SURVEY

Finding an efficient method to solve SuDoku puzzles is:

		8	6					
							6	
			4	8			2	3
		5		9				8
	4	9				2	1	
2				4		7		
3	6			2	9			
	1							
					5	1		

1: A waste of time
2: A decent spare time activity
3: A fundamental problem in computer science

SURVEY

Finding an efficient method to solve SuDoku puzzles is:

		8	6					
							6	
			4	8			2	3
		5		9				8
	4	9				2	1	
2				4		7		
3	6			2	9			
	1							
					5	1		

1: A waste of time
2: A decent spare time activity
3: A fundamental problem in computer science

Does every problem have efficient algorithms?

Halting Problem: Given program code, output whether program halts or not.

Theorem [Godel]: Halting cannot be solved by any algorithm.
Theorem: Integer Equations cannot be solved by any algorithm.
…

What about problems that have algorithms? Must they have efficient algorithms?

Theorem: There are problems that can be solved in exponential time, but not in polynomial time.

OK, but what about Set Cover, Vertex Cover, Shortest Spanning Path – all have brute force algorithms, but do they have efficient algorithms?

Decision Problems

Decision problem: Problems with "yes" or "no" answers.

Does a given set system have a set cover of size at most k?

Does a given graph have a vertex cover of size at most k?

Does a number have a non-trivial factorization?

Does a given graph have an MST of cost at most k?

Does a given flow network have a min-cut of capacity at most k?

Does a given sudoku problem have a solution?

Polynomial time. Algorithm A runs in poly-time if for every string x, A(x) terminates in at most p(|x|) "steps", where p is some polynomial.

length of x

P: The class of decision problems that can be solved in polynomial time.

PRIMES: X = { 2, 3, 5, 7, 11, 13, 17, 23, 29, 31, 37, …. }. Is input a prime?

Theorem [Agrawal-Kayal-Saxena, 2002] PRIMES is in P.

NP

Certification algorithm intuition.

- Certifier doesn't determine whether answer is "yes" on its own; rather, it checks a proposed proof t that answer is "yes".

Def. Algorithm C(x, t) is a certifier for problem X if for every string x, the answer is "yes" iff there exists a string t such that C(x, t) = `yes`.

"certificate" or "witness"

NP. Decision problems for which there exists a poly-time certifier.

C(x, t) is a poly-time algorithm and $|t| \leq p(|x|)$ for some polynomial p.

Remark. NP stands for nondeterministic polynomial-time.

Certifiers and Certificates: Composite

COMPOSITES. Given an integer x, is x composite?

Certificate. A nontrivial factor t of x. Note that such a certificate exists iff x is composite. Moreover $|t| \le |s|$.

Certifier.

```
boolean C(x, t) {
   if (t = 1 or t = x)
      return false
   else if (x is a multiple of t)
      return true
   else
      return false
}
```

Instance. x = 437,669.

Certificate. t = 541 or 809. ← 437,669 = 541 X 809

Conclusion. COMPOSITES is in NP.

Certifiers and Certificates: 3-Satisfiability

3SAT. Given a 3-CNF formula, is there a satisfying assignment?

Certificate. An assignment of truth values to the n boolean variables.

Certifier. Check that each clause has at least one true literal.

Ex.

$$\left(\overline{x_1} \vee x_2 \vee x_3\right) \wedge \left(x_1 \vee \overline{x_2} \vee x_3\right) \wedge \left(x_1 \vee x_2 \vee x_4\right) \wedge \left(\overline{x_1} \vee \overline{x_3} \vee \overline{x_4}\right)$$

instance s

$$x_1 = 1,\ x_2 = 1,\ x_3 = 0,\ x_4 = 1$$

certificate t

Conclusion. 3SAT is in NP.

