Stable Matching Problem

Goal. Given n companies and n applicants, find a "suitable" matching.

- Companies rate applicants, applicants rate companies.
- Each company lists applicants in order of preference from best to worst.

favorite ↓ (1st) … least favorite ↓ (3rd)

	1st	2nd	3rd
X	A	B	C
Y	B	A	C
Z	A	B	C

Company's Preference Profile

favorite ↓ (1st) … least favorite ↓ (3rd)

	1st	2nd	3rd
A	Y	X	Z
B	X	Y	Z
C	X	Y	Z

Applicant's Preference Profile

Stable Matching Problem

Perfect matching:

- Each company gets exactly one applicant.
- Each applicant gets exactly one company.

Stability: no incentive for some pair of participants to undermine assignment by joint action.

- In matching M, an unmatched pair c-a is unstable if company c and applicant a prefer each other to current matches.
- Unstable pair c-a could each improve by switching.

Stable matching: perfect matching with no unstable pairs.

Stable matching problem. Given the preference lists of n companies and n applicants, find a stable matching if one exists.

Stable Matching Problem

Q. Is assignment X-C, Y-B, Z-A stable?

	1st (favorite)	2nd	3rd (least favorite)
X	A	B	C
Y	B	A	C
Z	A	B	C

companies's Preference Profile

	1st (favorite)	2nd	3rd (least favorite)
A	Y	X	Z
B	X	Y	Z
C	X	Y	Z

applicants's Preference Profile

Stable Matching Problem

Q. Is assignment X-C, Y-B, Z-A stable?

A. No. B and X will defect.

favorite ↓ (1st) … least favorite ↓ (3rd)

	1st	2nd	3rd
X	A	B	C
Y	B	A	C
Z	A	B	C

companies's Preference Profile

favorite ↓ (1st) … least favorite ↓ (3rd)

	1st	2nd	3rd
A	Y	X	Z
B	X	Y	Z
C	X	Y	Z

applicants's Preference Profile

Stable Matching Problem

Q. Is assignment X-A, Y-B, Z-C stable?

A. Yes.

	1st (favorite)	2nd	3rd (least favorite)
X	A	B	C
Y	B	A	C
Z	A	B	C

companies's Preference Profile

	1st (favorite)	2nd	3rd (least favorite)
A	Y	X	Z
B	X	Y	Z
C	X	Y	Z

applicants's Preference Profile

Stable Roommate Problem

Q. Do stable matchings always exist?
A. Not obvious a priori.

Stable roommate problem.
- 2n people; each person ranks others from 1 to 2n-1.
- Assign roommate pairs so that no unstable pairs.

	1st	2nd	3rd
A	B	C	D
B	C	A	D
C	A	B	D
D	A	B	C

A-B, C-D ⇒ B-C unstable
A-C, B-D ⇒ A-B unstable
A-D, B-C ⇒ A-C unstable

Observation. Stable matchings do not always exist for stable roommate problem.

Propose-And-Reject Algorithm

Propose-and-reject algorithm. [Gale-Shapley 1962] Intuitive method that guarantees to find a stable matching.

```
Initialize each person to be free.
while (some company is free and hasn't proposed to every
applicant) {
    Choose such a company x
    a = 1st applicant on x's list to whom x has not yet
proposed
    if (a is free)
        assign x and a to each other
    else if (a prefers x to her current assignment y)
        assign a to x, and y to be free
    else
        a rejects x
}
```

Proof of Correctness: Termination

Observation 1. companies propose to applicants in decreasing order of preference.

Observation 2. Once an applicant is matched, she never becomes unmatched; she only "trades up."

Claim. Algorithm terminates after at most n^2 iterations of while loop.
Pf. Each time through the while loop a company proposes to a new applicant. There are only n^2 possible proposals. ▪

	1st	2nd	3rd	4th	5th
V	A	B	C	D	E
W	B	C	D	A	E
X	C	D	A	B	E
Y	D	A	B	C	E
Z	A	B	C	D	E

	1st	2nd	3rd	4th	5th
A	W	X	Y	Z	V
B	X	Y	Z	V	W
C	Y	Z	V	W	X
D	Z	V	W	X	Y
E	V	W	X	Y	Z

$n(n-1) + 1$ proposals required

Proof of Correctness: Perfection

Claim. All companies and applicants get matched.

Pf. (by contradiction)

- Suppose, for sake of contradiction, that Z is not matched upon termination of algorithm.
- Then some applicant, say A, is not matched upon termination.
- By Observation 2, A was never proposed to.
- But, Z proposes to everyone, since Z ends up unmatched. ▪

Proof of Correctness: Stability

Claim. No unstable pairs.

Pf. (by contradiction)

- Suppose A-Z is an unstable pair: each prefers each other to partner in Gale-Shapley matching S*.

- Case 1: Z never proposed to A. ← companies propose in decreasing order of preference
  - ⇒ Z prefers GS applicant to A.
  - ⇒ A-Z is stable.

