Graphs

Objects & Relationships

Facebook friends:

- Obj: People
- Rel: Two are related if they are friends

Cities and Roads:

- Obj: Cities
- Rel: Two are related if they have a road between them

Data flow in programs:

- Obj: Lines of the program
- Rel: Two are related if one line depends on the other

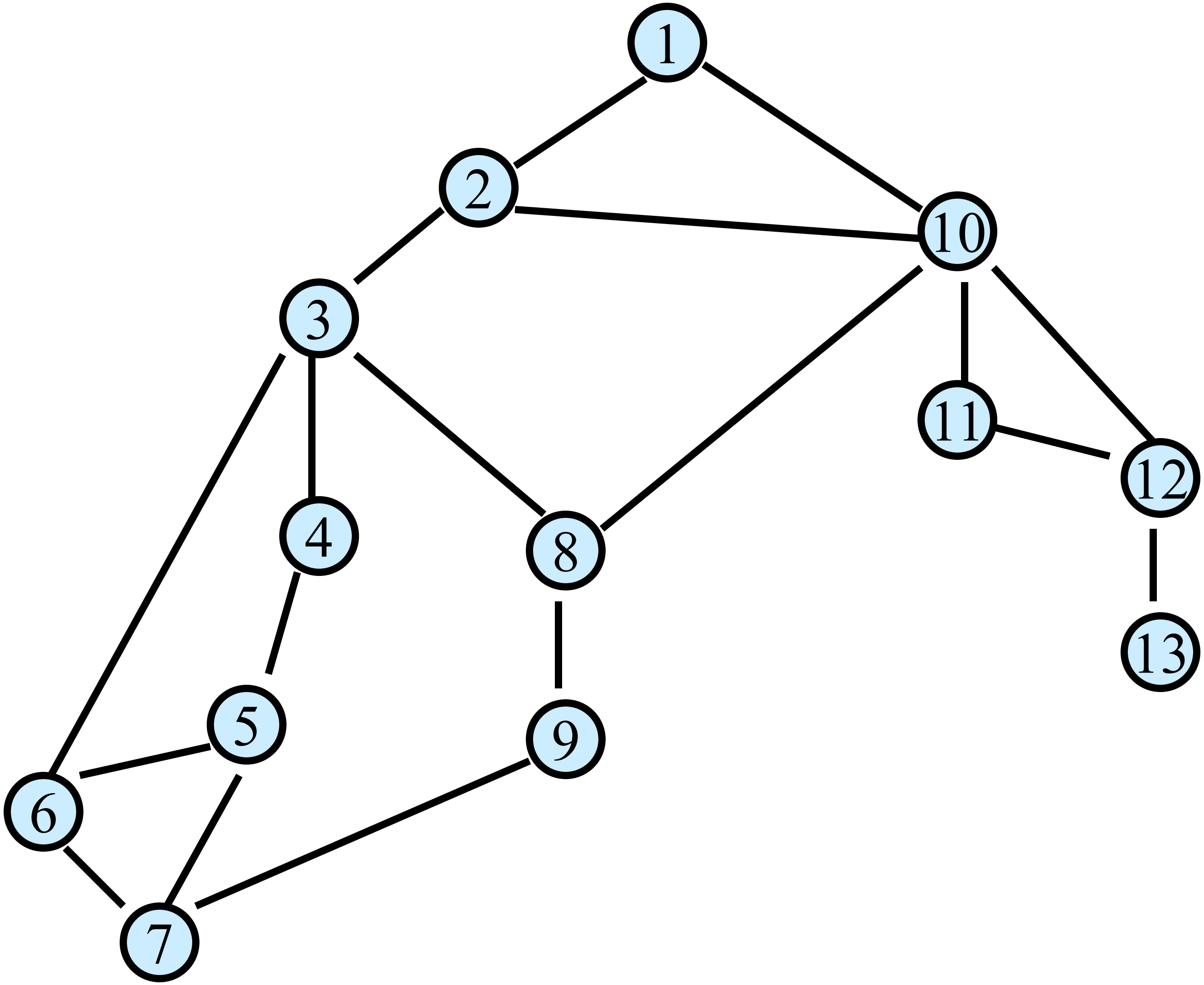
Graphs

Objects: "vertices," aka "nodes"

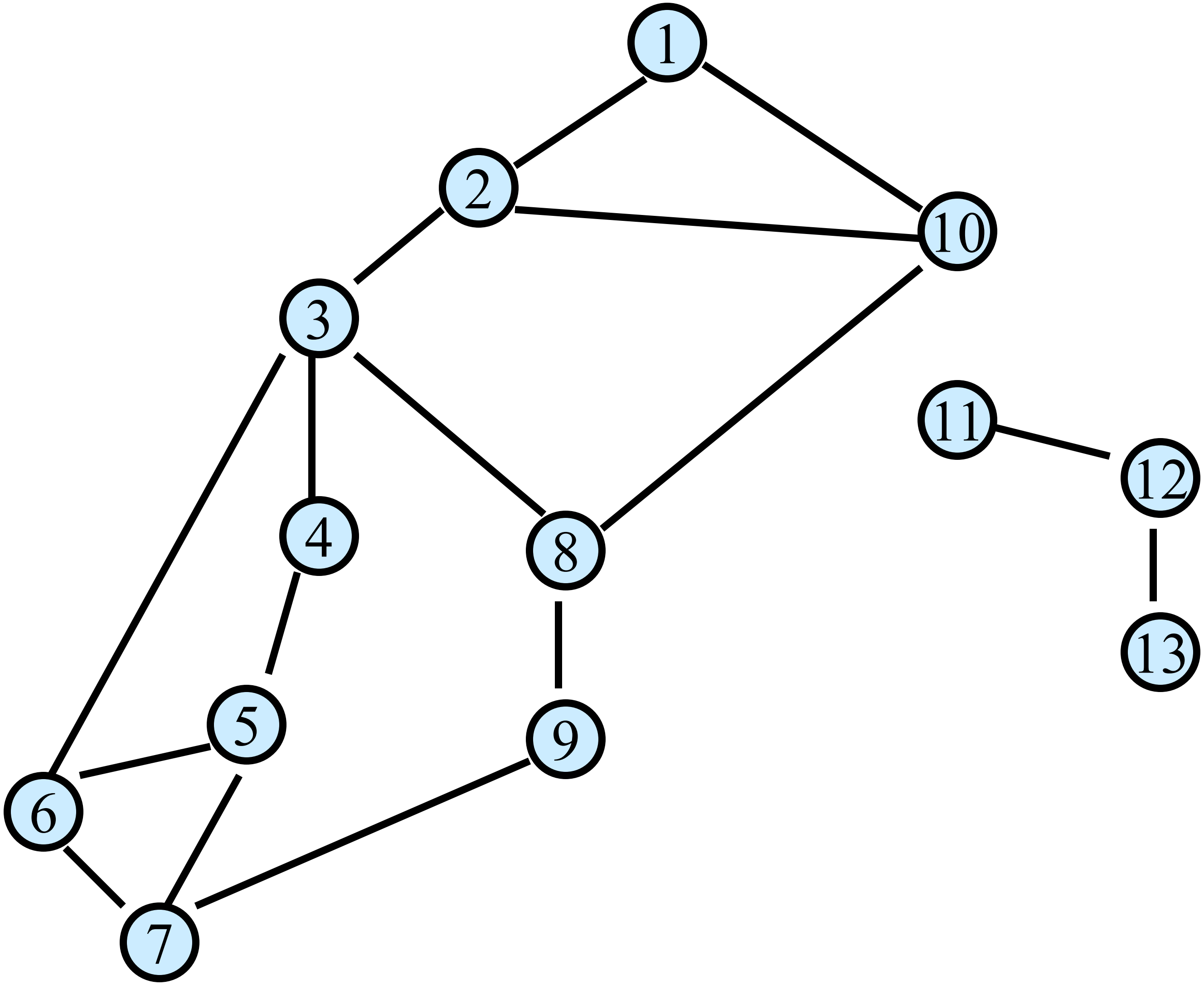
Relationships between pairs: "edges”

Formally, a graph G = (V, E) is a pair of sets, V the vertices and E the edges. Each edge is a set or tuple of two vertices.