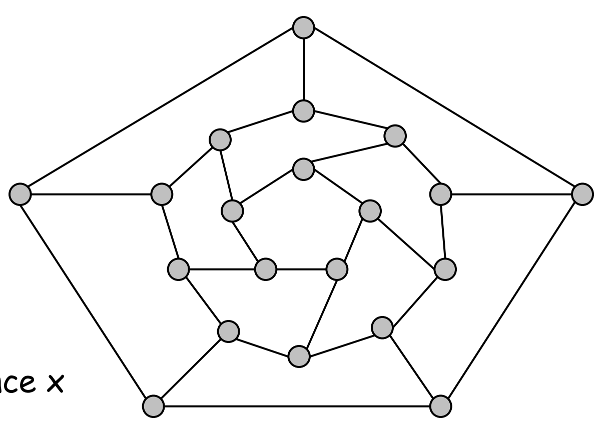
Certifiers and Certificates: Hamiltonian Cycle

HAM-CYCLE. Given an undirected graph $G = (V, E)$, does there exist a simple cycle C that visits every node?

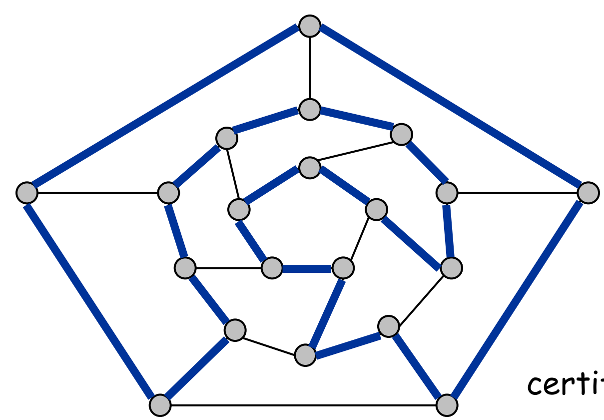
Certificate. A permutation of the n nodes.

Certifier. Check that the permutation contains each node in V exactly once, and that there is an edge between each pair of adjacent nodes in the permutation.

Conclusion. HAM-CYCLE is in NP.



Instance x



certificate t

Certifiers and Certificates: Min-Cut

MIN-CUT. Given a flow network, and a number k, does there exist a min-cut of capacity at most k?

Certificate. A min-cut T.

Certifier. Check that the capacity of the min-cut is at most T.

Conclusion. MIN-CUT is in NP.

Certifiers and Certificates: Min-Cut

MIN-CUT. Given a flow network, and a number k, does there exist a min-cut of capacity at most k?

Certificate. The empty string.

Certifier. Compute the min-cut of the graph and check whether its capacity is at most k.

Conclusion. MIN-CUT is in NP.

Examples of NP Problems

Eg: Does a given set system have a set cover of size at most k?

Certificate: A set cover of size at most k

Does a given graph have a vertex cover of size at most k?

Certificate: A vertex cover of size at most k.

Does a number have a non-trivial factorization?

Certificate: A non-trivial factorization

Does a given graph have an MST of cost at most k?

Certificate: An MST of cost at most k

Does a given flow network have a min-cut of capacity at most k?

Certificate: A min-cut of capacity at most k

Does a given sudoku problem have a solution?

Certificate: A valid solution.

P, NP, EXP

P. Decision problems for which there is a poly-time algorithm.
EXP. Decision problems for which there is an exponential-time algorithm.
NP. Decision problems for which there is a poly-time certifier.

Claim. P $\subseteq$ NP.
Pf. Consider any problem X in P.
- By definition, there exists a poly-time algorithm $A(x)$ that solves X.
- Certificate: t = empty string, certifier $C(x, t) = A(x)$. ▪

Claim. NP $\subseteq$ EXP.
Pf. Consider any problem X in NP.
- By definition, there exists a poly-time certifier $C(x, t)$ for X.
- To solve input x, run $C(x, t)$ on all strings t with $|t| \leq p(|x|)$ (running time of C).
- Return `yes`, if $C(x, t)$ returns `yes` for any of these. ▪

The Main Question: P Versus NP

Does P = NP? [Cook 1971, Edmonds, Levin, Yablonski, Gödel]

- Is the decision problem as easy as the certification problem?
- Clay $1 million prize.

If yes: Efficient algorithms for 3-COLOR, TSP, FACTOR, SAT, …

If no: No efficient algorithms possible for 3-COLOR, TSP, SAT, …

NP-Completeness

Punchline: If you find a way to solve sudoku in polynomial time, you will solve factoring in polynomial time!

NP-Completeness

Punchline: If you find a way to solve sudoku in polynomial time, you will solve set cover in polynomial time!

NP-Completeness

Punchline: If you find a way to solve sudoku in polynomial time, you will solve SAT in polynomial time!

