Lecture 2: The Product Rule, Permutations and Combinations

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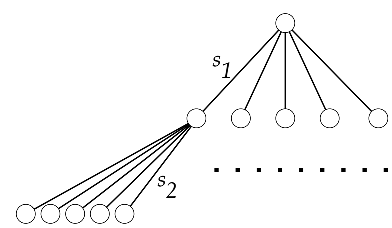
We begin our discussion of counting. Here we show how to use the factorial function and binomial coefficients to count.

Counting

Now, we turn our attention to counting the sizes of different kinds of sets. It is often useful to map the set we are interested in counting to another kind of object, such as a function or a sequence, to make it easier to count.

Example

Suppose S is a set that has n elements. How many elements does S^k have?



To count there elements in S^k we use the *product-rule*. One way to visualize this is to draw a rooted tree of depth k, where each internal node has n children that correspond to the elements of S. Each step in the tree corresponds to choosing an element of S, and the leaves of the tree correspond to elements of S^k. The number of leaves in the tree can be counted by noting that there are n choices for the first step, then n choices for the second step, and so on, so the total number of leaves is $n \times n \times \ldots = n^k$.

Example

Your brand new car comes with 5 possible color choices for the external color—red, blue, magenta, white and black. There are 3 choices for the internal trim—black leather, brown leather and vinyl. How many possible configurations can you car have?

This is a simple applicaiton of the product rule. There are 5 choices for the paint, and 3 choices for the trim, so there are a total of $5 \times 3 = 15$ configurations available.

Example

Consider the set $S = 2^{[n]}$ that consists of all subsets of $[n]$. How many elements does S have?

Here it is helpful to view the elements of S using their indicator vectors. Each element of S is a subset of $[n]$, so its indicator vector is the set of n-bit strings $\{0,1\}^n$. To count the number of n-bit strings, we again use the product rule: there are 2 options for the first coordinate, then 2 options for the second coordinate, and so on. So, the number of n-bit strings is $2 \times 2 \times \ldots = 2^n$. This shows that $|S|$ is also 2^n.

It is often easy to trick ourselves into using the product rule when it is not the right thing to do, or use it incorrectly. Consider the following example:

Example

CSE312 has 4 sections and 6 TA—Yifan, Yael, Yin Yin, Siva, Jainul and Su. Each TA can teach more than one section. Anup wants to assign the TAs to teach sections, in such a way that every section gets at least one TA. How many ways are there to do this?

We can start by modeling the sections using the set $[4]$, and let $A \subseteq [4]$ be the set of sections taught by Yael, and similarly let $B, C, D, E, F \subseteq [4]$ be the sets that encode the sections taught by each of the other TAs. We saw above that there are 2^4 options for choosing A. So, we might conclude that the number of ways to assign TAs to the sections is $2^4 \times 2^4 \times 2^4 \times 2^4 \times 2^4 \times 2^4 = 2^{24} = 16,777,216$. However, this calculation is incorrect.

Try to figure out why the calculation is incorrect before you read on.

The problem is that we have disregarded the fact that each section must get at least one TA. The above calculation counts all possible ways of assigning TAs to sections, including those that assign no TAs to some sections. The fix is to model the situation differently. Let W be the set of TAs assigned to the first section, and similarly let X, Y, Z be the set of TAs assigned to the other sections. Now the number of options for W is exactly $2^6 - 1 = 63$. This is because there are 2^6 possible subsets of the 6 TAs, and one of these subsets—the empty set—is not allowed. After making this change, we can use the product rule properly to conclude that the number of ways to assign the TAs while making sure that each section gets at least one TA is $(2^6 - 1)^4$, which is less than the incorrect calculation gave us.

What if we required that each section gets at least 2 TAs? What would the correct count be then?

Example

Jane has 3 children—Alice, Bob and Charlie. Jane has 5 books that she would like to distribute to her 3 children. How many ways are there for her to distribute the books?

