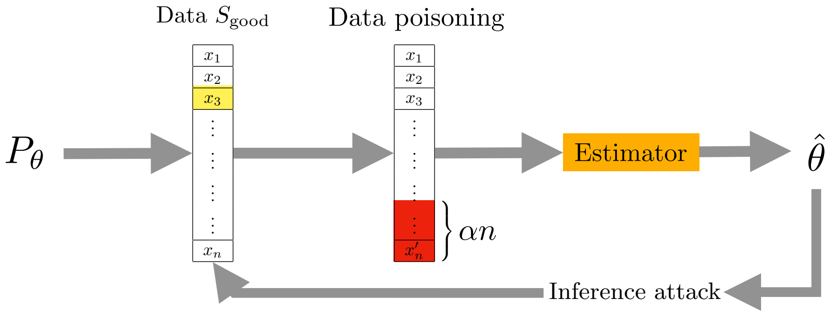
Statistical estimation, robustly and privately

- Statistics⇒ Robust estimation



Statistical estimation, robustly and privately

- Statistics⇒ Robust estimation⇒ Robust and private estimation



- This talk focuses on mean estimation
- Q. What is the extra cost (in the estimation error) we pay for {Robustness, Privacy, and Robustness+Privacy}

Mean estimation

- Estimate the mean μ from n i.i.d. samples
- For this talk,
  we assume sub-Gaussian distribution with identity covariance matrix
- Minimax error rate:

$$\min_{\hat{\mu} \in \mathcal{F}_{S_n}} \max_{P_\mu} \mathbb{E}\big[\, \|\hat{\mu}(S_n) - \mu\| \,\big] \;\propto\; \sqrt{\frac{d}{n}}$$

$\mathcal{F}_{S_n}$ is set of all estimators over n i.i.d. samples in $\mathbb{R}^d$ from P_μ,
P_μ is maximized over all sub-Gaussian distributions with identity covariance

Robust mean estimation

- Threat model
  - Adversarial corruption model:
    $\{x_i\}_{i=1}^n \sim P_\mu$ is drawn, then adversary replaces α-fraction arbitrarily

- Robust mean estimation:
  - Low dimensional:
    [Tukey,1960] [Huber,1964]
  - Computationally intractable methods in high dimension:
    [Donoho,Liu,1988], [ChenGaoRen,2015],[Zhu,Jiao,Steinhardt,2019]
  - Breakthroughs in polynomial time algorithms:
    [Lai,Rao,Vempala,2016],[Diakonikolas,Kamath,Kane,Li,Moitra,Stewart,2019]
  - Linear time algorithms:
    [Cheng,Dianikolas,Ge,2019], [Depersin,Lecué,2019],[Dong,Hopkins,Li,2019]

Robust mean estimation

- Threat model
  - Adversarial corruption model:
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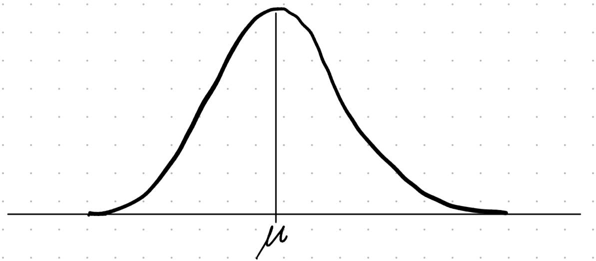
- Relatively easy to estimate mean robustly in low-dimensions

histogram of sub-Gaussian samples in 1D



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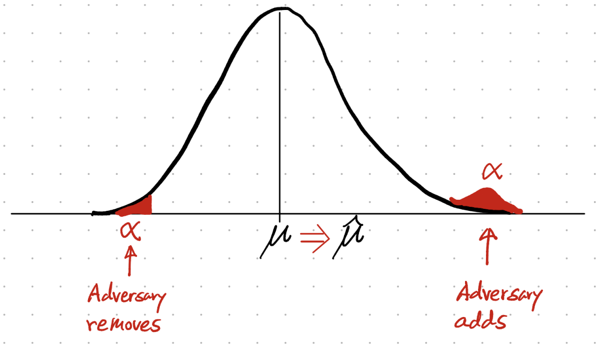

simple outlier detection achieves $|\hat{\mu} - \mu| \leq \alpha\sqrt{\log(1/\alpha)}$