NP-Completeness

Punchline: If you find a way to solve sudoku in polynomial time, you will solve all machine learning problems in polynomial time!

NP-Completeness

Punchline: If you find a way to solve sudoku in polynomial time,
you will solve every problem in NP in polynomial time!

NP-Completeness

Def. Problem X polynomial reduces to problem Y ($X \leq_p Y$) if arbitrary instances of problem X can be solved using:

- Polynomial number of standard computational steps, plus
- Polynomial number of calls to subroutine that solves problem Y.

NP-complete Problem. A problem Y in NP with the property that for every problem X in NP, $X \leq_p Y$.

Theorem. Suppose Y is an NP-complete problem. Then Y is solvable in poly-time iff P = NP.

Pf. $\Leftarrow$ If P = NP then Y can be solved in poly-time since Y is in NP.

Pf. $\Rightarrow$ Suppose Y can be solved in poly-time.

- Let X be any problem in NP. Since $X \leq_p Y$, we can solve X in poly-time. This implies NP $\subseteq$ P.
- We already know P $\subseteq$ NP. Thus P = NP. ▪

Fundamental question. Do there exist "natural" NP-complete problems?

Program Satisfiability

PROGRAM-SAT. Given a line program on inputs $x = x_1, x_2, \ldots, x_n$ is there a way to set the inputs so that the output is 1?

$l_1 = x_1$ AND x_2 ;
$l_2 = x_3$ OR x_5 ;
$l_3 =$ NOT x_6 AND x_8;
$l_4 = l_1$ XOR l_3 ;
$l_5 = l_2$ AND x_4 ;
$l_6 =$ NOT l_4 OR l_2 ;
...
$l_{m-2} = l_{17}$ AND l_{25} ;
$l_{m-1} = x_1$ XOR x_2 ;
$l_m = x_1$ XOR l_{m-2} ;
OUTPUT l_m

The "First" NP-Complete Problem

Theorem. PROGRAM-SAT is NP-complete. [Cook 1971, Levin 1973]

Pf. (sketch)

- Any polynomial time algorithm can be compiled into a poly-size program.
- If problem X has poly-time certifier C(x, t), to solve X, need to know if there exists a certificate t such that C(x, t) = `yes`.
- Let K(t) be poly-size program computing C(x, t)
- Program K(t) is satisfiable iff X(x) = `yes`.

Establishing NP-Completeness

Recipe to establish NP-completeness of problem Y.

- Step 1. Show that Y is in NP.
- Step 2. Choose an NP-complete problem X.
- Step 3. Prove that $X \leq_p Y$.

Justification. If X is an NP-complete problem, and Y is a problem in NP with the property that $X \leq_P Y$ then Y is NP-complete.

Pf. Let W be any problem in NP. Then $\underbrace{W \leq_P X}_{\text{by definition of NP-complete}} \underbrace{\leq_P Y}_{\text{by assumption}}$.