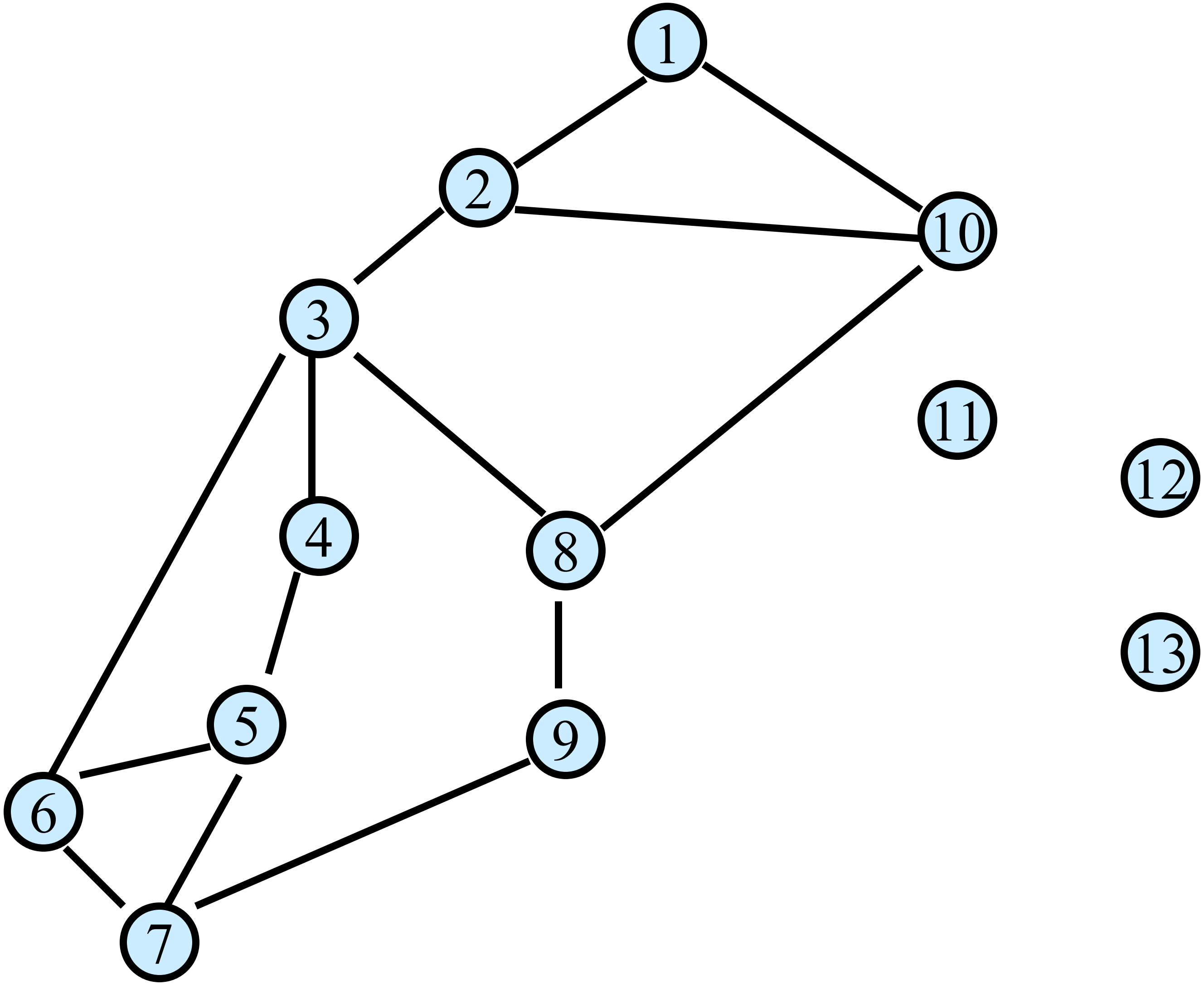
Undirected Graph G = (V,E)



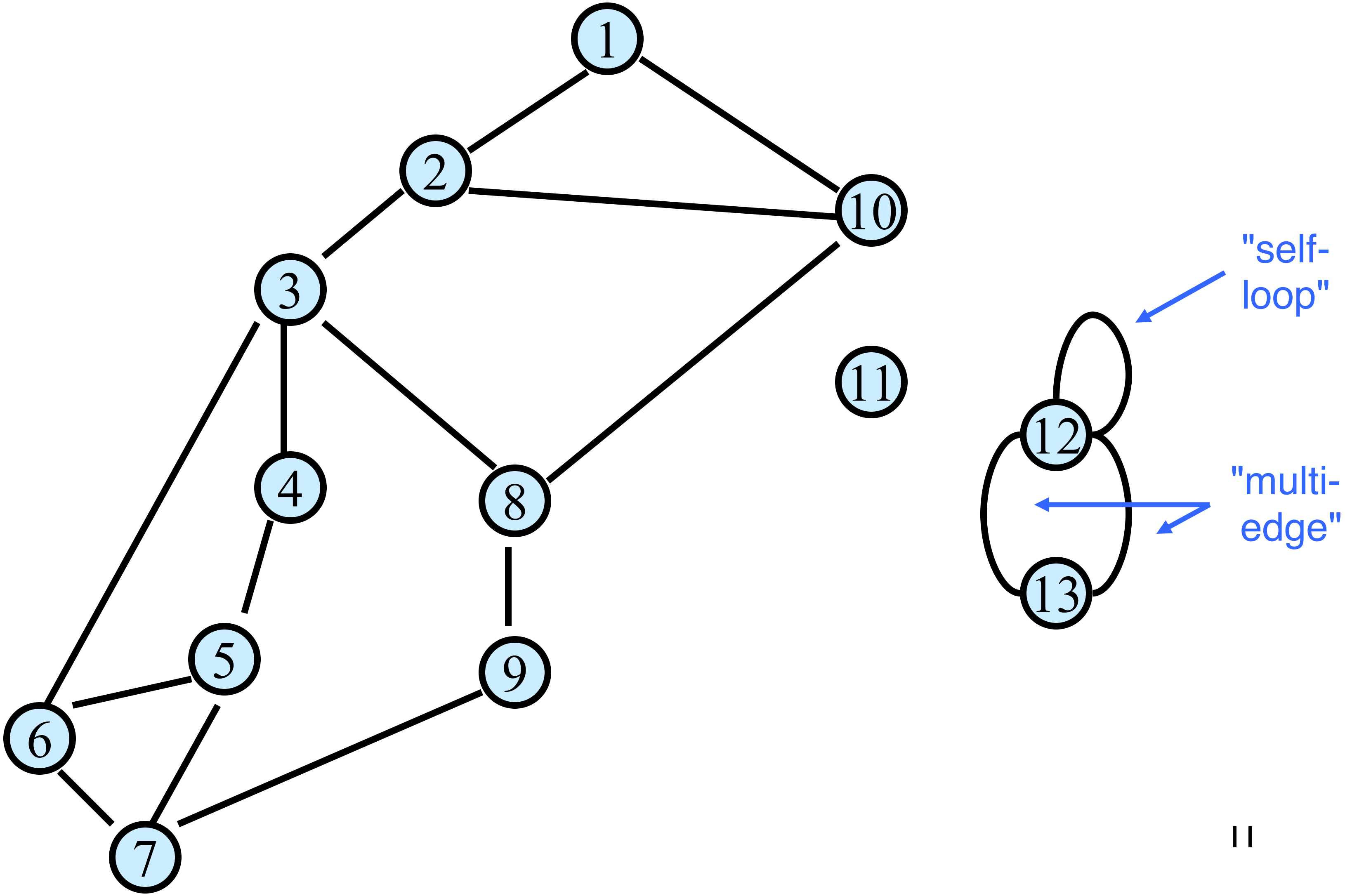
Undirected Graph G = (V,E)