Robust mean estimation

- Threat model
  - Adversarial corruption model:
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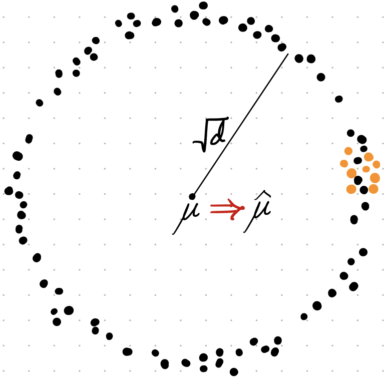
- Mean estimation becomes challenging in high-dimensions

scatter plot of sub-Gaussian samples in high-dimension



each corrupted sample looks uncorrupted and still $\|\hat{\mu} - \mu\| \geq \alpha\sqrt{d}$

Efficient algorithm: Filtering [Diakonikolas et al.,2017]

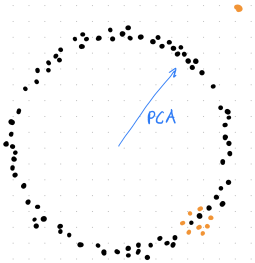
Geometric Lemma [Dong,Hopkins,Li,2019]

Given n i.i.d. samples from a sub-Gaussian distribution with identity covariance matrix, if at most αn samples are corrupted, then, w.h.p.

$$\|\mu_{\text{emp}}(S) - \mu\| \;\leq\; \sqrt{\frac{d}{n}} + \alpha\sqrt{\log(1/\alpha)} + \sqrt{\alpha\|\mathrm{Cov}(S) - \mathbf{I}\|}$$

- Repeat until $\|\mathrm{Cov}(S) - \mathbf{I}\|$ is $O(\alpha\log(1/\alpha))$
  - $v \leftarrow \arg\max_{v:\|v\|=1} v^T \mathrm{Cov}(S) v$
  - $S \leftarrow$ 1D-Filter$(\{\langle v, x_i - \mu_{\text{emp}}(S)\rangle^2\}_{i\in S})$

- Each step guarantees that
  - at least one sample is removed
  - if $\|\mathrm{Cov}(S) - \mathbf{I}\| > C\alpha\sqrt{\log(1/\alpha)}$ more corrupted samples removed than clean samples in expectation



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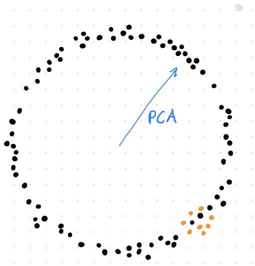

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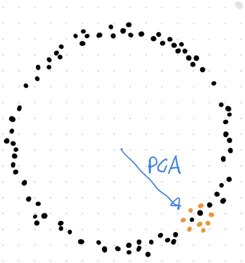

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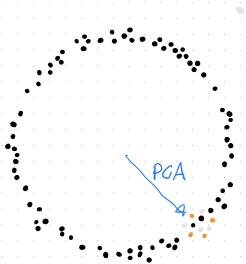

Robust mean estimation

- Minimax error rate under α-corruption

$$\min_{\hat{\mu}} \max_{P_\mu} \mathbb{E}\left[\, \|\hat{\mu}(S_{n,\alpha}) - \mu\| \,\right] \;\propto\; \underbrace{\sqrt{\frac{d}{n}}}_{\text{no corruption}} + \underbrace{\alpha}_{\alpha\text{-corruption}}$$

achieved by filtering algorithm of [Diakonikolas et al.,2017]

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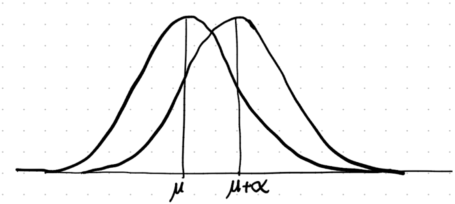
  achieved by filtering algorithm of [Diakonikolas et al.,2017]
- Lower bound [Chen,Gao,Ren,2015]
  - Even with infinite samples $\|\hat{\mu}(S) - \mu\| \geq \alpha$
    because we cannot tell if clean distribution is $\mathcal{N}(\mu + \alpha, 1)$
    or it was α-corrupted from $\mathcal{N}(\mu, 1)$



$$\mathrm{TV}(\mathcal{N}(\mu,1), \mathcal{N}(\mu+\alpha,1)) = \Theta(\alpha)$$

Minimax error rate for mean estimation under sub-Gaussian distributions with identity covariance