Let us try to model the question using the sets we have defined. The 5 books are 5 unique objects, so we can model them with the set $[5] = \{1, 2, 3, 4, 5\}$. The books Alice gets can be thought of a subset

$A \subseteq [5]$, and similarly, the books Bob and Charlie gets can be viewed as sets $B, C \subseteq [5]$. The number of options for the set A is exactly 2^5. So our first thought might be that the number of ways to distributed the books among Alice, Bob and Charlie is the number options for A times the number of options for B times the number of options for C, which is $2^5 \cdot 2^5 \cdot 2^5 = 2^{15} = 32768$. However, this calculation is not right.

Try to figure out why the calculation is incorrect before you read on.

The problem is that the same book cannot be given to both Alice and Bob. In other words, the sets A, B, C must be pairwise disjoint—they cannot share any elements. Jane cannot set $A = \{1\}, B = \{1,2\}, C = \{3,4,5\}$, while we did count such a setting in our count. Another problem is that every book must be distributed to someone. Jane cannot set $A = \{2\}, B = \{3,5\}, C = \{4\}$, because this leaves the book corresponding to 1 unassigned. We also counted such configurations in our count.

The issue can be resolved by picking a different model for the problem. We need to break down the assignment of books to children in terms of the books. Let b_1 be the name of the person that gets book 1, and similarly $b_2, \ldots, b_5$ be the names of the people that gets books $2, \ldots, 5$. Then we see that there are 3 options for b_1, 3 options for b_2 and so on. So the total number of ways to distribute the books is $3^5 = 243$.

This is much smaller than our erroneous count from earlier.

In the last lecture, introduced the product rule and used it to count some sets. The product rule is most useful when the number of choices available in each step is independent of the choices made in previous steps. However, even if this is not the case, it is possible to get counts on the number of options.

Introducing the Factorial Function

THE FACTORIAL FUNCTION IS VERY useful for counting many kinds of sets.

Example

How many ways are there to arrange the letters of the word GRAPEFXUIT?

This word has 10 letters. So there are 10 choices available for the first letter. After we pick the first letter, we can no longer use the first letter again. So there are only 9 choices available for the second letter, and so on. In this way, we compute the number of arrangements to be $10 \times 9 \times 8 \times \ldots \times 2 \times 1 = 10!$.

In general, we define

$$n! = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 2 \cdot 1.$$

By convention, we set $0! = 1$.

This is verbalized as *n factorial*.

Example

How many 5 letter words with distinct letters can be made using the English alphabet?

The number of choices for the first letter is 26. Given that first choice, we are reduced to 25 choices for the second letter, and so on. So the number of such words is $26 \times 25 \times 24 \times 23 \times 22 = \frac{26!}{23!}$.

In general, if we have a finite set S of n things, and want to know how many sequences of length k can be generated from distinct elements of S, the answer is

$$P(n,k) = \frac{n!}{(n-k)!}.$$

A slightly different way to arrive at the same count is by considering the *overcounting* when we just count permutations. There are $n!$ ways of permuting the n items. For each of these ways, we can just take the first k items in the order that they occur. This count certainly includes every sequence of k elements, but it counts each sequence many times. In fact, each sequence of k elements is counted exactly $(n-k)!$ times, because there are $(n-k)!$ ways to permute the elements that come after the first k elements. So, the correct count is again

$$P(n,k) = \frac{n!}{(n-k)!}.$$

The method of counting and then estimating the overcount is quite handy.

Example

How many ways are there to rearrange the letters CARAVAN?

Caravan has 7 letters, but 3 of them are the same. If all 7 letters were distinct, the answer would be 7!. Now consider what happens if we make the 3 As distinct by viewing them as A, A′, A″. Then, a particular anagram like VANACAR can be written in 6 different ways: VANA′CA″R, VANA″CA′R, VA′NACA″R, VA′NA″CAR, VA″NACA′R and VA″NA′CAR. These 6 ways correspond to the $3! = 6$ different ways of arranging the As. So, 7! overcounts by a factor of 3!. The answer is thus $\frac{7!}{3!}$.

Example

How many ways are there to rearrange the letters AABBBCCCC?

Using exactly the same reasoning above, there are 9! ways to arrange the string, but this overcounts each string by $2! \times 3! \times 4!$. So the correct count is $\frac{9!}{2! \cdot 3! \cdot 4!}$.