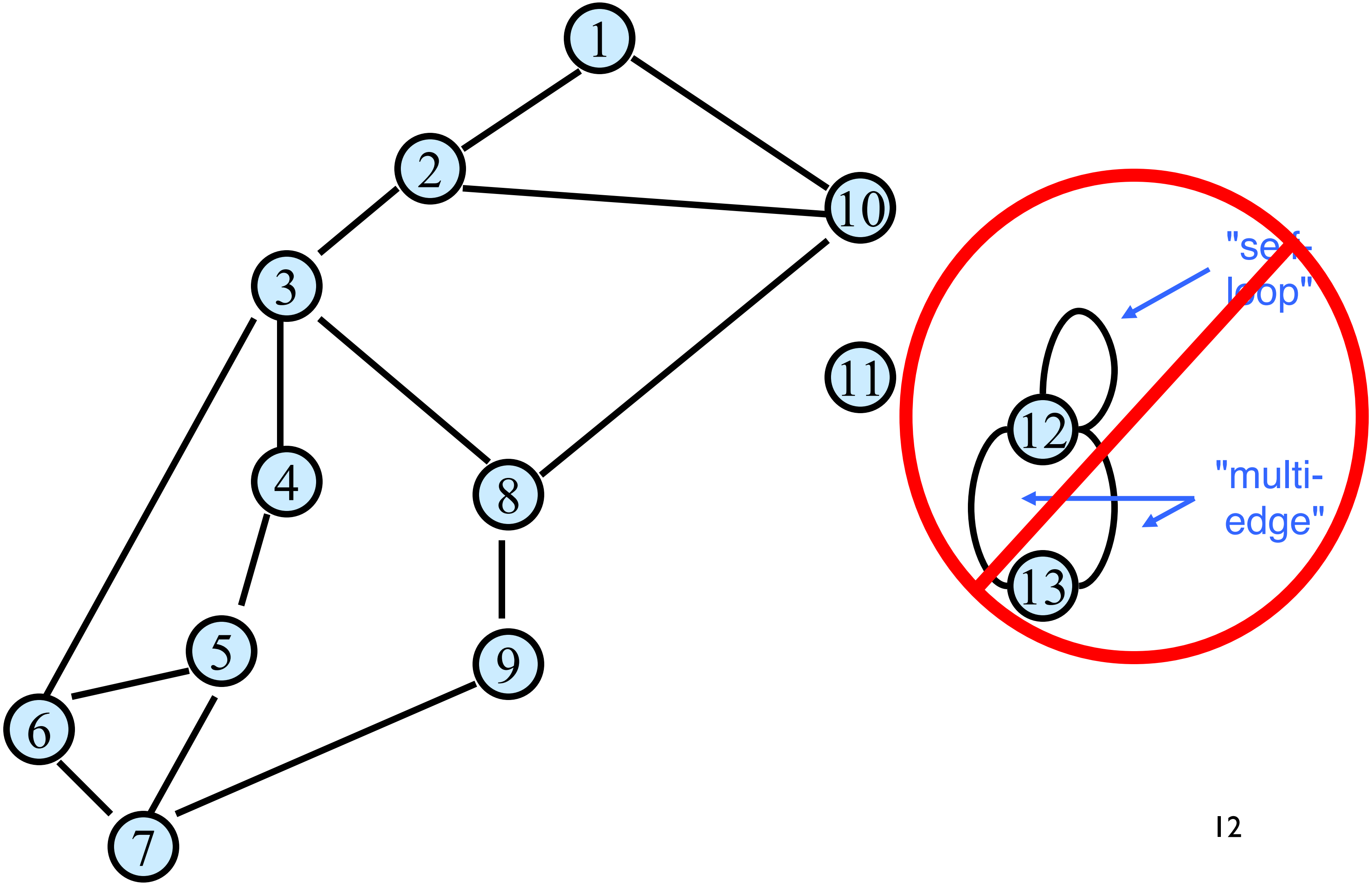
Undirected Graph G = (V,E)



Undirected Graph G = (V,E)

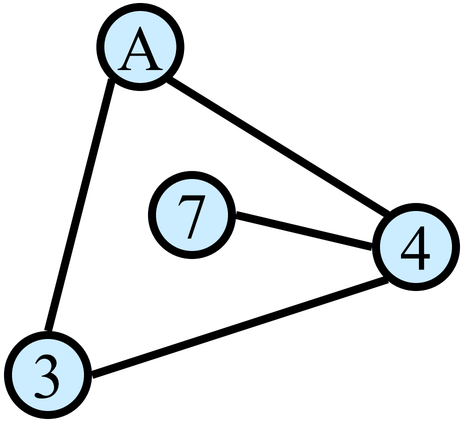


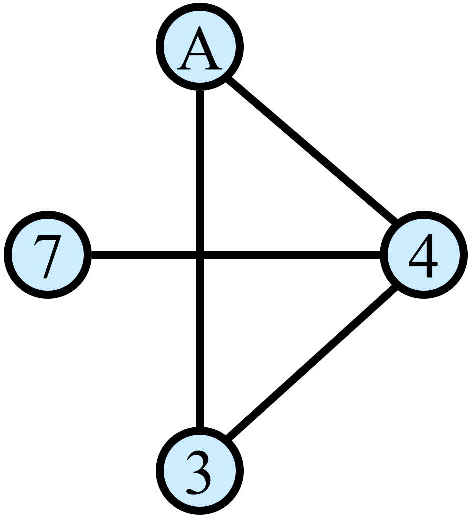
Undirected Graph G = (V,E)

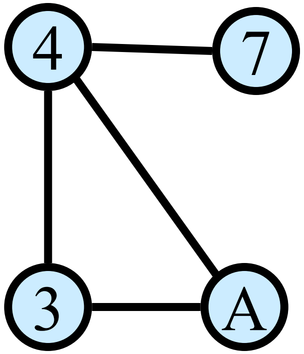


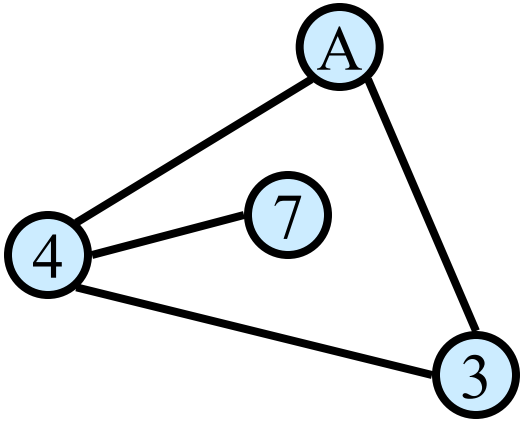
Graphs don't live in Flatland

Geometrical drawing is mentally convenient, but mathematically irrelevant: 4 drawings, 1 graph.