	Error $\|\hat{\mu} - \mu\|$
no corruption or privacy	$\sqrt{\frac{d}{n}}$
α-corruption	$\sqrt{\frac{d}{n}} + \alpha$ [Diakonikolas et al.,2017]
(ε, δ)-DP	
α-corruption and (ε, δ)-DP	

Differential Privacy provably ensures plausible deniability

- Goal: a strong adversary who knows all the other entries in the database except for yours, should not be able to identify whether you participated in that database or not

- Definition*: For two databases S and S' that differ by only one entry, a randomized output to a query is (ε, δ)-differentially private if

$$\mathbb{P}(\text{query\_output}(S) \in A) \;\leq\; e^{\varepsilon}\, \mathbb{P}(\text{query\_output}(S') \in A) + \delta$$

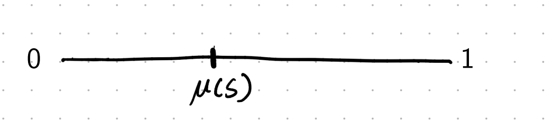
- smaller $\varepsilon, \delta \;\Rightarrow\;$ Testing S or S' fails $\;\Rightarrow\;$ inference attack fails

---

* [Dwork,McSherry,Nissim,Smith,2006]

(ε, δ)-differentially private mean estimation

S
0
1
0
0
0
1
$\vdots$
0



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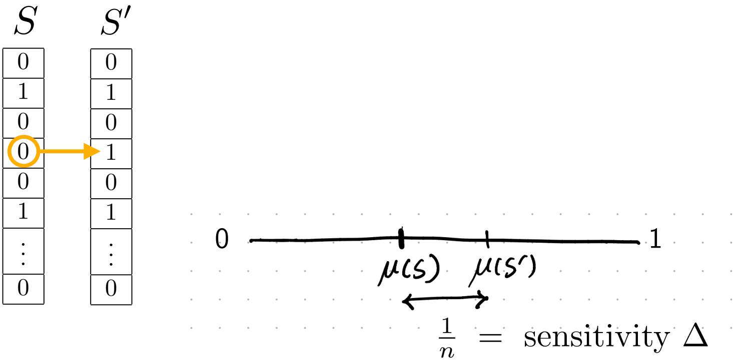

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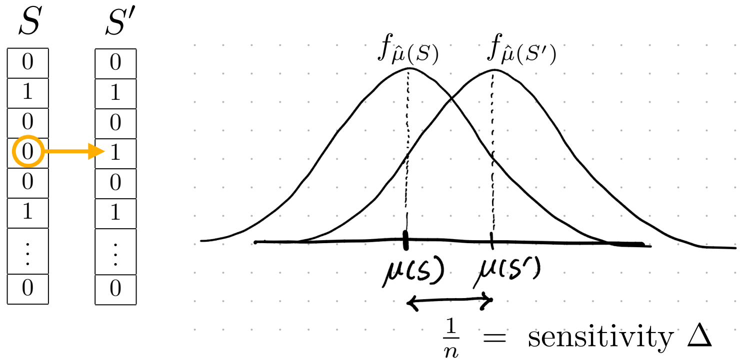
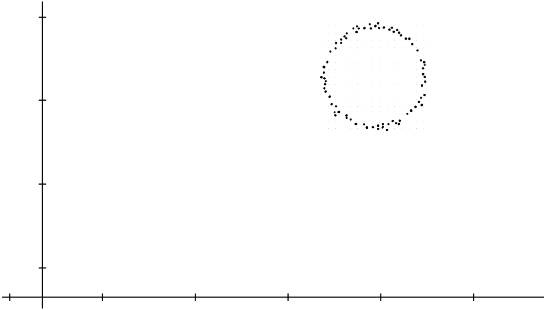


$$\hat{\mu}(S) = \mu(S) + \mathcal{N}\left(0, \left(\frac{\Delta\sqrt{\log 1/\delta}}{\varepsilon}\right)^2\right)$$

