# Problem Set 1

599i: Communication Complexity

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*Note: This is an optional homework assignment; however, I will grade it for those who choose to submit. Please send your submissions by email.*

**Problem 1.** Suppose Alice gets a string $x \in \{0,1\}^n$ that has more 0s than 1s, and Bob gets a string $y \in \{0,1\}^n$ that has more 1s than 0s. They wish to communicate to find a coordinate $i$ where $x_i \neq y_i$. Show that at least $2 \log n$ bits of communication are required.

**Problem 2.** Let $f : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$ be a function with the property that for every $x \in \{0,1\}^n$, there are exactly $t$ choices for $y$ with $f(x,y) = 1$, and for every $y \in \{0,1\}^n$, there are exactly $t$ choices for $x$ with $f(x,y) = 1$. Our goal is to show that $f$ must admit a 0-cover of size at most $(2n)^t$.

To do this, we use Hall's matching theorem. Let $L, R$ be disjoint sets of the same size. Given a bipartite graph with vertex set $L \cup R$, where every edge belongs to the set $L \times R$, Hall's theorem says that if every subset $S \subseteq L$ is connected to at least $|S|$ vertices in $R$, then the graph must contain a matching, namely $|L|$ disjoint edges.

- Use Hall's theorem to prove that there are permutations $\pi_1, \ldots, \pi_t$ such that $f(x,y) = 1$ if and only if $y = \pi_r(x)$ for some $r$.
- Give a 0-cover of size $(2n)^t$ for $f$.

**Problem 3.** The matrices we are working with have 0/1 entries, so one can view these entries as coming from any field – for instance, we can view them as real numbers, rational numbers, or elements of the finite field $\mathbb{F}_2$. This is important because the value of the rank may depend on the field used.

1. Give an example of a $3 \times 3$ matrix whose rank over $\mathbb{F}_2$ is not the same as its rank over the reals.
2. Prove that if $M$ has 0/1 entries, then the rank of $M$ over the reals is equal to its rank over the rationals, which is at least as large as its rank over $\mathbb{F}_2$.
3. For every $r$, show that there is a matrix whose $\mathbb{F}_2$-rank and communication complexity are both $r$. Conclude that the log-rank conjecture is false over $\mathbb{F}_2$.

**Problem 4.** Show that there is a Boolean matrix of rank at most $2r$ with $2^r$ distinct rows and $2^r$ distinct columns.

**Problem 5.** Prove that John's theorem is tight.