

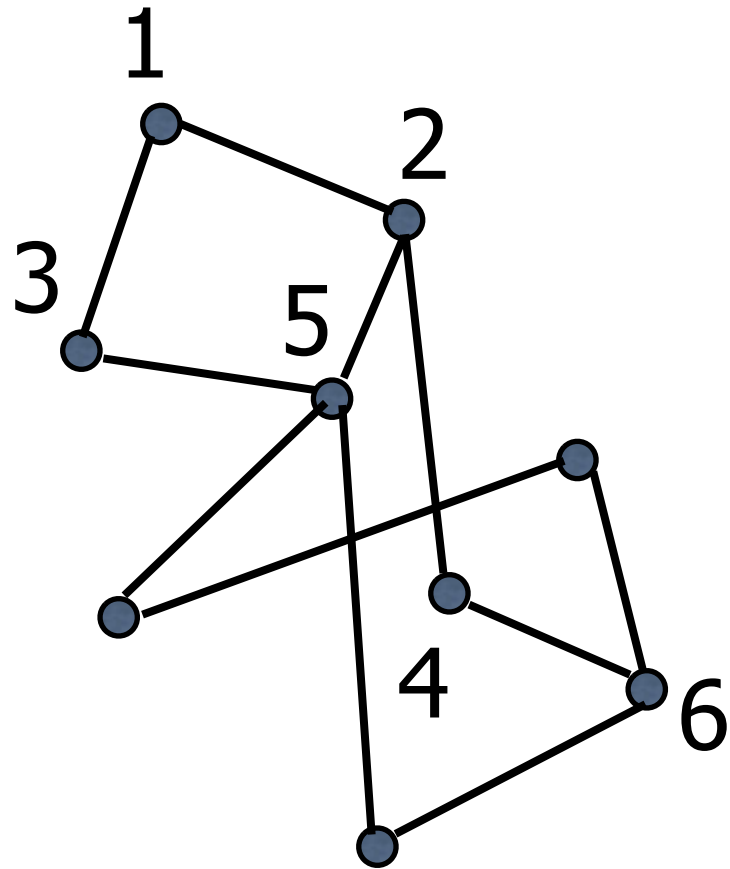
degrees

Claim: If a graph has m edges, then

$$\sum_v \deg(v) = 2m.$$

Proof: Every edge $\{u,v\}$ contributes exactly 2 to the left hand side, 1 to $\deg(u)$, and 1 to $\deg(v)$.

Claim: If a graph has no cycles, it must have a vertex v of degree ≤ 1 .



Path: A sequence of distinct vertices where each vertex is connected to the next by an edge.

Cycle: A path of length > 1 such that the first vertex is connected to the last one.

