Differential Privacy Meets Robust Statistics

Sewoong Oh

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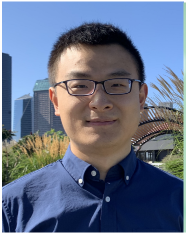
joint work with

Xiyang Liu

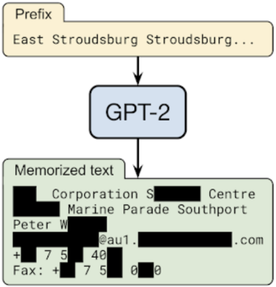


Weihao Kong

Sham Kakade

What can go wrong when training on shared data?

- Increasingly more models are being trained on shared data
- Sensitive information should not be revealed by the trained model
- Membership inference attacks can identify individual's sensitive data used in the training



[*]

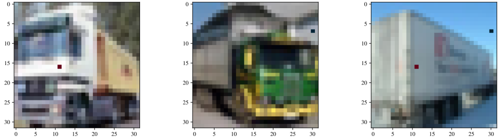
- Potential defense: Differentially Private Stochastic Gradient Descent[†] when computing the average of the gradients in the mini-batch, use differentially private mean estimation

[*] [Carlini et al.,2020]

[†] [Chaudhuri,Monteleoni,Sarwate,2011], [Abadi et al.,2016]

What can go wrong when training on shared data?

- When training on shared data, not all participants are trusted
- Malicious users can easily inject corrupted data
- Data poisoning attacks can create backdoors on the trained model such that any sample with the trigger will be predicts as ‘deer’



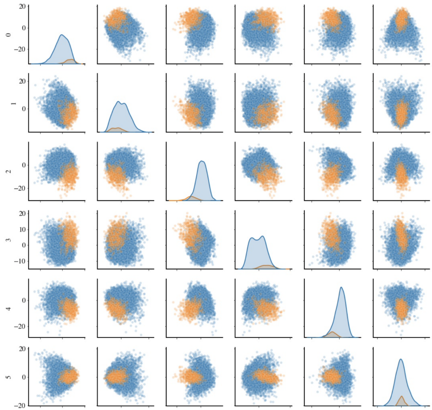
$y_i =$ ‘deer’

- Strong defense: Robust estimation*
- Insight: successful backdoor attacks leave a path of activations in the trained model that are triggered only by the corrupted samples

*[Hayase,Kong,Somani,O.,2021,ICML] inspired by [Tran,Li,Madry,2018]

Middle layer of a model trained with corrupted data

- All samples with label 'deer': CLEAN and POISONED
- Top-6 PCA projection of node activations at a middle layer
- Can we separate POISONED from CLEAN?



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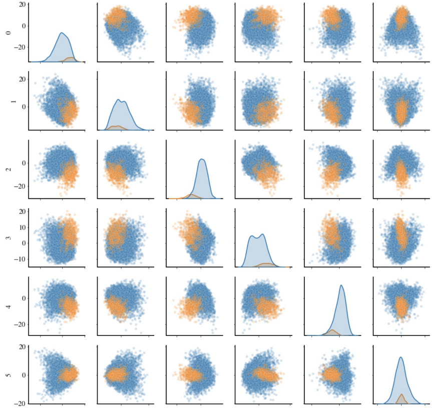
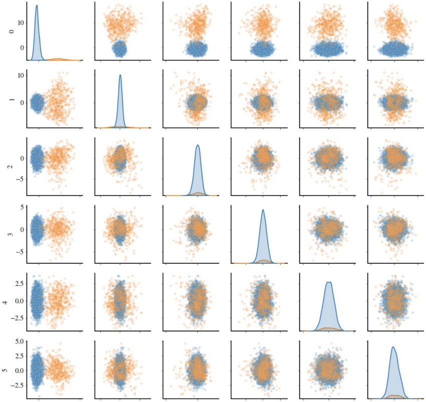


After whitening with
the covariance of CLEAN



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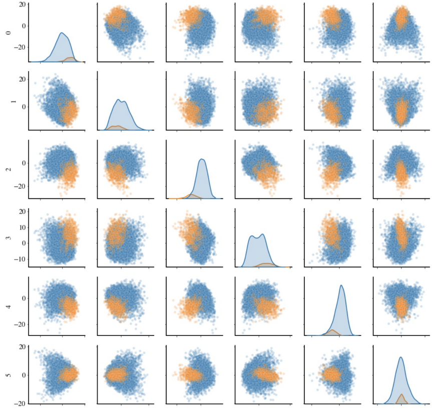
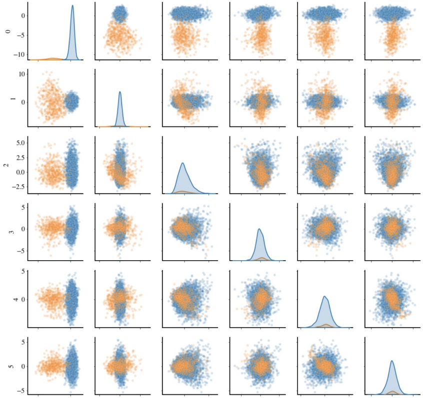
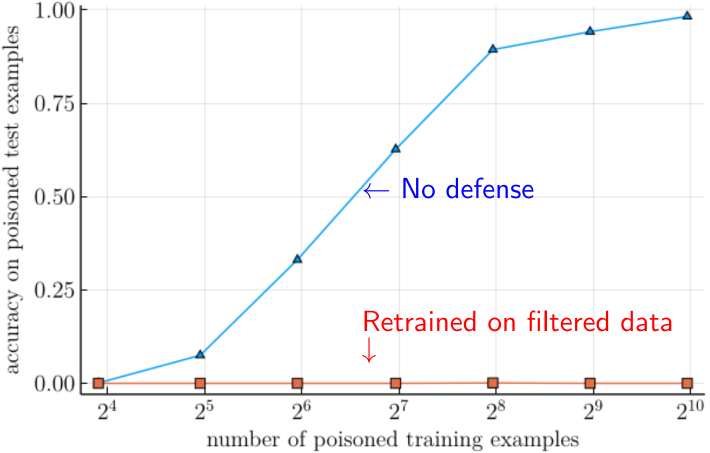