- extra error due to (ε, δ)-DP is

$$|\hat{\mu}(S) - \mu(S)| \simeq \frac{\Delta}{\varepsilon} = \frac{1}{n\,\varepsilon}$$

(ε, δ)-differentially private mean estimation*



*[Karwa,Vadhan,2017], [Kamath,Li,Singhal,Ullman,2019]

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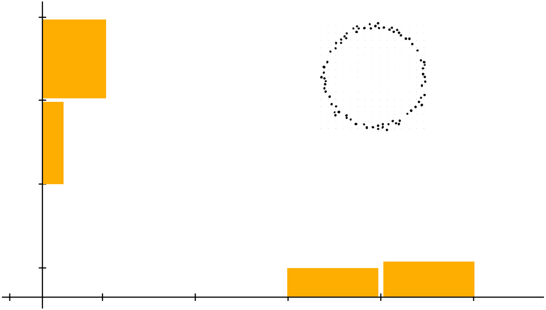

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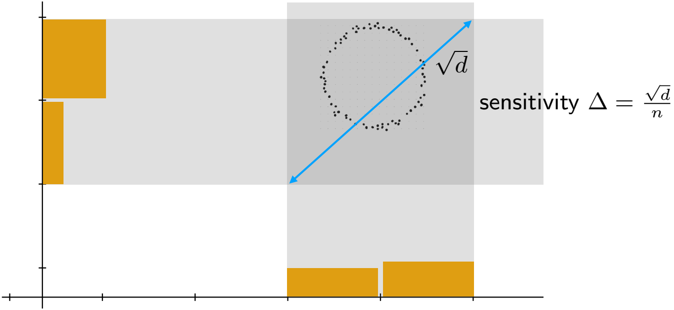

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α-corruption and (ε, δ)-DP		

Two main challenges in making filtering algorithms private

- (non-private) robust mean estimation [Diakonikolas et al.,2017]
- Repeat until $\|\mathrm{Cov}(S) - \mathbf{I}\| = O(\alpha \log(1/\alpha))$
  - $v \leftarrow \arg \max_{v:\|v\|=1} v^T \mathrm{Cov}(S) v$
  - $S \leftarrow$ 1D-Filter$(\{\langle v, x_i - \mu_{\mathrm{emp}}(S)\rangle^2\}_{i \in S})$

- First challenge:
  - in the worst case, the filter runs for $O(d)$ iterations
  - this happens if corrupted sample are spread out in orthogonal directions
  - because the filter only checks 1-dimensional subspace at a time
- This is particularly damaging for privacy, as more iterations mean more privacy leakage

Two main challenges in making filtering algorithms private

- (non-private) quantum robust mean estimation [Dong,Hopkins,Li,2019]
- Repeat until $\|\text{Cov}(S) - \mathbf{I}\| = O(\alpha \log(1/\alpha))$
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- If $\beta = \infty$, this recovers top PCA and uses only one-dimensional subspace
- If $\beta = 0$, this filters on $\|x_i - \mu_{\mathrm{emp}}(S)\|^2$ treating all directions equally
- For appropriate β, iterations reduce from $O(d)$ to $O((\log d)^2)$

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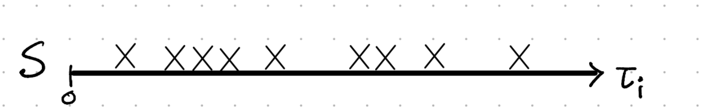
- Second challenge:
  - 1D-Filter has high sensitivity
  - each sample is independently filtered with probability proportional to $\tau_i \triangleq (x_i - \mu_{\mathrm{emp}}(S))^T V (x_i - \mu_{\mathrm{emp}}(S))$



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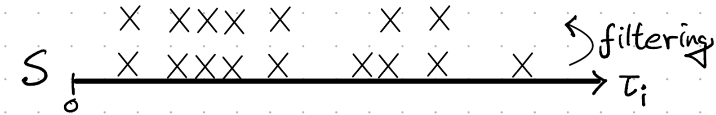