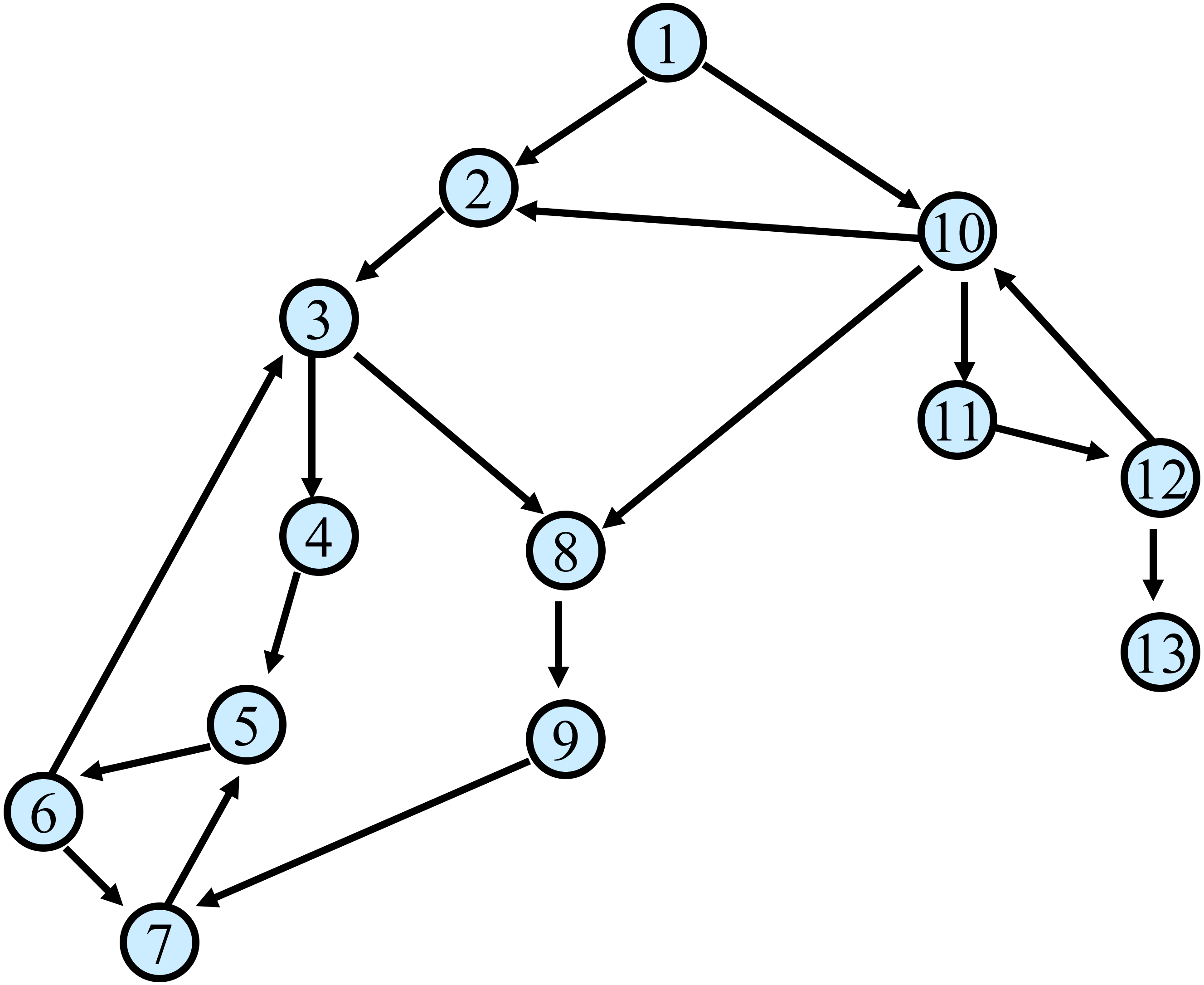




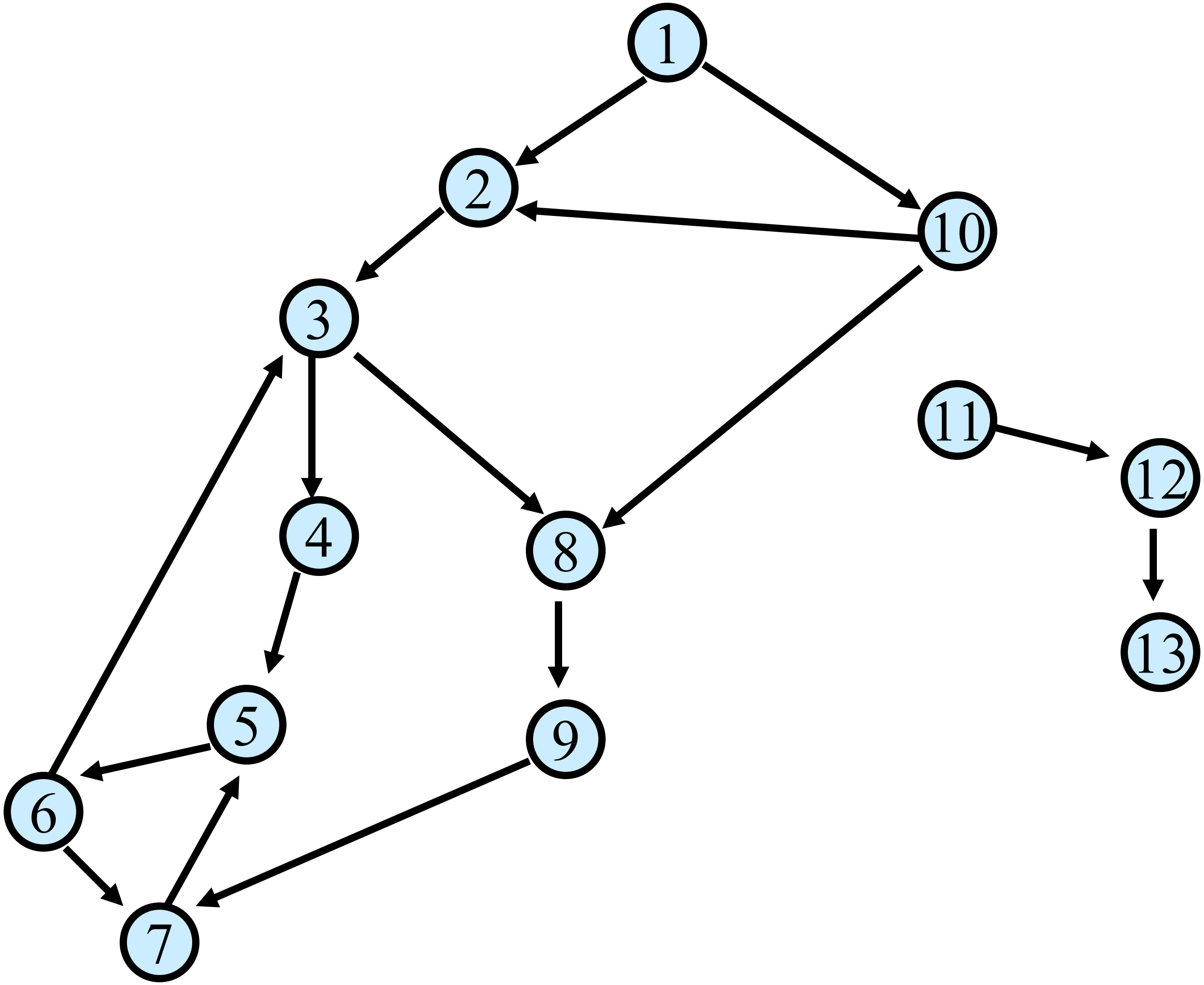




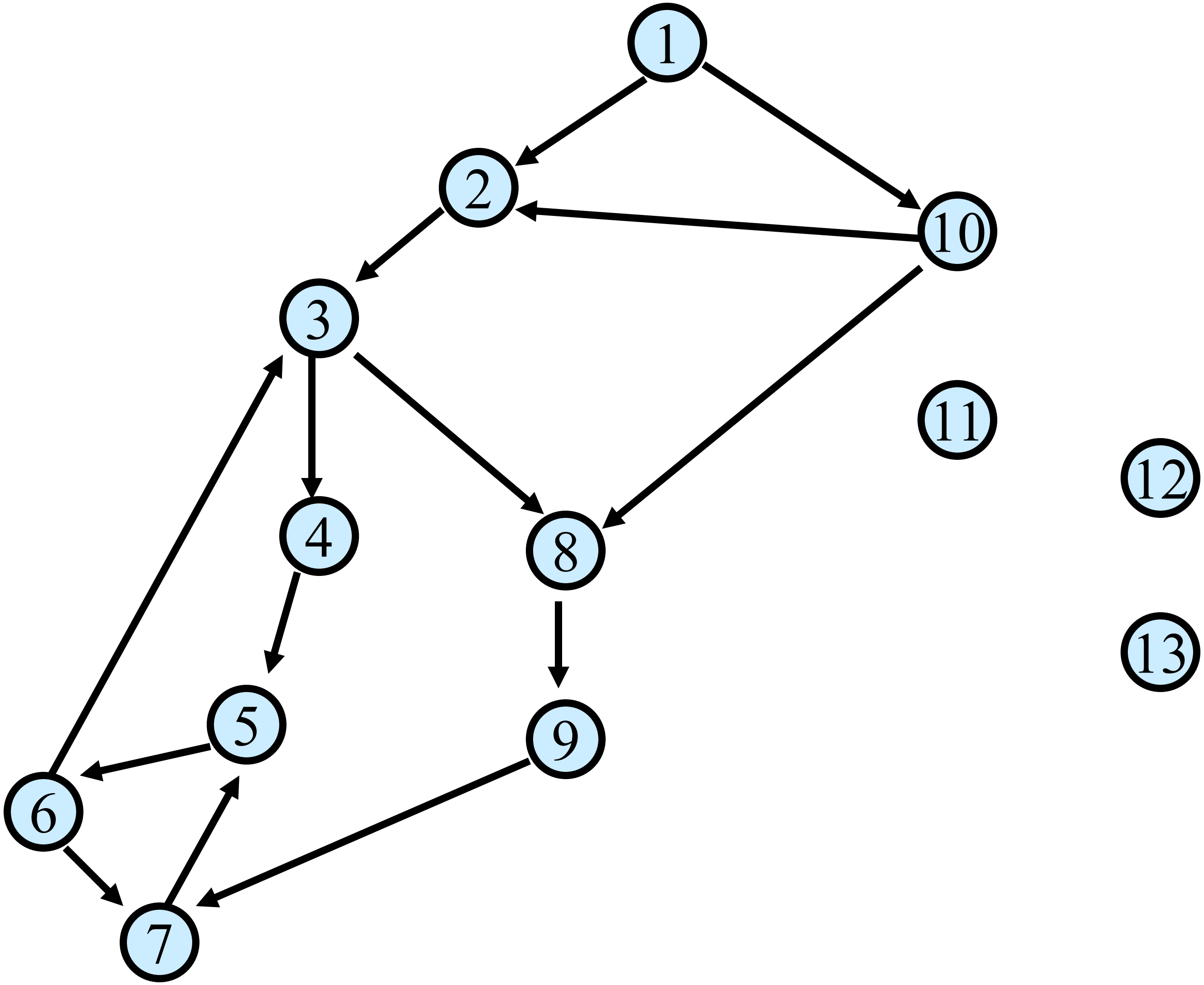
Directed Graph G = (V,E)