After whitening with
estimated **robust** mean and covariance



SPECTRE: Defense against backdoor attacks

[Hayase,Somani,Kong,O.,2021,ICML]‡



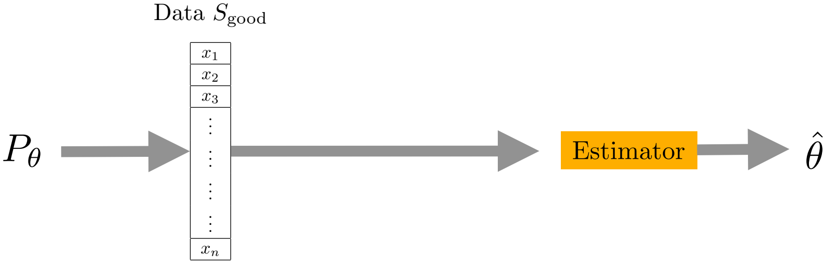
‡ `https://github.com/SewoongLab/backdoor-suite`

We need privacy and robustness, simultaneously

- When learning from shared data
  - Differential privacy is crucial in defending against inference attacks
  - Robust estimation is crucial in defending against data poisoning attacks
- Critical components are mean/covariance estimation
  - DP-SGD relies on DP mean estimation
  - Backdoor defense relies on robust mean/covariance estimation
- We provide the first efficient estimators that are provably differentially private and robust against data corruption

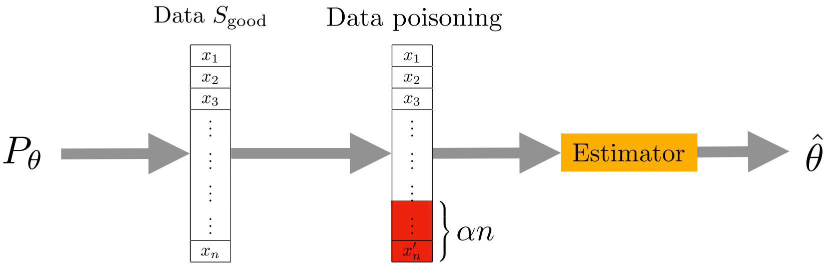
Statistical estimation, robustly and privately

- Statistics



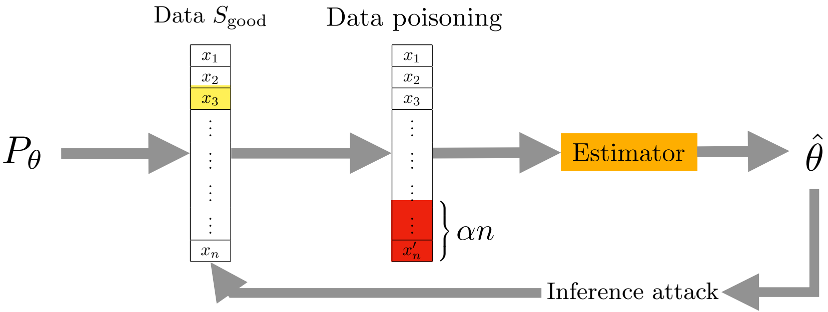
Statistical estimation, robustly and privately

- Statistics$\Rightarrow$ Robust estimation



Statistical estimation, robustly and privately

- Statistics$\Rightarrow$ Robust estimation$\Rightarrow$ Robust and private estimation



- This talk focuses on mean estimation
- Q. What is the extra cost (in the estimation error) we pay for {Robustness, Privacy, and Robustness+Privacy}

Mean estimation

- Estimate the mean μ from n i.i.d. samples
- For this talk,
  we assume sub-Gaussian distribution with identity covariance matrix
- Minimax error rate:

$$\min_{\hat{\mu}\in\mathcal{F}_{S_n}} \max_{P_\mu} \mathbb{E}\left[\, \|\hat{\mu}(S_n) - \mu\| \,\right] \propto \sqrt{\frac{d}{n}}$$

  $\mathcal{F}_{S_n}$ is set of all estimators over n i.i.d. samples in $\mathbb{R}^d$ from P_μ,
  P_μ is maximized over all sub-Gaussian distributions with identity covariance
- In this talk, I will ignore all constant and logarithmic factors

Robust mean estimation

- Threat model
  - Adversarial corruption model:
    $\{x_i\}_{i=1}^n \sim P_\mu$ is drawn, then adversary replaces α-fraction arbitrarily

- Robust mean estimation:
  - Low dimensional:
    [Tukey,1960] [Huber,1964]
  - Computationally intractable methods in high dimension:
    [Donoho,Liu,1988], [ChenGaoRen,2015],[Zhu,Jiao,Steinhardt,2019]
  - Breakthroughs in polynomial time algorithms:
    [Lai,Rao,Vempala,2016],[Diakonikolas,Kamath,Kane,Li,Moitra,Stewart,2019]
  - Linear time algorithms:
    [Cheng,Dianikolas,Ge,2019], [Depersin,Lecué,2019],[Dong,Hopkins,Li,2019]