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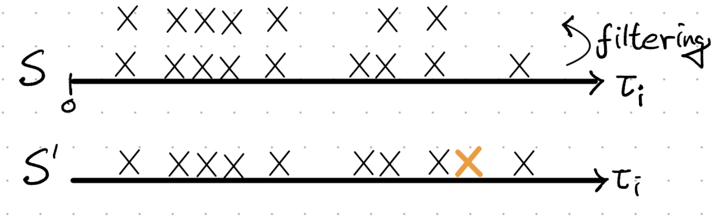

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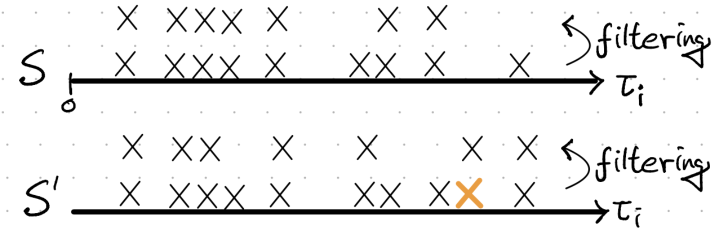
  - $S \leftarrow$ 1D-Filter$(\{(x_i - \mu_{\mathrm{emp}}(S))^T V (x_i - \mu_{\mathrm{emp}}(S))\}_{i \in S})$

- Second challenge:
  - 1D-Filter has high sensitivity
  - each sample is independently filtered with probability proportional to $\tau_i \triangleq (x_i - \mu_{\mathrm{emp}}(S))^T V (x_i - \mu_{\mathrm{emp}}(S))$



Two datasets lead to independent filtering, and sensitivity blows up

Two main challenges in making filtering algorithms private

- (non-private) quantum robust mean estimation [Dong,Hopkins,Li,2019]
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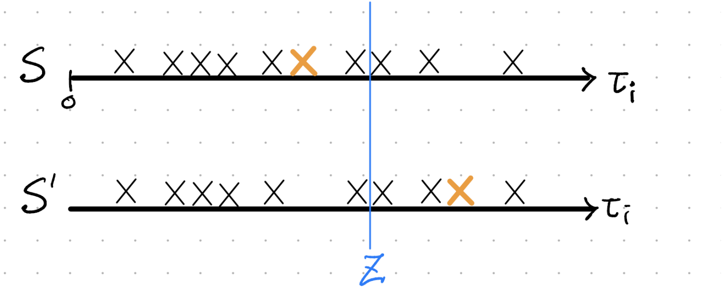
- Solution:
  - Use a single random threshold $Z \sim \mathrm{Uniform}[0, \rho]$, and filter samples above Z
  - this preserves the sensitivity to be one



After filtering, two sets differ only by one sample

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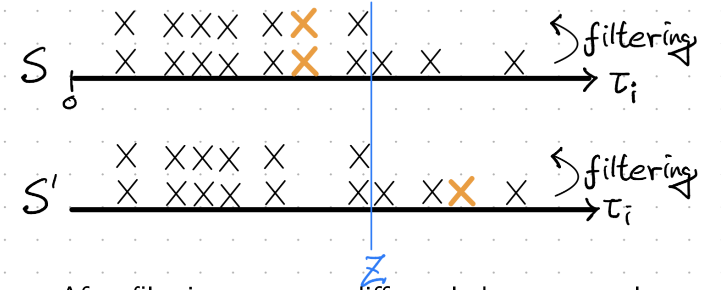

After filtering, two sets differ only by one sample

PRIME: Private and robust Mean Estimation [Liu,Kong,Kakade,O.,2021]