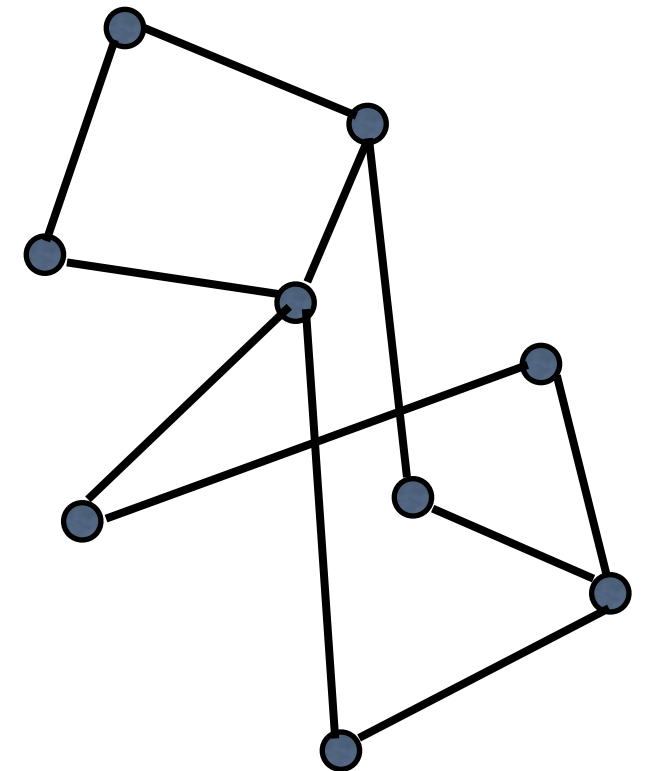
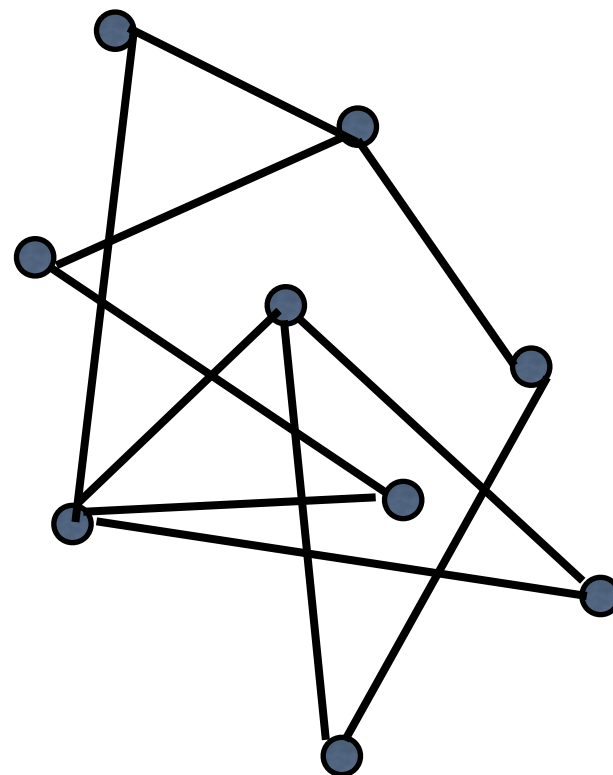
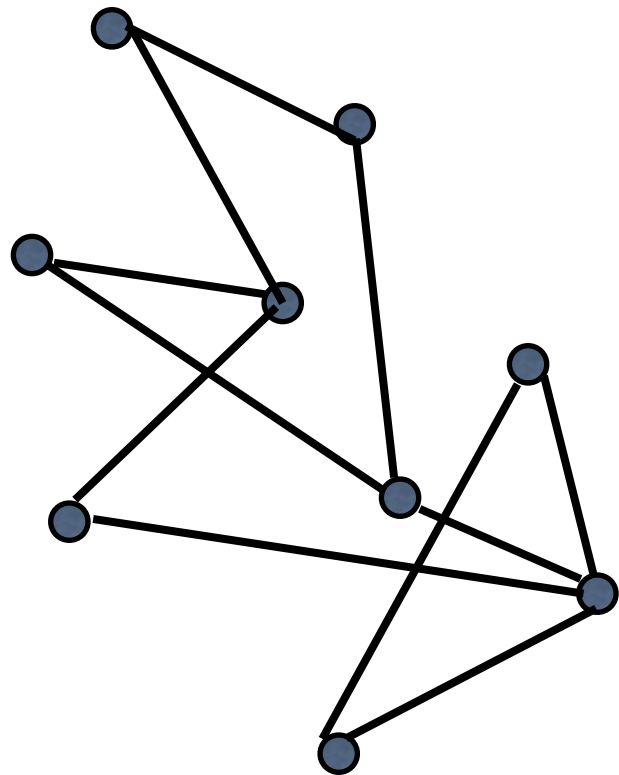
1,2,5,3 is a cycle

6,4,2,1,3,5,2,4 is not a cycle

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EQUIVALENT TO

Claim: If every vertex has degree > 1 , the graph has a cycle.