Robust mean estimation

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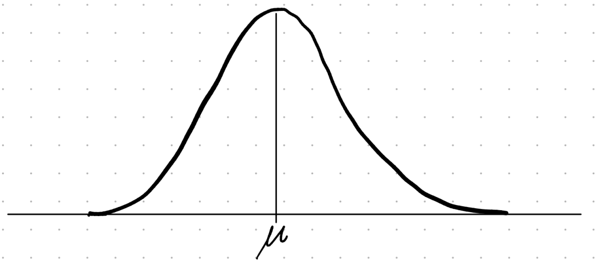
- Relatively easy to estimate mean robustly in low-dimensions

histogram of sub-Gaussian samples in 1D



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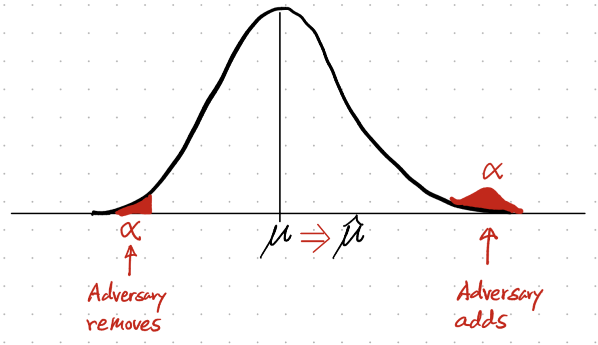

simple outlier detection achieves $|\hat{\mu} - \mu| \leq \alpha$

Robust mean estimation

- Threat model
  - Adversarial corruption model:
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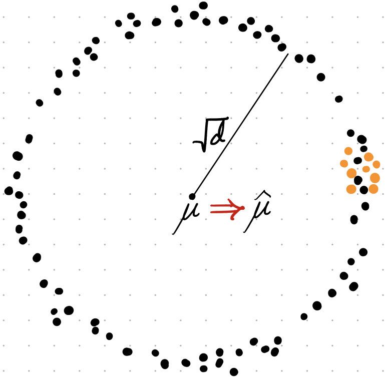
- Mean estimation becomes challenging in high-dimensions

scatter plot of sub-Gaussian samples in high-dimension



each corrupted sample looks uncorrupted and still $\|\hat{\mu} - \mu\| \geq \alpha\sqrt{d}$

Efficient algorithm: Filtering [Diakonikolas et al.,2017]

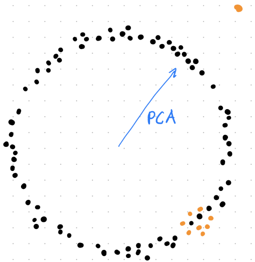
Geometric Lemma [Dong,Hopkins,Li,2019]

Given n i.i.d. samples from a sub-Gaussian distribution with identity covariance matrix, if at most αn samples are corrupted, then, w.h.p.

$$\|\mu_{\mathrm{emp}}(S) - \mu\| \ \leq \ \sqrt{\frac{d}{n}} + \alpha + \sqrt{\alpha \|\mathrm{Cov}(S) - \mathbf{I}\|}$$

- While $\|\mathrm{Cov}(S) - \mathbf{I}\| > c\,\alpha$
  - $v \leftarrow \arg\max_{v:\|v\|=1} v^T \mathrm{Cov}(S) v$
  - $S \leftarrow$ 1D-Filter$(\{\langle v, x_i - \mu_{\mathrm{emp}}(S)\rangle^2\}_{i\in S})$

- Each step guarantees that
  - at least one sample is removed
  - more corrupted samples removed than clean samples in expectation



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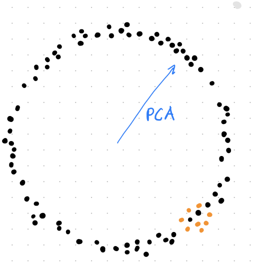

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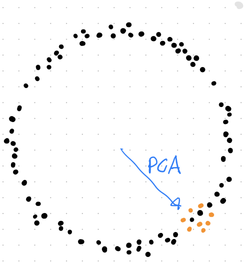

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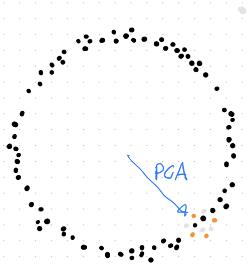

Robust mean estimation

- Minimax error rate under α-corruption

$$\min_{\hat{\mu}} \max_{P_\mu} \mathbb{E}\big[\,\|\hat{\mu}(S_{n,\alpha}) - \mu\|\,\big] \;\propto\; \underbrace{\sqrt{\frac{d}{n}}}_{\text{no corruption}} + \underbrace{\alpha}_{\alpha\text{-corruption}}$$

achieved by filtering algorithm of [Diakonikolas et al.,2017]
information-theoretic lower bound from [Chen,Gao,Ren,2015]

Minimax error rate for mean estimation under sub-Gaussian distributions with identity covariance

	Error $\|\hat{\mu} - \mu\|$
no corruption or privacy	$\sqrt{\frac{d}{n}}$
α-corruption	$\sqrt{\frac{d}{n}} + \alpha$ [Diakonikolas et al.,2017]
(ε, δ)-DP	
α-corruption and (ε, δ)-DP	