- By transitivity, $W \leq_P Y$.
- Hence Y is NP-complete. ▪

3-SAT is NP-Complete

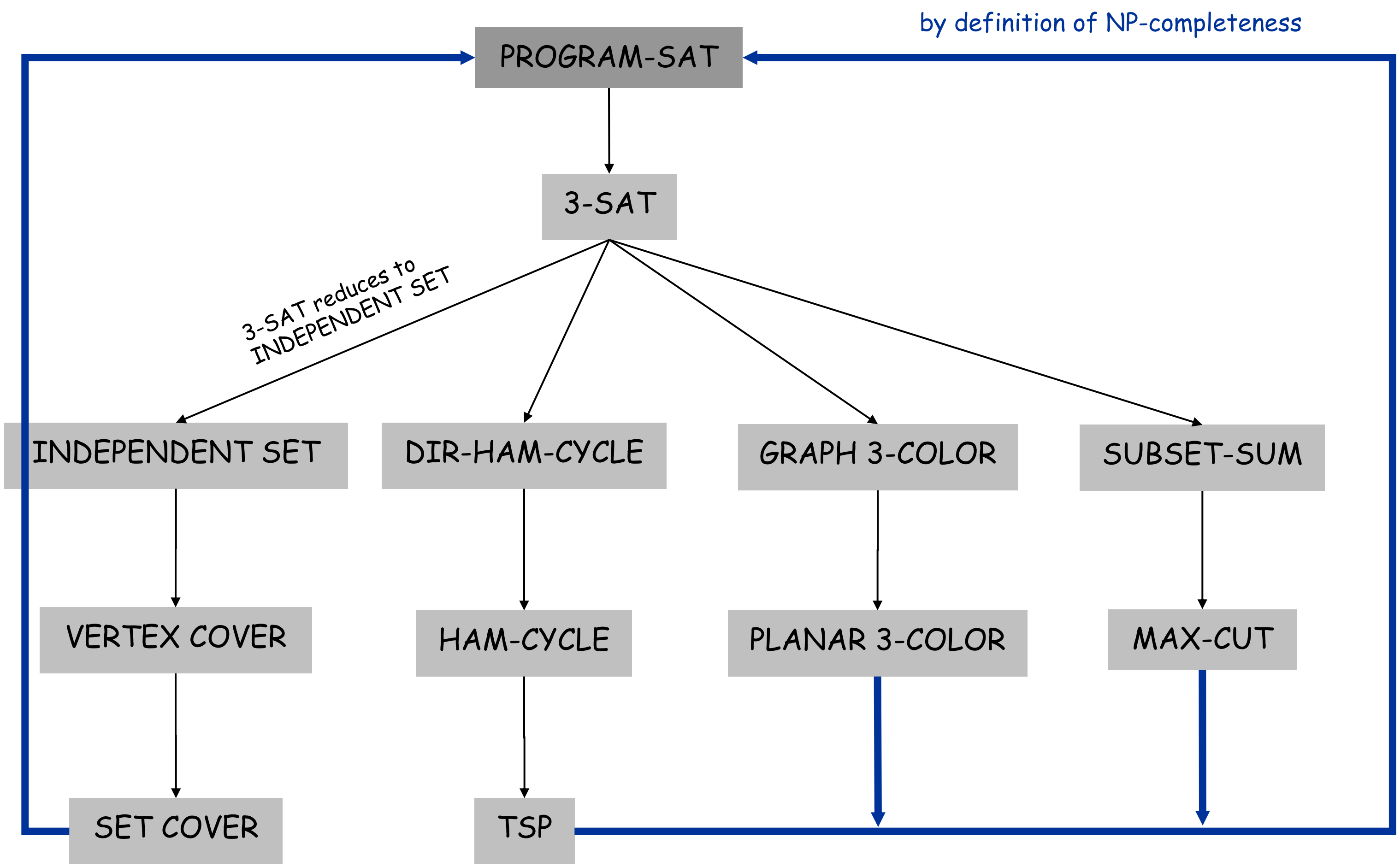
Theorem. 3-SAT is NP-complete.

Pf. Suffices to show that PROGRAM-SAT $\leq_P$ 3-SAT since 3-SAT is in NP.

- Let K be any line program.
- Create a 3-SAT variable l_i for each line i.
- Make variables compute correct values at each node:
  - $l_i = l_4$ AND x_5 add 4 clauses: (l_i OR not l_4 OR not x_5) AND (l_i OR not l_4 OR x_5) AND (l_i OR l_4 OR not x_5) AND (not l_i OR l_4 OR x_5)
- 3SAT formula is satisfiable if and only if K is satisfiable.

NP-Completeness

Observation. All problems below are NP-complete and polynomial reduce to one another!



More NP-Complete Computational Problems

Aerospace engineering: optimal mesh partitioning for finite elements.
Biology: protein folding.
Chemical engineering: heat exchanger network synthesis.
Civil engineering: equilibrium of urban traffic flow.
Economics: computation of arbitrage in financial markets with friction.
Electrical engineering: VLSI layout.
Environmental engineering: optimal placement of contaminant sensors.
Financial engineering: find minimum risk portfolio of given return.
Game theory: find Nash equilibrium that maximizes social welfare.
Genomics: phylogeny reconstruction.
Mechanical engineering: structure of turbulence in sheared flows.
Medicine: reconstructing 3-D shape from biplane angiocardiogram.
Operations research: optimal resource allocation.
Physics: partition function of 3-D Ising model in statistical mechanics.
Politics: Shapley-Shubik voting power.
Pop culture: Minesweeper consistency.
Statistics: optimal experimental design.