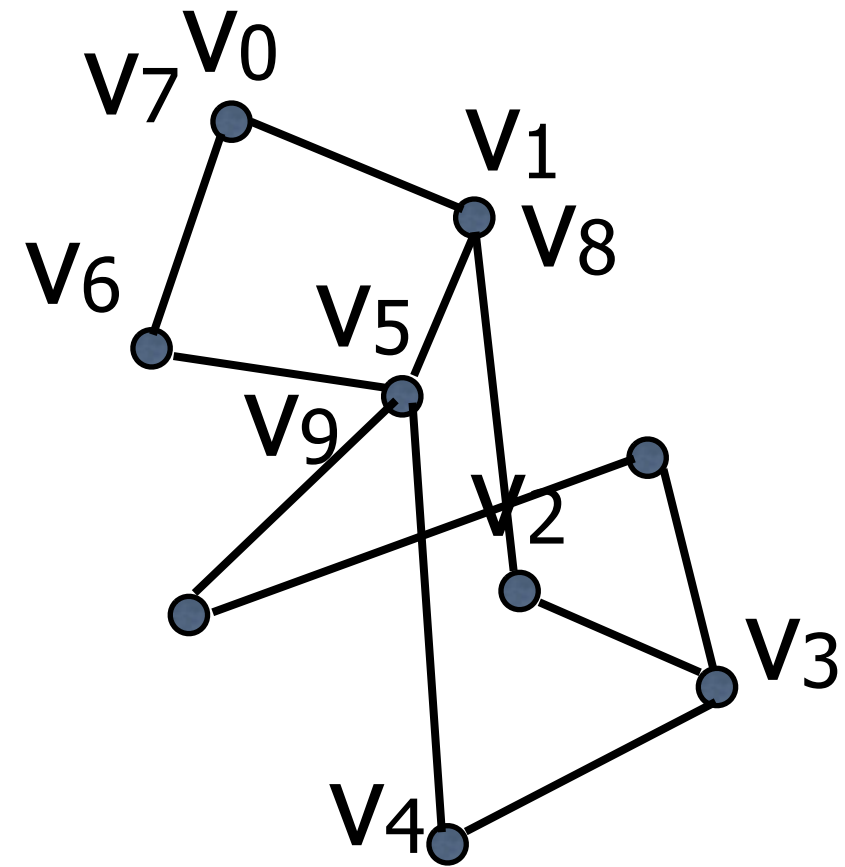
Claim: If a graph has no cycles, it must have a vertex v of degree ≤ 1 .

Proof:

Suppose not.

Then there is a graph G that has no cycles, and yet every vertex in the graph has degree > 1 .

There must be v_0, v_1, \dots, v_n , a sequence of $n+1$ vertices such that $v_{i-1} \neq v_{i+1}$, for all i , and adjacent vertices have an edge.



Intuition: this should not be possible! Let's use the degrees of the vertices to find a cycle.

Note: The argument should work for every such graph!

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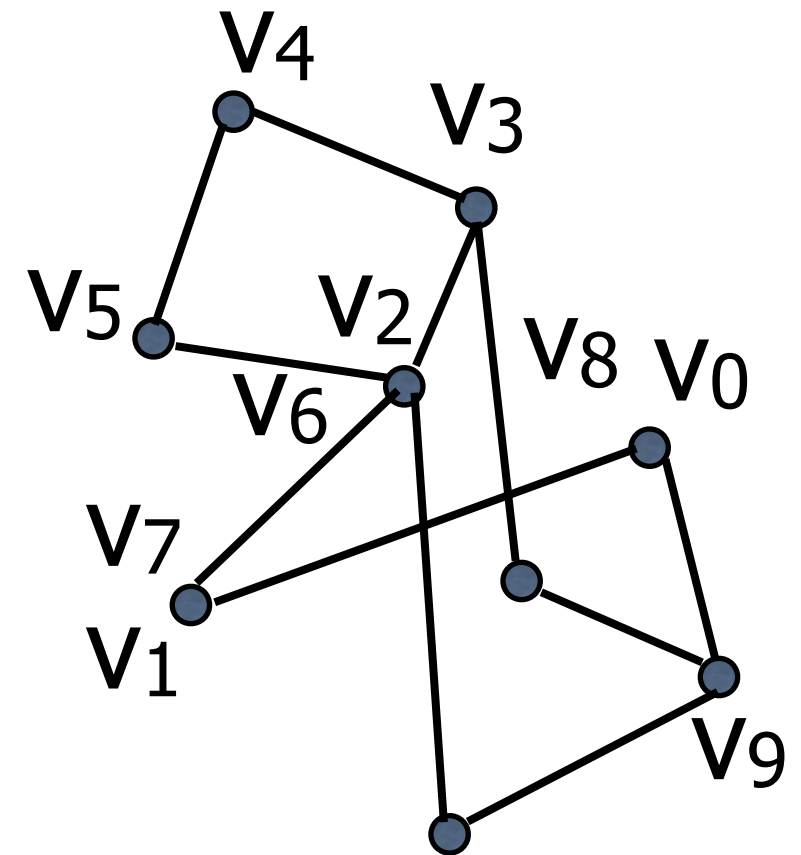
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By the pigeonhole principle, there must $i < j$ s.t. $v_i = v_j$.