Differential Privacy provably ensures plausible deniability

- Goal: a strong adversary who knows all the other entries in the database except for yours, should not be able to identify whether you participated in that database or not

- Definition*: For two databases S and S' that differ by only one entry, a randomized output to a query is (ε, δ)-differentially private if

$$\mathbb{P}(\text{query\_output}(S) \in A) \;\leq\; e^{\varepsilon}\, \mathbb{P}(\text{query\_output}(S') \in A) + \delta$$

- smaller $\varepsilon, \delta \;\Rightarrow\;$ Testing S or S' fails $\;\Rightarrow\;$ inference attack fails

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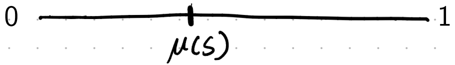
*[Dwork,McSherry,Nissim,Smith,2006]

(ε, δ)-differentially private mean estimation

S
0
1
0
0
0
1
$\vdots$
0



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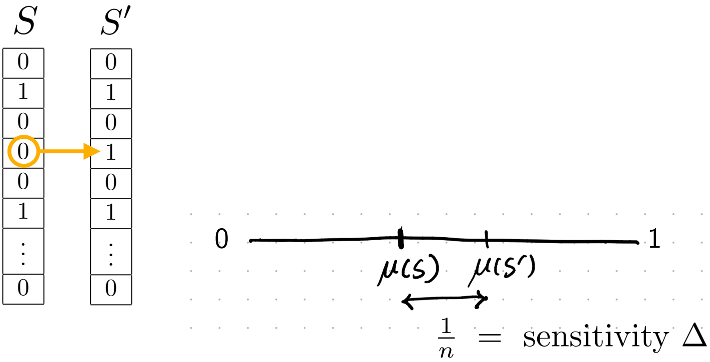

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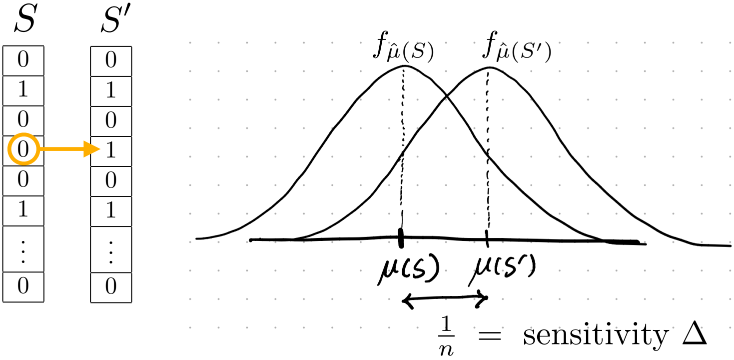
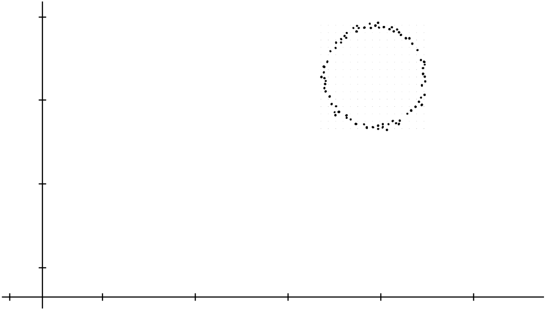


$$\hat{\mu}(S) = \mu(S) + \mathcal{N}\left(0, \left(\frac{\Delta\sqrt{\log 1/\delta}}{\varepsilon}\right)^2\right)$$

- extra error due to (ε, δ)-DP is

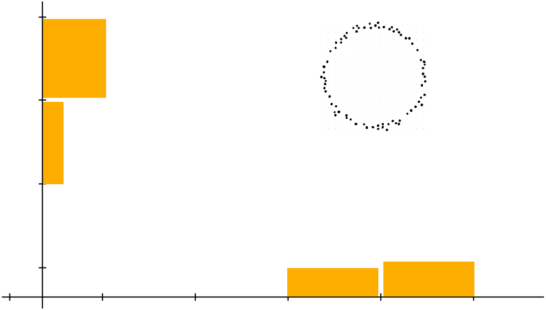
$$|\hat{\mu}(S) - \mu(S)| \simeq \frac{\Delta}{\varepsilon} = \frac{1}{n\,\varepsilon}$$

(ε, δ)-differentially private mean estimation*



*[Karwa,Vadhan,2017], [Kamath,Li,Singhal,Ullman,2019]

(ε, δ)-differentially private mean estimation*



- step 1. privately find a bounding hypercube

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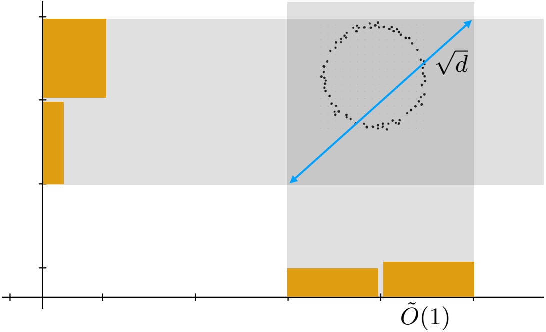