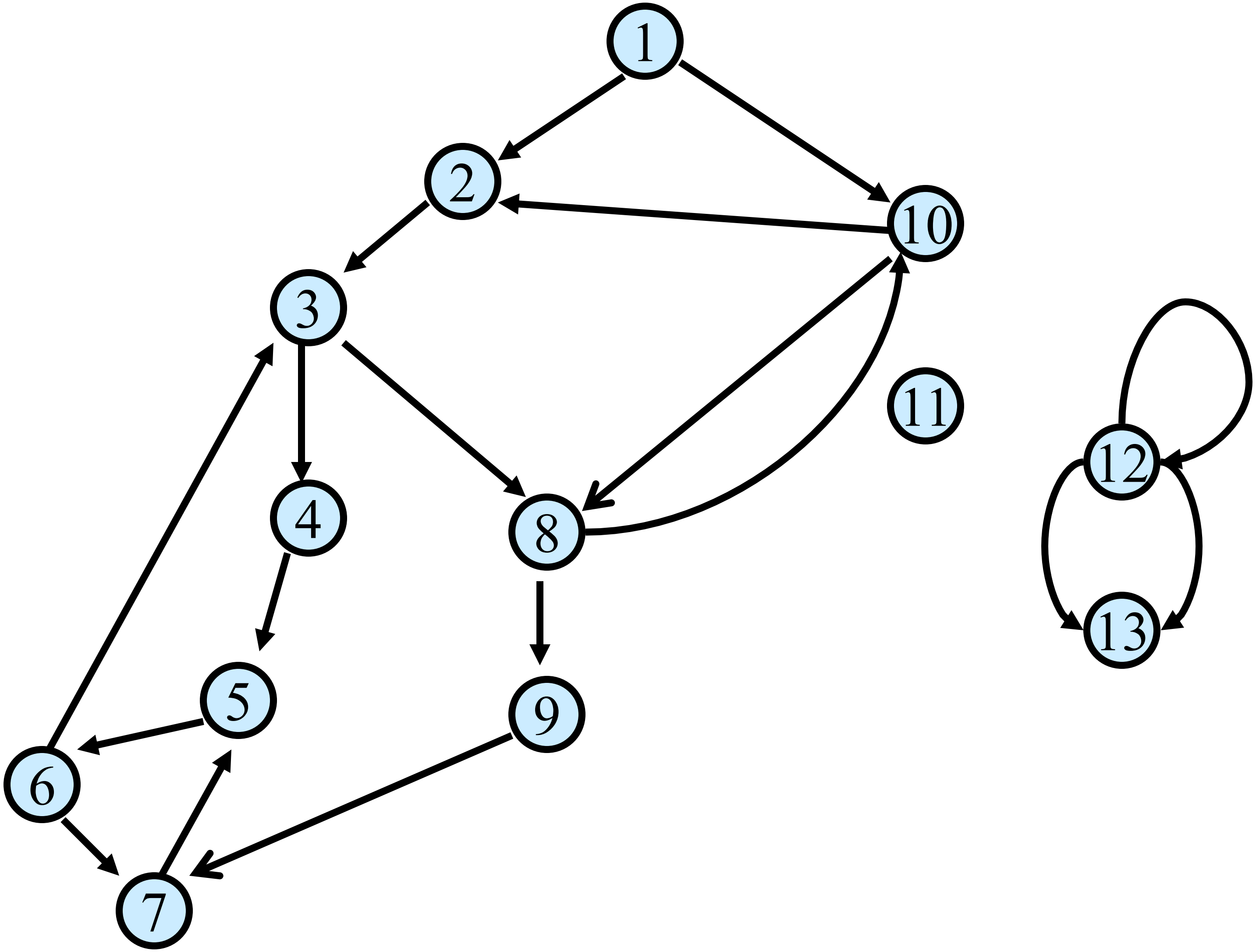
Directed Graph G = (V,E)



Directed Graph G = (V,E)



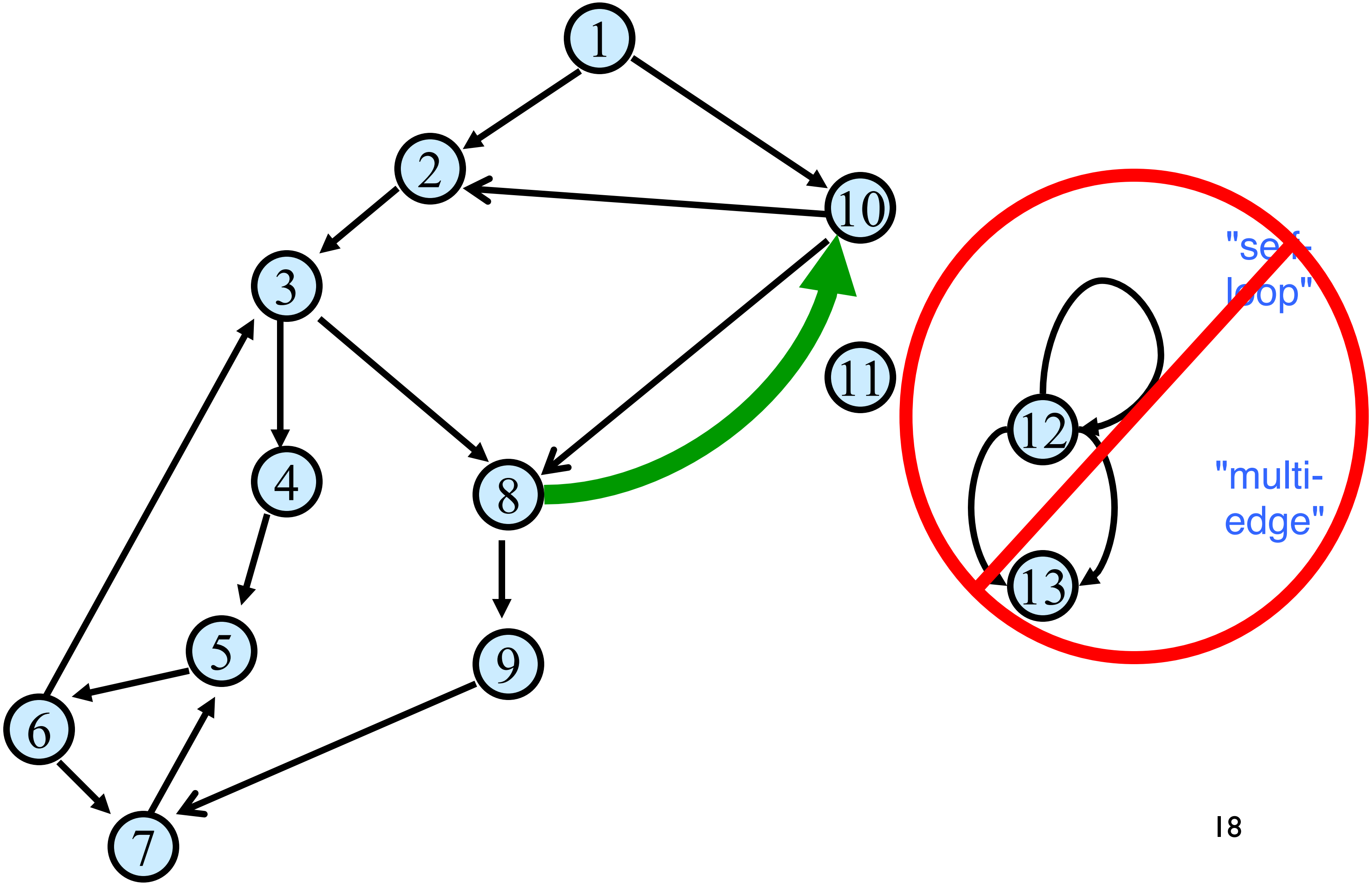
Directed Graph G = (V,E)



"self-loop"

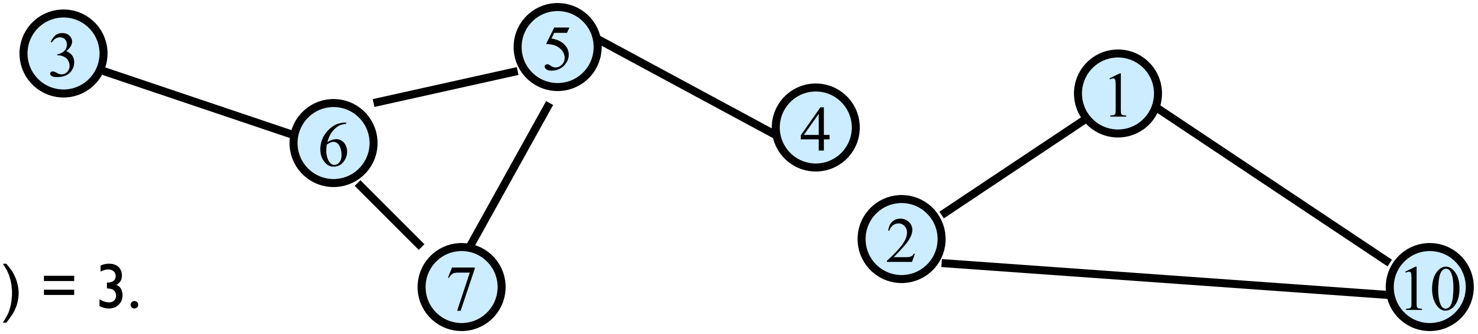
"multi-edge"

Directed Graph G = (V,E)



Graphs

Degree of a vertex, deg(v): # edges that touch that vertex



deg(6) = 3.