- Run private histogram to get a bounding box with side length $O(\sqrt{\log n})$
- Repeat until $\|\tilde{\Sigma} - \mathbf{I}\| = O(\alpha \log(1/\alpha))$
  - $\tilde{\mu} \leftarrow \mu_{\text{emp}}(S) + \mathcal{N}\left(0, \left(\frac{d^{1/2}\sqrt{\log(1/\delta)}}{n\varepsilon}\right)^2 \mathbf{I}_{d\times d}\right)$
  - $\tilde{\Sigma} \leftarrow \text{Cov}(S) + \mathcal{N}\left(0, \left(\frac{d\sqrt{\log(1/\delta)}}{n\varepsilon}\right)^2 \mathbf{I}_{d^2\times d^2}\right)$
  - $V \leftarrow \frac{1}{\text{Trace}(\exp\{\beta\tilde{\Sigma}\})} \exp\{\beta\tilde{\Sigma}\}$
  - $\rho \leftarrow$ DP-threshold$(\{(x_i - \tilde{\mu})^T V (x_i - \tilde{\mu})\}_{i\in S})$
  - $Z \leftarrow \text{Uniform}[0, \rho]$
  - $S \leftarrow$ 1D-Filter$(\{(x_i - \tilde{\mu})^T V (x_i - \tilde{\mu})\}_{i\in S}, Z)$

Mean estimation under sub-Gaussian distributions with identity covariance

	Error $\|\hat{\mu} - \mu\|$	
no corruption or privacy	$\sqrt{\frac{d}{n}}$	
α-corruption	$\sqrt{\frac{d}{n}} + \alpha$	[Diakonikolas et al.,2017]
(ε, δ)-DP	$\sqrt{\frac{d}{n}} + \frac{d}{\varepsilon\, n}$	[KamathLiSinghalUllman.,2019]
α-corruption and (ε, δ)-DP	$\sqrt{\frac{d}{n}} + \alpha + \frac{d^{3/2}}{\varepsilon\, n}$ (SVD-time)	[LiuKongKakadeO.,2021]

There is a $d^{1/2}$ gap between PRIME and lower bound!

Where does $\frac{d^{1.5}}{\varepsilon n}$ come from?

- Sample complexity bottleneck: we need to compute

$$V \;\leftarrow\; \frac{1}{Z}\exp\{\beta\, \mathsf{Cov}(S)\}$$

  privately, at least once
- Best known algorithm adds i.i.d. entry Gaussian matrix $W \in \mathbb{R}^{d\times d}$ with $\mathcal{N}(0, (\frac{d\sqrt{\log 1/\delta}}{\varepsilon n})^2)$ to the covariance matrix
- The spectral norm perturbation is $\|W\|_{\text{spectral}} = O(\frac{d^{1.5}}{\varepsilon n})$

Minimax optimal mean estimation

	Error $\|\hat{\mu} - \mu\|$
no corruption or privacy	$\sqrt{\frac{d}{n}}$
α-corruption	$\sqrt{\frac{d}{n}} + \alpha$ [Diakonikolas et al.,2017]
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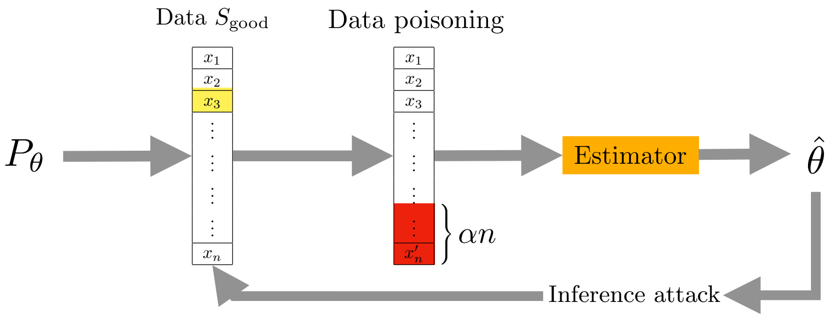
There is no extra *statistical* cost in requiring robustness and privacy simultaneously.

High-dimensional Propose-Test-Release



What is the fundamental connection between robust estimators and DP estimators?