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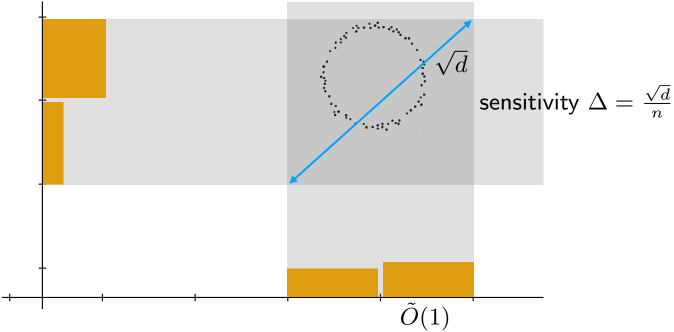

- step 1. privately find a bounding hypercube
- step 2. add Gaussian noise: $\hat{\mu}(S) = \mu(S) + \mathcal{N}\left(0, \left(\frac{\Delta\sqrt{\log 1/\delta}}{\varepsilon}\right)^2 \mathbf{I}_{d\times d}\right)$
- extra error due to (ε, δ)-DP is

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α-corruption	$\sqrt{\frac{d}{n}} + \alpha$	[Diakonikolas et al.,2017]
(ε, δ)-DP	$\sqrt{\frac{d}{n}} + \frac{d}{\varepsilon\, n}$	[Kamath,Li,Singhal,Ullman,2019]
α-corruption and (ε, δ)-DP		

Two main challenges in making filtering algorithms private

Algorithm (non-private) robust mean estimation [Diakonikolas et al.,2017]

1: **while** $\|\mathrm{Cov}(S) - \mathbf{I}\| > c\,\alpha$ **do**

2: $\quad v \leftarrow \arg\max_{v:\|v\|=1} v^T \mathrm{Cov}(S) v$

3: $\quad S \leftarrow$ 1D-Filter$(\{\langle v, x_i - \mu_{\mathrm{emp}}(S)\rangle^2\}_{i\in S})$

- First challenge:
  - in the worst case, the filter runs for $O(d)$ iterations
  - this happens if corrupted sample are spread out in orthogonal directions
  - because the filter only checks 1-dimensional subspace at a time
- This is particularly damaging for privacy, as more iterations mean more privacy leakage

Two main challenges in making filtering algorithms private

Algorithm Quantum robust mean estimation [Dong,Hopkins,Li,2019]

1: **while** $\|\mathrm{Cov}(S) - \mathbf{I}\| > c\,\alpha$ **do**

2: $\quad V \leftarrow \frac{1}{\mathrm{Trace}(\exp\{\beta \mathrm{Cov}(S)\})} \exp\{\beta \mathrm{Cov}(S)\}$

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- If $\beta = \infty$, this recovers top PCA and uses only one-dimensional subspace
- If $\beta = 0$, this filters on $\|x_i - \mu_{\mathrm{emp}}(S)\|^2$ treating all directions equally
- For appropriate β, iterations reduce from $O(d)$ to $O((\log d)^2)$

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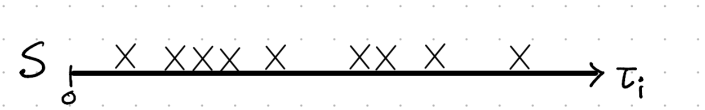
- Second challenge:
  - 1D-Filter has high sensitivity
  - each sample is independently filtered with probability proportional to $\tau_i \triangleq (x_i - \mu_{\mathrm{emp}}(S))^T V (x_i - \mu_{\mathrm{emp}}(S))$



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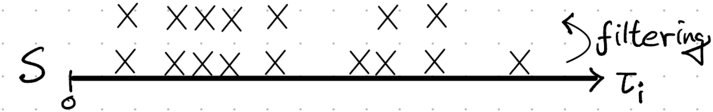

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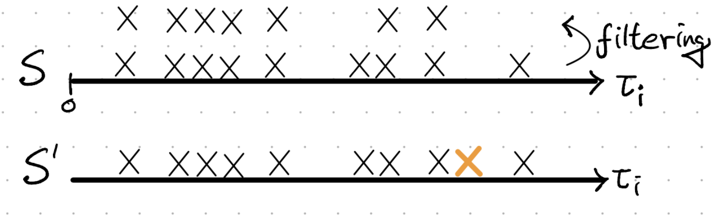

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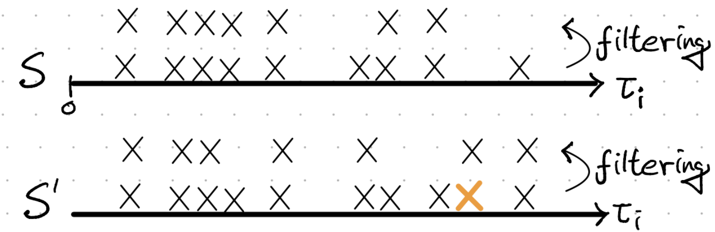

Two datasets lead to independent filtering, and sensitivity blows up

Two main challenges in making filtering algorithms private

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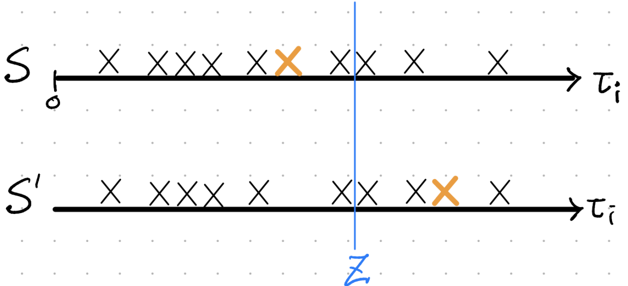
$\quad V \leftarrow \frac{1}{\mathrm{Trace}(\exp\{\beta \mathrm{Cov}(S)\})} \exp\{\beta \mathrm{Cov}(S)\}$