Let $i < j$ be the closest such pair. Then $v_i, v_{i+1}, \dots, v_{j-1}$ form a cycle.



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Lemma: Every tree on n vertices has exactly $n-1$ edges.

Tree: A connected graph that has no cycles.

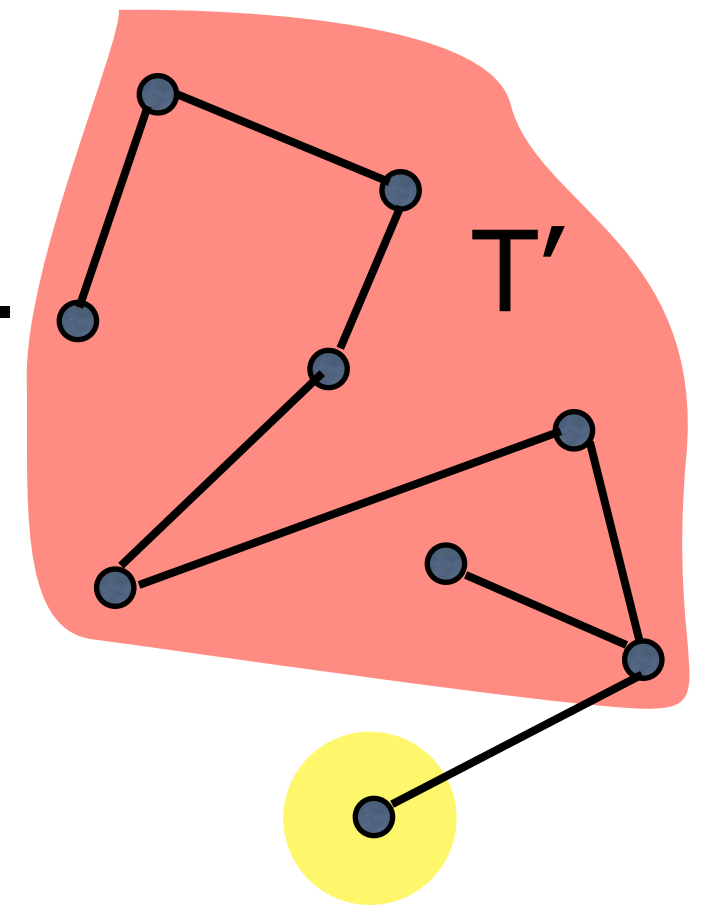
Proof: By induction on n .

Base case: $n=1$. Every graph with 1 vertex has $1-1 = 0$ edges. The claim holds.

Inductive step: $n>1$. Let T be a tree with n vertices.

Since T has no cycles, there is a vertex v , $\deg(v) \leq 1$ by previous claim. $\deg(v)=1$ since T is connected means there are no degree 0 vertices.

$T' = T-v$ is connected and has no cycles, so it is a tree. T' has $n-1$ vertices. So by induction, T' has $n-2$ edges. Thus T has $n-1$ edges.