High-dimensional Propose-Test-Release

- General framework for solving (inefficiently) statistical estimation problems with (ε, δ)-DP guarantee
- as a byproduct, we get robustness against α-corruption for free
- gives optimal sample complexity for mean estimation, covariance estimation, linear regression, and principal component analysis

HPTR step 1: design the score function

- Problem instance:
  mean estimation with i.i.d. samples from a sub-Gaussian distribution with mean μ and covariance Σ with error metric
  $$\|\Sigma^{-1/2}(\hat{\mu} - \mu)\|$$

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- Problem instance:
  mean estimation with i.i.d. samples from a sub-Gaussian distribution with mean μ and covariance Σ with error metric

$$\|\Sigma^{-1/2}(\hat{\mu} - \mu)\|$$

- Efficient algorithm [Kamath,Li,Singhal,Ullman,2019]:
  if $\mathbf{I} \preceq \Sigma \preceq \kappa \mathbf{I}$ and $n \geq d^{3/2}\sqrt{\log \kappa}/\varepsilon$

$$\|\Sigma^{-1/2}(\hat{\mu} - \mu)\| \;\leq\; \sqrt{\frac{d}{n}} + \frac{d}{\varepsilon n}$$

- Exponential-time [Brown,Gaboardi,Smith,Ullman,Zakynthinou,2021]:

$$\|\Sigma^{-1/2}(\hat{\mu} - \mu)\| \;\leq\; \sqrt{\frac{d}{n}} + \frac{d}{\varepsilon^2 n}$$

- Lower bound [Barber,Duchi,2014]:

$$\min_{\hat{\mu} \in \mathcal{F}_{\varepsilon,\delta}} \; \max_{P_{\mu,\Sigma}} \; \mathbb{E}\big[\, \|\Sigma^{-1/2}(\hat{\mu} - \mu)\| \,\big] \;\geq\; \sqrt{\frac{d}{n}} + \frac{d}{\varepsilon n}$$

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- Problem instance:
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$$\|\Sigma^{-1/2}(\hat{\mu} - \mu)\|$$

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- Problem instance:
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$$\begin{aligned}
\|\Sigma^{-1/2}(\hat{\mu} - \mu)\| &= \max_{\|v\|=1} v^T \Sigma^{-1/2}(\hat{\mu} - \mu) \\
&= \max_{\|v\|=1} \frac{v^T \hat{\mu} - \overbrace{v^T \mu}^{\mu_v}}{\underbrace{\sqrt{v^T \Sigma v}}_{\sigma_v}}
\end{aligned}$$

HPTR step 1: design the score function

- Problem instance:
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$$
\begin{aligned}
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&= \max_{\|v\|=1} \frac{v^T\hat{\mu} - \overbrace{v^T\mu}^{\mu_v}}{\underbrace{\sqrt{v^T\Sigma v}}_{\sigma_v}}
\end{aligned}
$$

- Design empirical loss function:

$$
D_S(\hat{\mu}) = \max_{\|v\|=1} \frac{v^T\hat{\mu} - \mu_v^{\text{robust}}}{\sigma_v^{\text{robust}}}
$$

HPTR step 2: sensitivity analysis

We want to minimize the loss function:

$$D_S(\hat{\mu}) \;=\; \max_{\|v\|=1} \frac{v^T\hat{\mu} - \mu_v^{\text{robust}}}{\sigma_v^{\text{robust}}}$$

- To stochastically minimize this robust empirical loss, we want to sample from (exponential mechanism*)

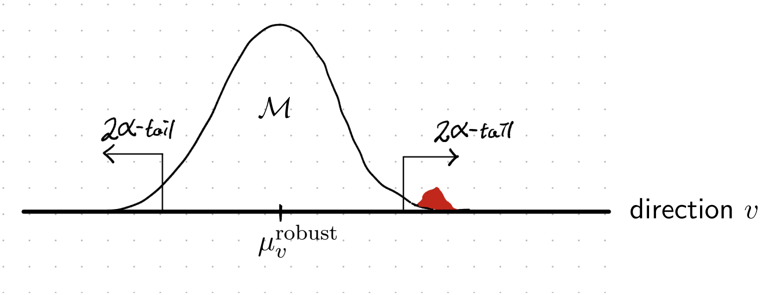
$$\hat{\mu} \;\sim\; \frac{1}{Z}\exp\Big\{-\frac{\varepsilon}{2\Delta}D_S(\hat{\mu})\Big\}$$

- If Δ is the sensitivity, then this is $(\varepsilon, 0)$-differentially private
- **The sensitivity of $D_S(\hat{\mu})$ dramatically reduces if we use 1-d robust statistics**
- Key ingredient is resilience property

*[McSherry,Talwar,2007]