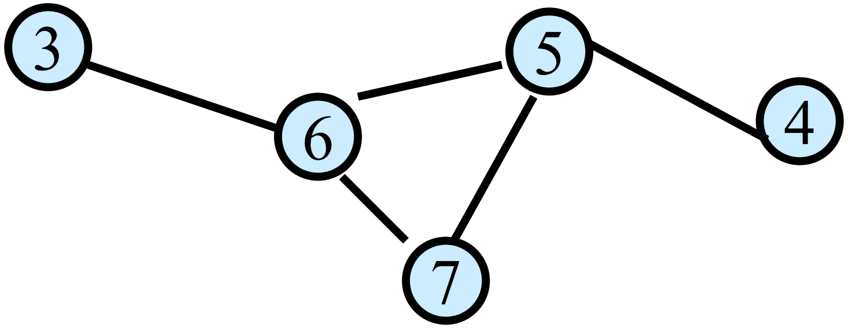
Path: sequence of distinct vertices s.t. each vertex is connected to the next vertex with an edge

Eg: 3,6,5,4

Connected: Graph is connected if there is a path between every two vertices

Connected component: Maximal set of connected vertices

Cycle: Path of length > 1 that has the same start and end. Eg: 6,5,7



Tree: A connected graph with no cycles

Vertices vs # Edges

Let G be an undirected graph with *n* vertices and *m* edges. How are *n* and *m* related?

Vertices vs # Edges

Let G be an undirected graph with *n* vertices and *m* edges. How are *n* and *m* related?

Since

every edge connects two different vertices (no loops), and no two edges connect the same two vertices (no multi-edges),

it must be true that:

$$0 \leq m \leq n(n-1)/2 = O(n^2)$$

More Cool Graph Lingo

A graph is called *sparse* if $m \ll n^2$, otherwise it is *dense*

- Boundary is somewhat fuzzy; $O(n)$ edges is certainly sparse, $\Omega(n^2)$ edges is dense.

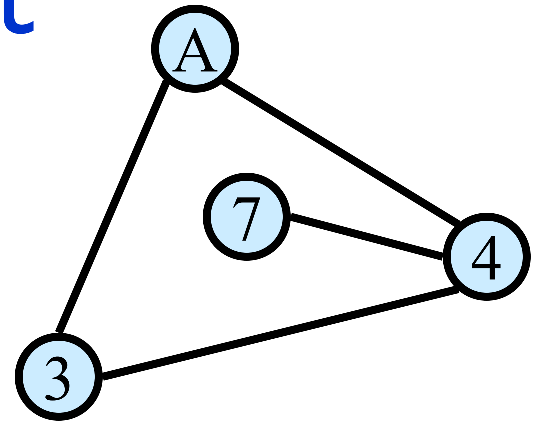
Sparse graphs are common in practice

- E.g., all planar graphs are sparse ($m \le 3n-6$, for $n \ge 3$)

Q: which is a better run time, $O(n+m)$ or $O(n^2)$?

A: $n+m = O(n^2)$, but $n+m$ usually way better!

Specifying undirected graphs as input



What are the vertices?

Explicitly list them: {"A", "7", "3", "4"}

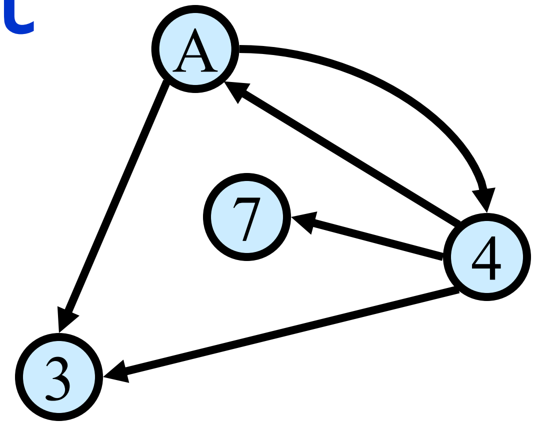
What are the edges?

Either, set of edges {{A,3}, {7,4}, {4,3}, {4,A}}

Or, (symmetric) adjacency matrix:

$$\begin{array}{c|cccc} & A & 7 & 3 & 4 \\ \hline A & 0 & 0 & 1 & 1 \\ 7 & 0 & 0 & 0 & 1 \\ 3 & 1 & 0 & 0 & 1 \\ 4 & 1 & 1 & 1 & 0 \end{array}$$

Specifying directed graphs as input



What are the vertices?

Explicitly list them:
{"A", "7", "3", "4"}

What are the edges?

Either, set of directed edges:
{(A,4), (4,7), (4,3), (4,A), (A,3)}

Or, (nonsymmetric) adjacency matrix:

$$\begin{array}{c|cccc} & A & 7 & 3 & 4 \\ \hline A & 0 & 0 & 1 & 1 \\ 7 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 \\ 4 & 1 & 1 & 1 & 0 \end{array}$$

Representing Graph $G = (V,E)$

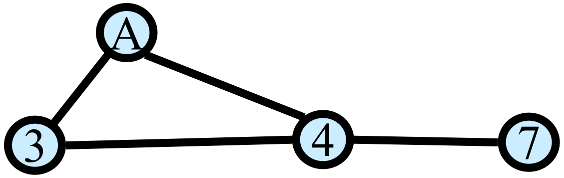
internally, indp of input format

Vertex set $V = \{v_1, \ldots, v_n\}$

Adjacency Matrix A

$A[i,j] = 1$ iff $(v_i,v_j) \in E$

Space is n^2 bits



	A	7	3	4
A	0	0	1	1
7	0	0	0	1
3	1	0	0	1
4	1	1	1	0

Advantages:

$O(1)$ test for presence or absence of edges.

Disadvantages: inefficient for sparse graphs, both in storage and access

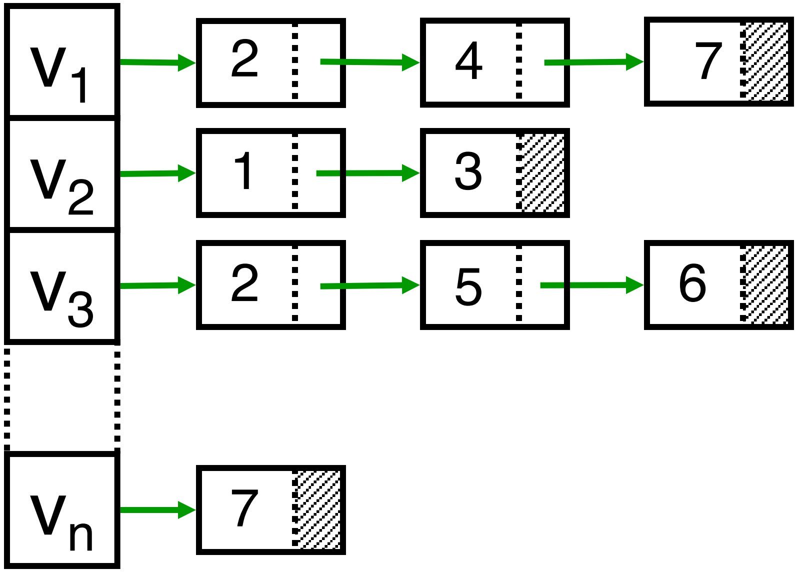
$m \ll n^2$

Representing Graph G=(V,E)

n vertices, m edges

Adjacency List:

O(n+m) words



Advantages:

- Compact for sparse graphs
- Easily see all edges

Disadvantages

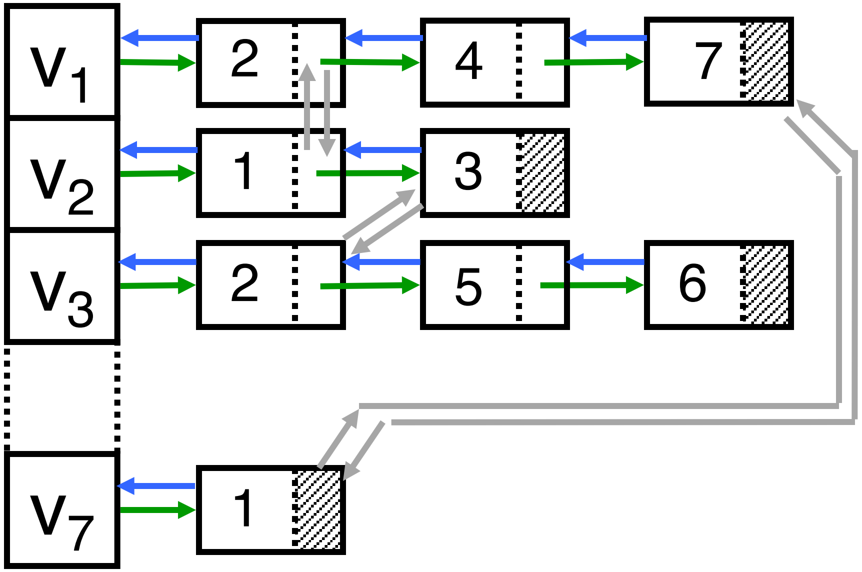
- More complex data structure
- no O(1) edge test

Representing Graph G=(V,E)

n vertices, m edges

Adjacency List:

O(n+m) words



Back- and cross pointers more work to build, but allow easier traversal and deletion of edges, *if needed*, (don't bother if not)

Graph Traversal

Learn the basic structure of a graph

"Walk," *via edges*, from a fixed starting vertex s to all vertices reachable from s

Being *orderly* helps. Two common ways:

- Breadth-First Search: order the nodes in successive layers based on distance from s
- Depth-First Search: more natural approach for exploring a maze; many efficient algs build on it.