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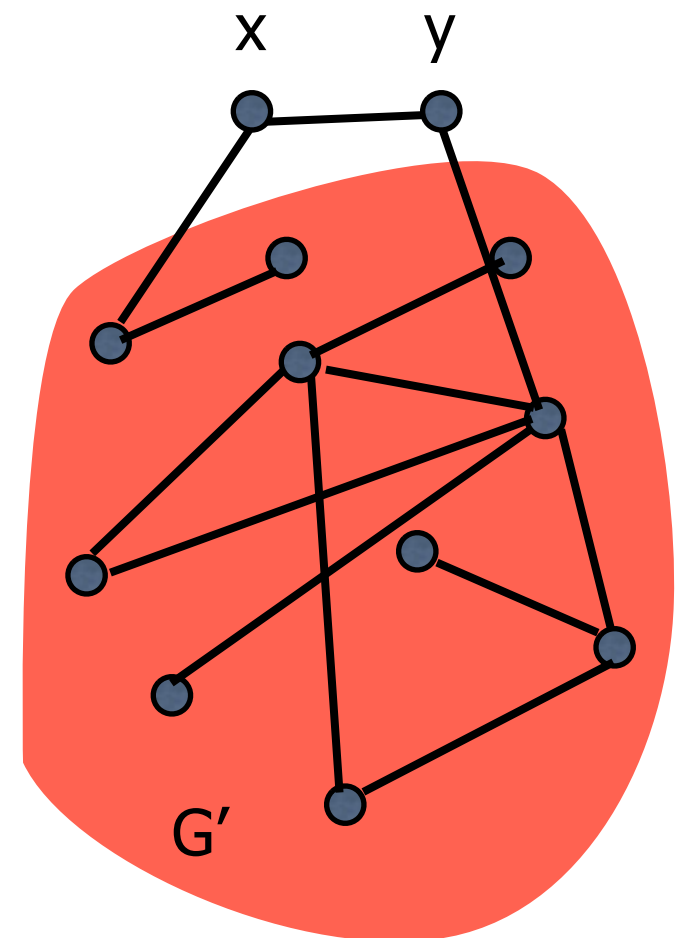
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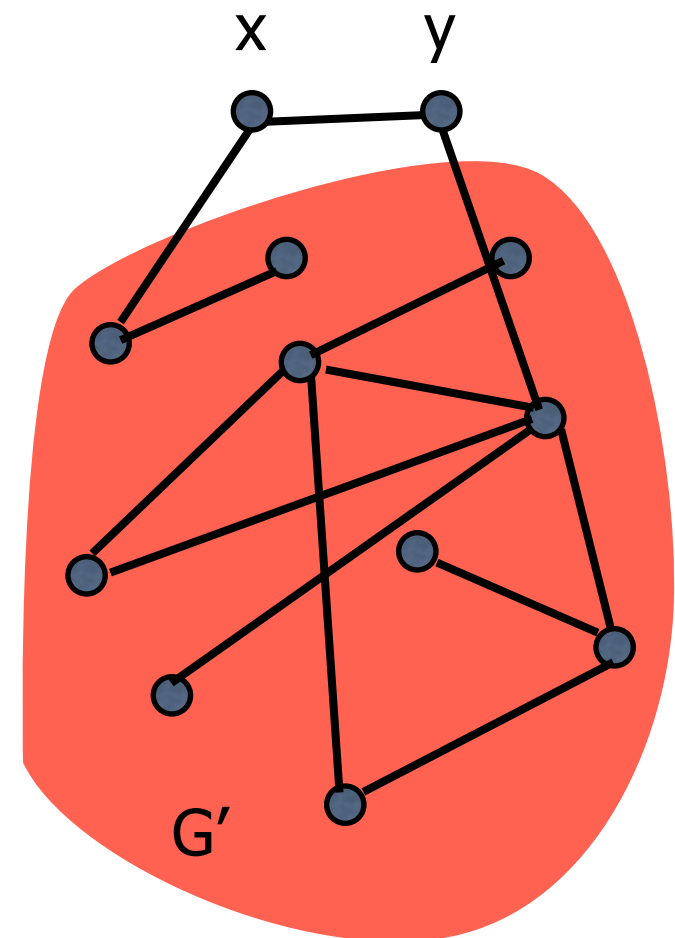
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Then by the induction hypothesis, G' has a triangle.



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Let $\{x,y\}$ be an edge of the graph, and let G' denote the subgraph using the rest of the vertices.

Case 1: At least $(n-1)^2+1$ edges are in G' . Then by the induction hypothesis, G' has a triangle.

Case 2: At most $(n-1)^2$ edges are in G' . There is one edge between x,y , so #edges from $\{x,y\}$ to $G' \geq n^2-(n-1)^2 = 2(n-1)+1$. But G' has $2(n-1)$ vertices, so there is vertex z such that $\{x,z\},\{y,z\}$ are edges. Then x,y,z is a triangle.