HPTR step 2: sensitivity analysis

- $\mu_v^{\text{robust}} = \frac{1}{|\mathcal{M}|} \sum_{\mathcal{M}} v^T x_i$ has sensitivity $\Delta = \frac{\sigma_v \sqrt{\log(1/\alpha)}}{n}$



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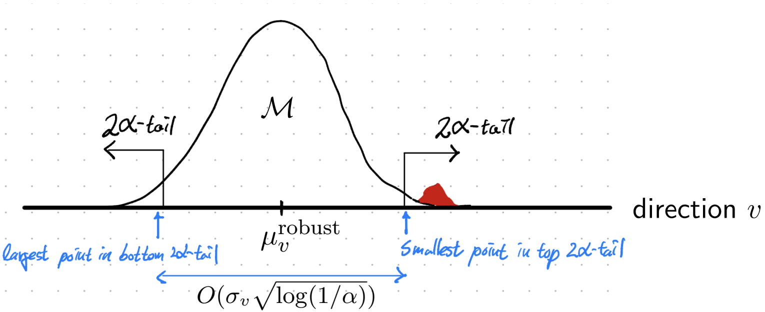

Resilience property of sub-Gaussian samples [Steinhardt,Charikar,Valiant,2018]

Given n i.i.d. sub-Gaussian samples S with $n \geq d/\alpha^2$, for all $S' \subset S$ of size at least αn,

$$\left| v^T(\mu(S) - \mu(S')) \right| \leq \sigma_v\sqrt{\log(1/\alpha)} .$$

High-dimensional Propose-Test-Release*

- HPTR(S)

Propose : Propose $\Delta = O(1/n)$ based on the resilience of the distribution

Test : Privately test the sensitivity for all neighboring dataset S'

Release : If S passes the test, release $\hat{\mu}$ sampled from

$$\hat{\mu} \quad \sim \quad \frac{1}{Z}\exp\Big\{ -\frac{\varepsilon}{2\Delta}D_S(\hat{\mu})\Big\}$$

*inspired by original PTR [Dwork,Lei,2009] and a more advanced PTR [Brown,Gaboardi,Smith,Ullman,Zakynthinou,2021]

Generality of HPTR

- HPTR can be applied to any statistical estimation problem to achieve the optimal sample complexity
  - sub-Gaussian mean estimation
  - k-th moment bounded mean estimation
  - sub-Gaussian linear regression
  - Gaussian covariance estimation
  - sub-Gaussian principal component analysis
- and other cases achieve the state-of-the-art sample complexity, but no matching lower bounds yet
  - k-th moment bounded linear regression
  - k-th moment bounded PCA

Minimax error rate for mean estimation under sub-Gaussian distributions with identity covariance

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There is no extra *statistical* cost in requiring robustness and privacy simultaneously.

Open questions

- New directions at the intersection of robustness and privacy
  - Mean (sub-Gaussian/Covariance bounded) [Liu,Kong,Kakade,O.2021]
  - Covariance (Gaussian)
  - Mean (bounded k-th moment)
  - Principal Component Analysis
  - Linear regression
  - Convex optimization

- Different settings
  - User-level robustness and privacy
  - Discrete distributions

Conclusion

- We characterize the minimax error rate of a fundamental statistical task of mean estimation under adversarial corruption and differential privacy, and show its optimality

$$\|\hat{\mu} - \mu\| \simeq \sqrt{\frac{d}{n}} + \alpha + \frac{d}{\varepsilon\, n}$$

- We give the first efficient algorithm that achieves

$$\|\hat{\mu} - \mu\| \leq \sqrt{\frac{d}{n}} + \alpha + \frac{d^{1.5}}{\varepsilon\, n}$$

- arXiv:2102.09159 Xiyang Liu, Weihao Kong, Sham Kakade, Sewoong Oh "Robust and Differentially Private Mean Estimation"
- working paper, Xiyang Liu, Weihao Kong, Sewoong Oh "Differential Privacy and Robust Statistics in High Dimensions"
- arXiv:2104.11315 Jonathan Hayase, Weihao Kong, Raghav Somani, S. Oh "SPECTRE: Defending Against Backdoor Attacks Using Robust Covariance Estimation"