Breadth-First Search

Completely explore the vertices in order of their distance from *s*

Naturally implemented using a queue

Graph Traversal: Implementation

Learn the basic structure of a graph

"Walk," via edges, from a fixed starting vertex s to all vertices reachable from s

Three states of vertices

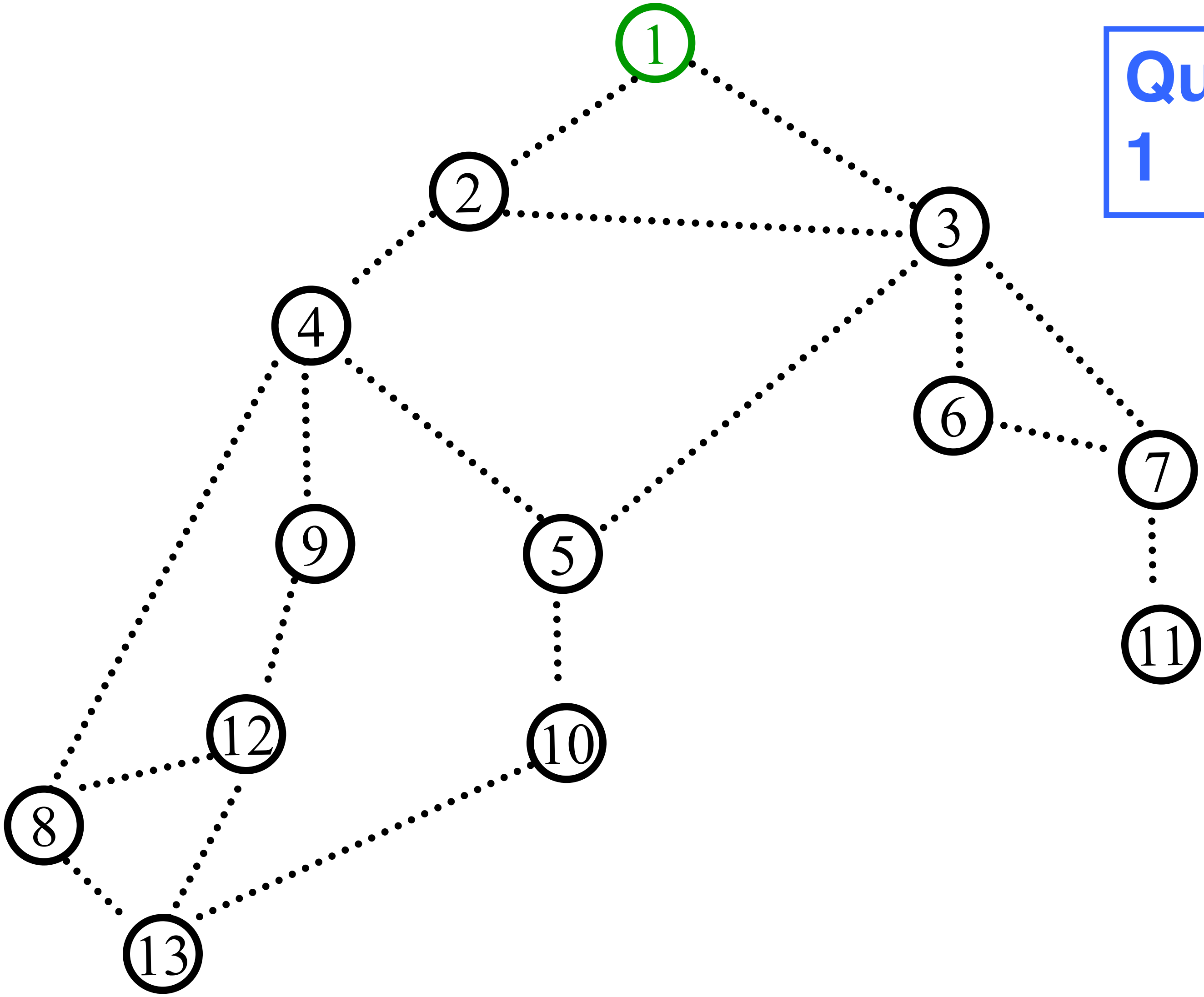
- *undiscovered*
- *discovered*
- *fully-explored*

BFS(s) Implementation

Global initialization: mark all vertices "undiscovered"

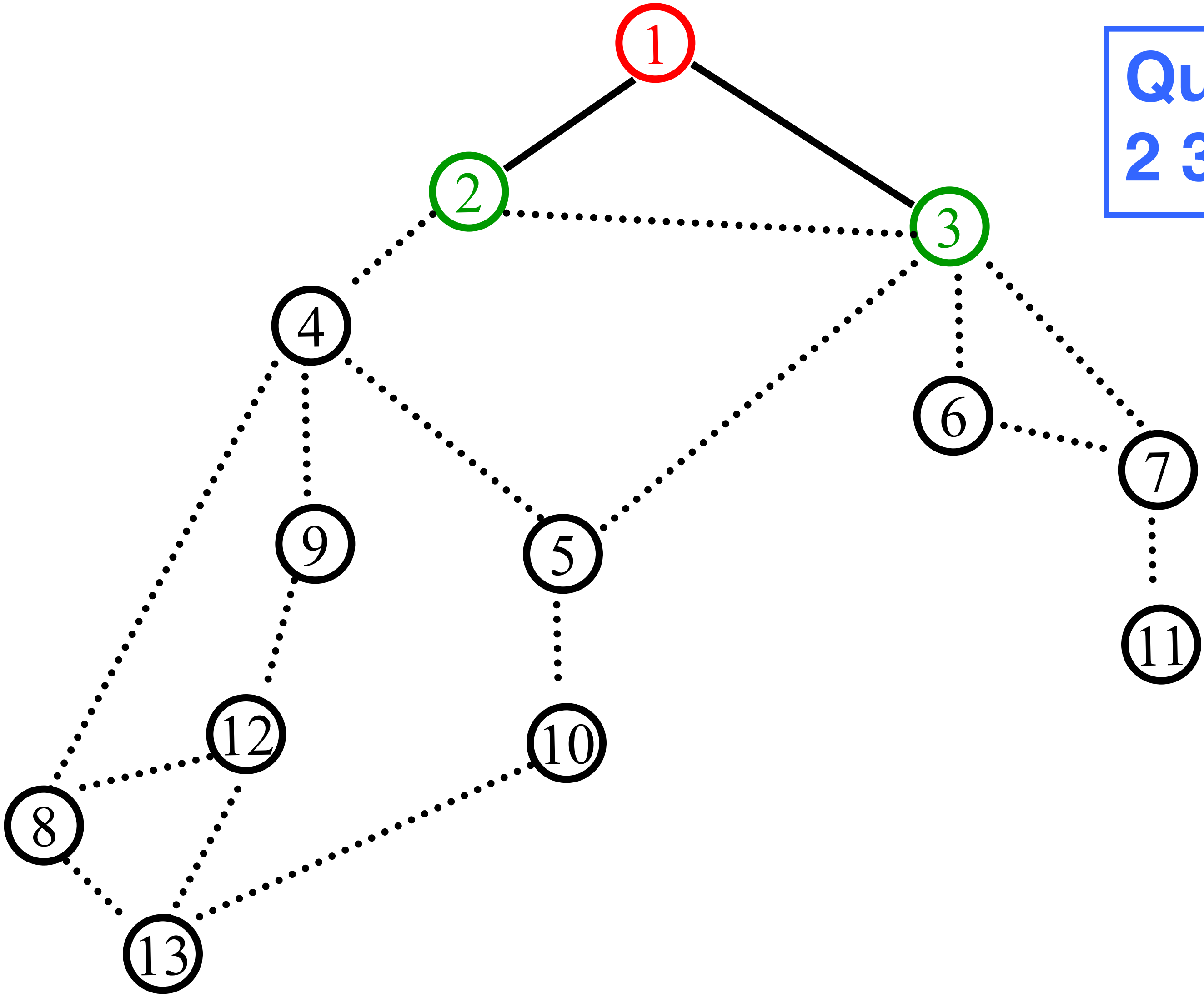
```
BFS(s)
    mark  s "discovered"
    queue = { s }
    while queue not empty
        u = remove_first(queue)
        for each edge {u,x}
            if (x is undiscovered)
                mark x discovered
                append x on queue
        mark u fully explored
```