- Case 2: Z proposed to A.
  - ⇒ A rejected Z (right away or later)
  - ⇒ A prefers her GS company to Z. ← applicants only trade up
  - ⇒ A-Z is stable.

- In either case A-Z is stable, a contradiction. ▪

S*
A-Y
B-Z
. . .

Summary

Stable matching problem. Given n companies and n applicants, and their preferences, find a stable matching if one exists.

Gale-Shapley algorithm. Guarantees to find a stable matching for any problem instance.

Q. How to implement GS algorithm efficiently?

Q. If there are multiple stable matchings, which one does GS find?

Efficient Implementation

Efficient implementation. We describe $O(n^2)$ time implementation.

Note: this is linear in the size of the input.

Representing companies and applicants.

- Assume companies are named 1, …, n.
- Assume applicants are named 1', …, n'.

Queues.

- Maintain a list of free companies, e.g., in a queue.
- Maintain two arrays `applicant[c]`, and `company[a]`.
  - set entry to `0` if unmatched
  - if c matched to a then `applicant[c]=a` and `company[a]=c`

companies proposing.

- For each company, maintain a list of applicants, ordered by preference.
- Maintain an array `count[c]` that counts the number of proposals made by company `c`.

Efficient Implementation

applicants rejecting/accepting.

- Does applicant `a` prefer company `c` to company `c'`?
- For each applicant, create inverse of preference list of companies.
- Constant time access for each query after O(n) preprocessing.

A	1st	2nd	3rd	4th	5th	6th	7th	8th
Pref	8	3	7	1	4	5	6	2

A	1	2	3	4	5	6	7	8
Inverse	4th	8th	2nd	5th	6th	7th	3rd	1st

A prefers company 3 to 6
since `inverse[3]` < `inverse[6]`

2 7

```
for i = 1 to n
   inverse[pref[i]] = i
```

Understanding the Solution

Q. For a given problem instance, there may be several stable matchings. Do all executions of Gale-Shapley yield the same stable matching? If so, which one?

An instance with two stable matchings.

- A-X, B-Y, C-Z.
- A-Y, B-X, C-Z.

	1st	2nd	3rd
X	A	B	C
Y	B	A	C
Z	A	B	C

	1st	2nd	3rd
A	Y	X	Z
B	X	Y	Z
C	X	Y	Z

Understanding the Solution

Q. Do all executions of Gale-Shapley yield the same stable matching?

Def. company m is a valid partner of applicant w if there exists some stable matching in which they are matched.

	1st	2nd	3rd
X	A	C	B
Y	A	B	C
Z	A	B	C

	1st	2nd	3rd
A	Y	Z	X
B	Y	Z	X
C	Y	X	Z

Q. Are X-A valid partners?

Understanding the Solution

Q. Do all executions of Gale-Shapley yield same stable matching?

Def. company c is valid partner of applicant a if exists some stable matching in which they are matched.

company-optimal assignment. Each company receives best valid partner.

Claim. All executions of GS yield company-optimal assignment, which is a stable matching!

- No reason a priori to believe that company-optimal assignment is a matching, let alone stable.
- Simultaneously best for every company.

company Optimality

Claim. GS matching S* is company-optimal.

Pf. (by contradiction)

- Suppose some company is paired with someone other than best partner. companies propose in decreasing order of preference ⇒ some company is rejected by valid partner.
- Let Y be first such company, and let A be first valid applicant that rejects it.
- Let S be a stable matching where A and Y are matched.
- When Y is rejected, A forms (or reaffirms) engagement with a company, say Z, whom she prefers to Y.
- Let B be Z's partner in S. B is a valid partner of Z.
- Z matched to A and not yet rejected by any valid partner at the point when Y is rejected by A. Thus, Z prefers A to B.↑
- But A prefers Z to Y.
- Thus A-Z is unstable in S. ▪

since this is first rejection by a valid partner of anyone

S
A-Y
B-Z
. . .

Stable Matching Summary

Stable matching problem. Given preference profiles of n companies and n applicants, find a stable matching.

no company and applicant prefer to be with each other than assigned partner

Gale-Shapley algorithm. Finds a stable matching in $O(n^2)$ time.

company-optimality. In version of GS where companies propose, each company receives best valid partner.

w is a valid partner of m if there exist some stable matching where m and w are paired

Q. Does company-optimality come at the expense of the applicants?

applicant Pessimality

applicant-pessimal assigncompaniest. Each applicant receives worst valid partner.

Claim. GS finds applicant-pessimal stable matching S*.

Pf.

- Suppose A-Z matched in S*, but Z is not worst valid partner for A.
- There exists stable matching S in which A is paired with a company, say Y, whom she likes less than Z.
- Let B be Z's partner in S. company-optimality
- Z prefers A to B.
- Thus, A-Z is an unstable in S. ▪

S
A-Y
B-Z
. . .

Lessons Learned

Powerful ideas

- Isolate underlying structure of problem.
- Create useful and efficient algorithms.

Potentially deep social ramifications.

Moral: Be the one doing the proposing!