$\quad S \leftarrow$ 1D-Filter$(\{(x_i - \mu_{\mathrm{emp}}(S))^T V (x_i - \mu_{\mathrm{emp}}(S))\}_{i \in S})$

- Solution:
  - Use a single random threshold $Z \sim \mathrm{Uniform}[0, \rho]$, and filter samples above Z
  - this preserves the sensitivity to be one



Two main challenges in making filtering algorithms private

Algorithm Quantum robust mean estimation [Dong,Hopkins,Li,2019]

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2: $\quad V \leftarrow \frac{1}{\mathrm{Trace}(\exp\{\beta \mathrm{Cov}(S)\})} \exp\{\beta \mathrm{Cov}(S)\}$
3: $\quad S \leftarrow$ 1D-Filter$(\{(x_i - \mu_{\mathrm{emp}}(S))^T V (x_i - \mu_{\mathrm{emp}}(S))\}_{i \in S})$

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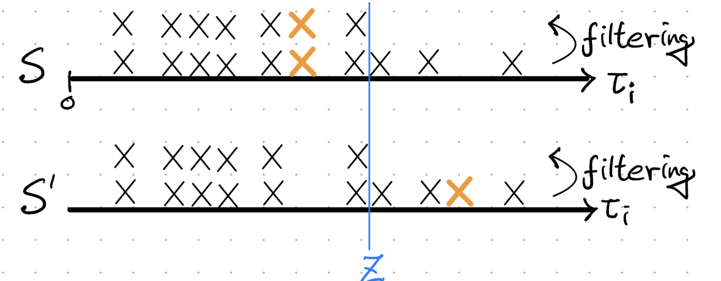


PRIME: PRIvate and robust Mean Estimation

- Run private histogram to get a bounding hypercube
- While $\|\tilde{\Sigma} - \mathbf{I}\| > c\,\alpha$
  - $\tilde{\mu} \leftarrow \mu_{\text{emp}}(S) + \mathcal{N}\left(0, \left(\frac{d^{1/2}\sqrt{\log(1/\delta)}}{n\varepsilon}\right)^2 \mathbf{I}_{d\times d}\right)$
  - $\tilde{\Sigma} \leftarrow \text{Cov}(S) + \mathcal{N}\left(0, \left(\frac{d\sqrt{\log(1/\delta)}}{n\varepsilon}\right)^2 \mathbf{I}_{d^2\times d^2}\right)$
  - $V \leftarrow \frac{1}{\text{Trace}(\exp\{\beta\tilde{\Sigma}\})} \exp\{\beta\tilde{\Sigma}\}$
  - $\rho \leftarrow$ DP-threshold$(\{(x_i - \tilde{\mu})^T V (x_i - \tilde{\mu})\}_{i\in S})$
  - $Z \leftarrow \text{Uniform}[0, \rho]$
  - $S \leftarrow$ 1D-Filter$(\{(x_i - \tilde{\mu})^T V (x_i - \tilde{\mu})\}_{i\in S}, Z)$

Theorem. [Liu,Kong,Kakade,O.,2021,NeurIPS]

PRIME is (ε, δ)-differentially private. For an α-corruption of n i.i.d. samples from a sub-Gaussian distribution with identity covariance matrix, with high probability

$$\|\hat{\mu} - \mu\| \;\lesssim\; \sqrt{\frac{d}{n}} + \alpha + \frac{d^{3/2}}{\varepsilon\, n}.$$

Mean estimation under sub-Gaussian distributions with identity covariance

	Error $\|\hat{\mu} - \mu\|$	
no corruption or privacy	$\sqrt{\frac{d}{n}}$	
α-corruption	$\sqrt{\frac{d}{n}} + \alpha$	[Diakonikolas et al.,2017]
(ε, δ)-DP	$\sqrt{\frac{d}{n}} + \frac{d}{\varepsilon\, n}$	[KamathLiSinghalUllman.,2019]
α-corruption and (ε, δ)-DP	$\sqrt{\frac{d}{n}} + \alpha + \frac{d^{3/2}}{\varepsilon\, n}$ (SVD time)	[LiuKongKakadeO.,2021]

There is a $d^{1/2}$ gap between PRIME and lower bound!

Where does $\frac{d^{3/2}}{\varepsilon n}$ come from?

- Sample complexity bottleneck: we need to privately compute

$$\tilde{\Sigma} \leftarrow \mathsf{Cov}(S) + W$$

- Best known algorithm adds i.i.d. entry Gaussian matrix $W \in \mathbb{R}^{d \times d}$ with $\mathcal{N}(0, (\frac{d\sqrt{\log 1/\delta}}{\varepsilon n})^2)$ to the covariance matrix
- The spectral norm perturbation is $\|W\|_{\text{spectral}} = O(\frac{d^{3/2}}{\varepsilon n})$

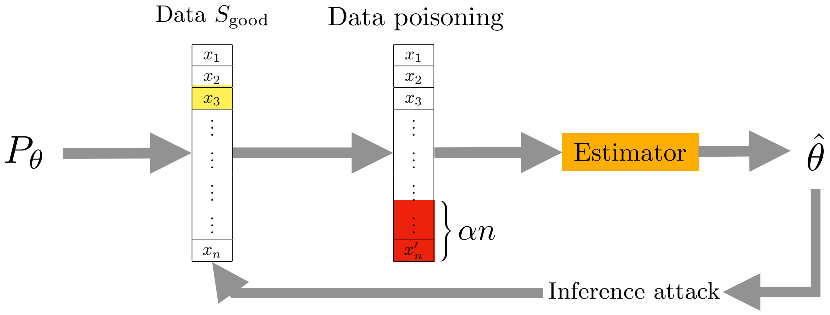
- In general, this cannot be improved as it matches a known lower bound [Dwork,Talwar,Thakurta,Zhang,2014]

Minimax optimal mean estimation

	Error $\|\hat{\mu} - \mu\|$	
no corruption or privacy	$\sqrt{\frac{d}{n}}$	
α-corruption	$\sqrt{\frac{d}{n}} + \alpha$	[Diakonikolas et al.,2017]
(ε, δ)-DP	$\sqrt{\frac{d}{n}} + \frac{d}{\varepsilon\, n}$	[KamathLiSinghalUllman.,2019]
α-corruption and (ε, δ)-DP	$\sqrt{\frac{d}{n}} + \alpha + \frac{d^{3/2}}{\varepsilon\, n}$ (SVD time) $\sqrt{\frac{d}{n}} + \alpha + \frac{d}{\varepsilon\, n}$ (exponential time)	[LiuKongKakadeO.,2021]

There is no extra *statistical* cost in requiring robustness and privacy simultaneously.

High-dimensional Propose-Test-Release



What is the fundamental connection between robust estimators and DP estimators?

High-dimensional Propose-Test-Release

- General framework for solving (inefficiently) statistical estimation problems with (ε, δ)-DP guarantee
- as a byproduct, we get robustness against α-corruption for free
- gives optimal sample complexity for mean estimation, covariance estimation, linear regression, and principal component analysis

HPTR step 1: design the score function

- Problem instance:
  mean estimation with i.i.d. samples from a sub-Gaussian distribution with mean μ and covariance Σ with error metric
  $$\|\Sigma^{-1/2}(\hat{\mu} - \mu)\|$$

HPTR step 1: design the score function

- Problem instance:
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$$\|\Sigma^{-1/2}(\hat{\mu} - \mu)\|$$

- Polynomial-time [Kamath,Mouzakis,Singhal,Steinke,Ullman,2021]:
  if $n \geq d^{5/2}/\varepsilon$

$$\|\Sigma^{-1/2}(\hat{\mu} - \mu)\| \ \leq \ \sqrt{\frac{d}{n}} + \frac{d}{\varepsilon n}$$

- Exponential-time [Brown,Gaboardi,Smith,Ullman,Zakynthinou,2021]:

$$\|\Sigma^{-1/2}(\hat{\mu} - \mu)\| \ \leq \ \sqrt{\frac{d}{n}} + \frac{d}{\varepsilon^2 n}$$

- Lower bound [Barber,Duchi,2014]:

$$\min_{\hat{\mu} \in \mathcal{F}_{\varepsilon,\delta}} \ \max_{P_{\mu,\Sigma}} \mathbb{E}\big[\, \|\Sigma^{-1/2}(\hat{\mu} - \mu)\| \,\big] \ \geq \ \sqrt{\frac{d}{n}} + \frac{d}{\varepsilon n}$$

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HPTR step 1: design the score function

- Problem instance:
  mean estimation with i.i.d. samples from a sub-Gaussian distribution with mean μ and covariance Σ with error metric

$$
\begin{aligned}
\|\Sigma^{-1/2}(\hat{\mu}-\mu)\| &= \max_{\|v\|=1} v^T \Sigma^{-1/2}(\hat{\mu}-\mu) \\
&= \max_{\|v\|=1} \frac{v^T \hat{\mu} - \overbrace{v^T \mu}^{\mu_v}}{\underbrace{\sqrt{v^T \Sigma v}}_{\sigma_v}}
\end{aligned}
$$

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\end{aligned}
$$

- Design empirical loss function:

$$
D_S(\hat{\mu}) = \max_{\|v\|=1} \frac{v^T \hat{\mu} - \mu_v^{\text{robust}}}{\sigma_v^{\text{robust}}}
$$

HPTR step 2: sensitivity analysis

We want to minimize the loss function:

$$D_S(\hat{\mu}) \;=\; \max_{\|v\|=1} \frac{v^T \hat{\mu} - \mu_v^{\text{robust}}}{\sigma_v^{\text{robust}}}$$

- To stochastically minimize this robust empirical loss, we want to sample from (exponential mechanism*)

$$\hat{\mu} \;\sim\; \frac{1}{Z} \exp\Big\{ -\frac{\varepsilon}{2\Delta} D_S(\hat{\mu}) \Big\}$$

- If Δ is the sensitivity, then this is $(\varepsilon, 0)$-differentially private
- **The sensitivity of $D_S(\hat{\mu})$ dramatically reduces if we use 1-d robust statistics**
- Key ingredient is resilience property

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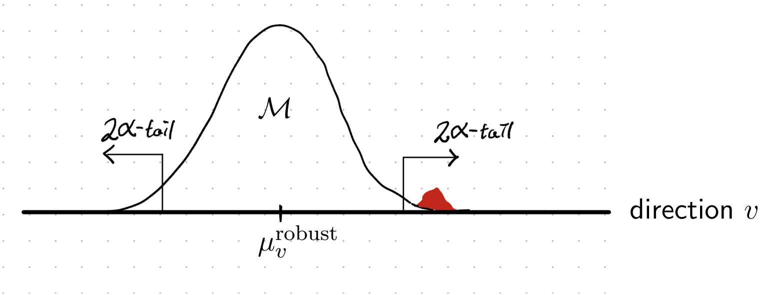
* [McSherry,Talwar,2007]

HPTR step 2: sensitivity analysis

- $\mu_v^{\text{robust}} = \frac{1}{|\mathcal{M}|}\sum_{\mathcal{M}} v^T x_i$ has sensitivity $\Delta = \frac{\sigma_v}{n}$



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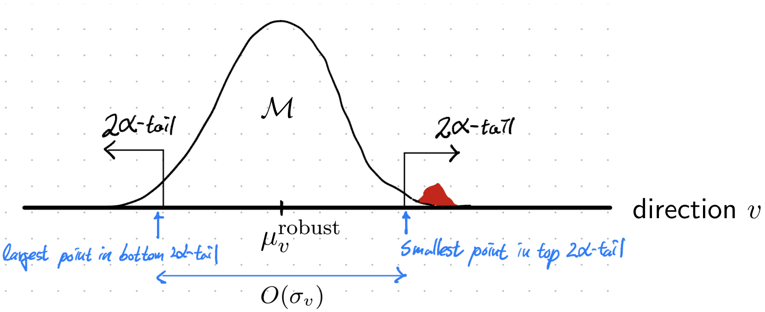

Resilience property for sub-Gaussian [Steinhardt,Charikar,Valiant,2018]

Given n i.i.d. sub-Gaussian samples S with $n \geq d/\alpha^2$, for all $S' \subset S$ of size at least αn,

$$\left| v^T(\mu(S) - \mu(S')) \right| \;\leq\; \sigma_v \;.$$

High-dimensional Propose-Test-Release*

- HPTR(S)

Propose : Propose $\Delta = O(1/n)$ based on the resilience of the distribution

Test : Privately test the sensitivity for all neighboring dataset S'

Release : If S passes the test, release $\hat{\mu}$ sampled from

$$\hat{\mu} \quad \sim \quad \frac{1}{Z} \exp\Big\{ -\frac{\varepsilon}{2\Delta} D_S(\hat{\mu}) \Big\}$$

*inspired by original PTR [Dwork,Lei,2009] and a more advanced PTR [Brown,Gaboardi,Smith,Ullman,Zakynthinou,2021]

Generality of HPTR

- HPTR can be applied to any statistical estimation problem to achieve the near-optimal error rate under (ε, δ)-DP
  - sub-Gaussian mean estimation:
  $$\|\Sigma^{-1/2}(\hat{\mu} - \mu)\| = O\Big(\sqrt{\frac{d}{n}} + \frac{d}{\varepsilon n}\Big)$$
  - k-th moment bounded mean estimation:
  $$\|\Sigma^{-1/2}(\hat{\mu} - \mu)\| = O\Big(\sqrt{\frac{d}{n}} + \Big(\frac{d}{\varepsilon n}\Big)^{1-\frac{1}{k}}\Big)$$
  - sub-Gaussian linear regression:
  $$\|\Sigma^{1/2}(\hat{\beta} - \beta)\| = O\Big(\sqrt{\frac{d}{n}} + \frac{d}{\varepsilon n}\Big)$$
  - Gaussian covariance estimation:
  $$\|\Sigma^{-1/2}\hat{\Sigma}\Sigma^{-1/2} - \mathbf{I}\|_F = O\Big(\sqrt{\frac{d^2}{n}} + \frac{d^2}{\varepsilon n}\Big)$$
  - sub-Gaussian principal component analysis:
  $$1 - \frac{\hat{v}^\top \Sigma \hat{v}}{\|\Sigma\|} = O\Big(\sqrt{\frac{d}{n}} + \frac{d}{\varepsilon n}\Big)$$

Conclusion and open questions

- First half of the talk, we gave the first efficient algorithm that achieves both differential privacy and robustness:

$$\|\hat{\mu} - \mu\| \;\le\; \sqrt{\frac{d}{n}} + \alpha + \frac{d^{1.5}}{\varepsilon\, n}$$

$$\|\Sigma^{-1/2}\hat{\Sigma}\Sigma^{-1/2} - \mathbf{I}\|_F \;\le\; \sqrt{\frac{d^2}{n}} + \alpha + \frac{d^3}{\varepsilon\, n}$$

  - Can we have an efficient algorithm that closes the $d^{1/2}$ gap (for mean)?
  - Can we use it to make DP-SGD robust?
  - Can we use it to make defense against backdoor attacks (such as SPECTRE) also private?
  - Can we design efficient algorithms for other problems:
    - Principal component analysis, linear regression, convex optimization

Conclusion and open questions

- Second half of the talk, we introduced HPTR that achieves optimal error rate on mean estimation, covariance estimation, linear reression, and PCA
  - Characterize fundamental tradeoffs in structured data (sparsity and low-rank)
  - Characterize fundamental tradeoffs in discrete or graph data

- arXiv:2102.09159, Xiyang Liu, Weihao Kong, Sham Kakade, Sewoong Oh "Robust and Differentially Private Mean Estimation"
- arxiv:2111.06578, Xiyang Liu, Weihao Kong, Sewoong Oh "Differential Privacy and Robust Statistics in High Dimensions"
- arXiv:2104.11315, Jonathan Hayase, Weihao Kong, Raghav Somani, S. Oh "SPECTRE: Defending Against Backdoor Attacks Using Robust Covariance Estimation"