BFS(v)



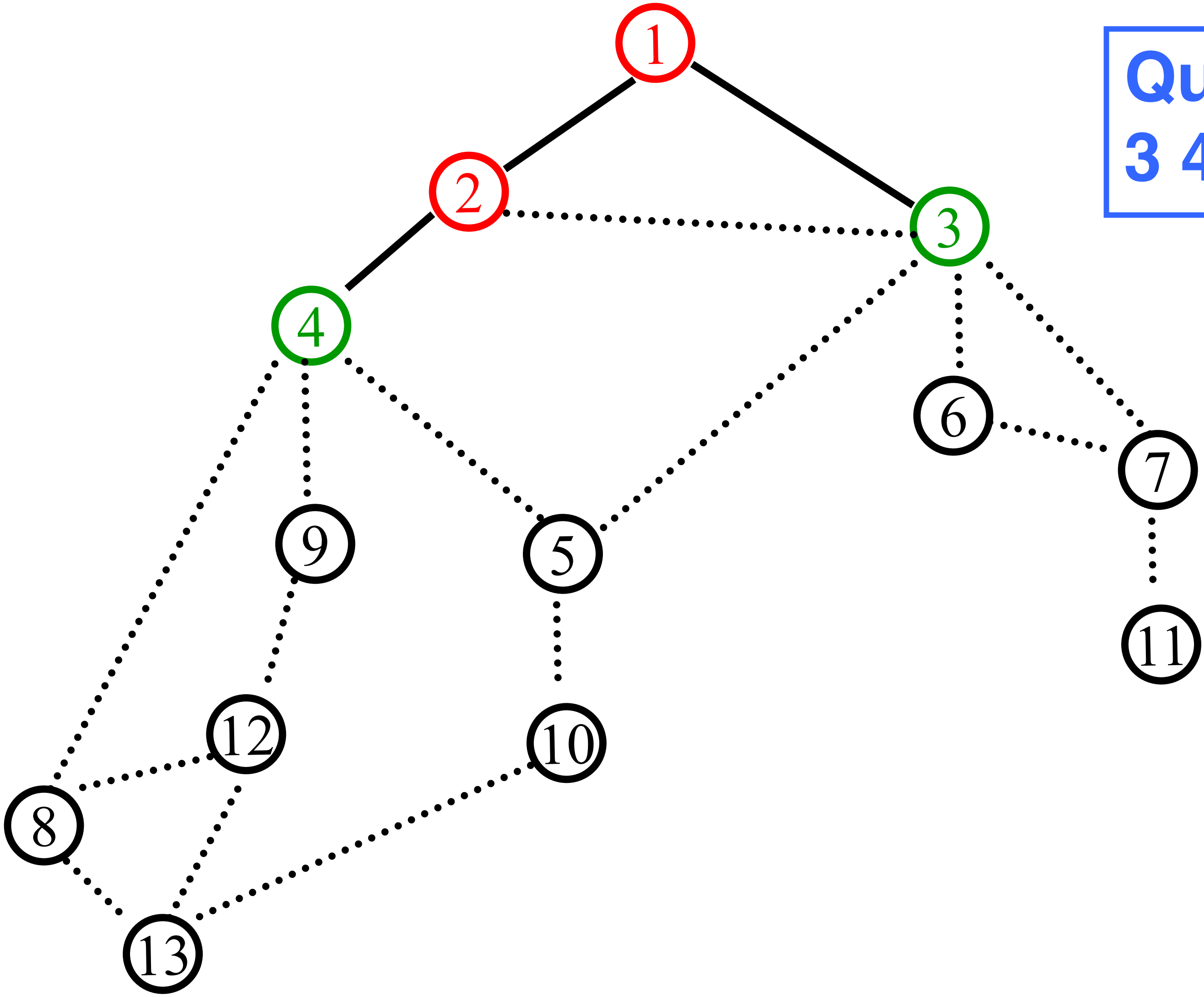
Queue:
1

BFS(v)



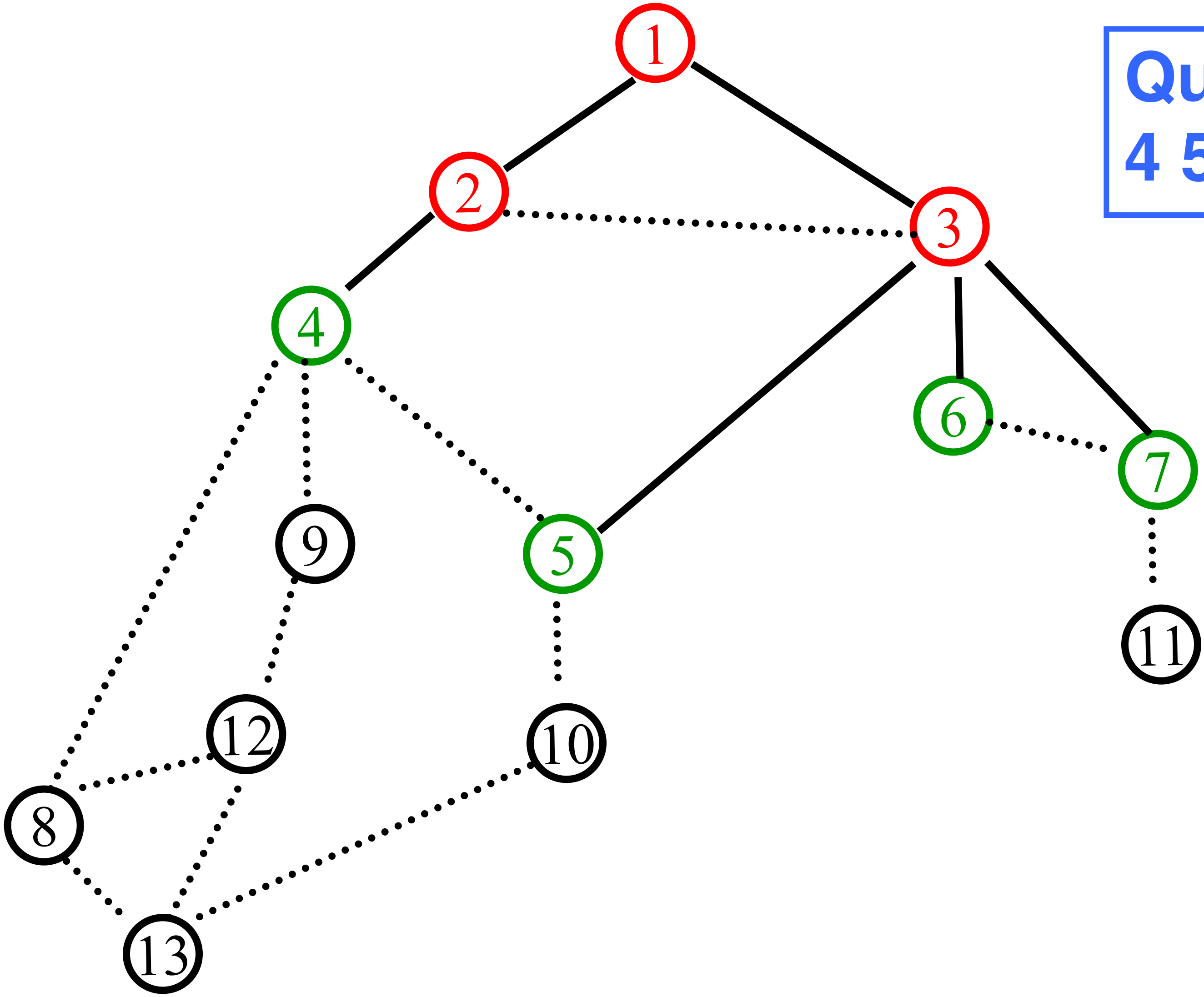
Queue:
2 3

BFS(v)



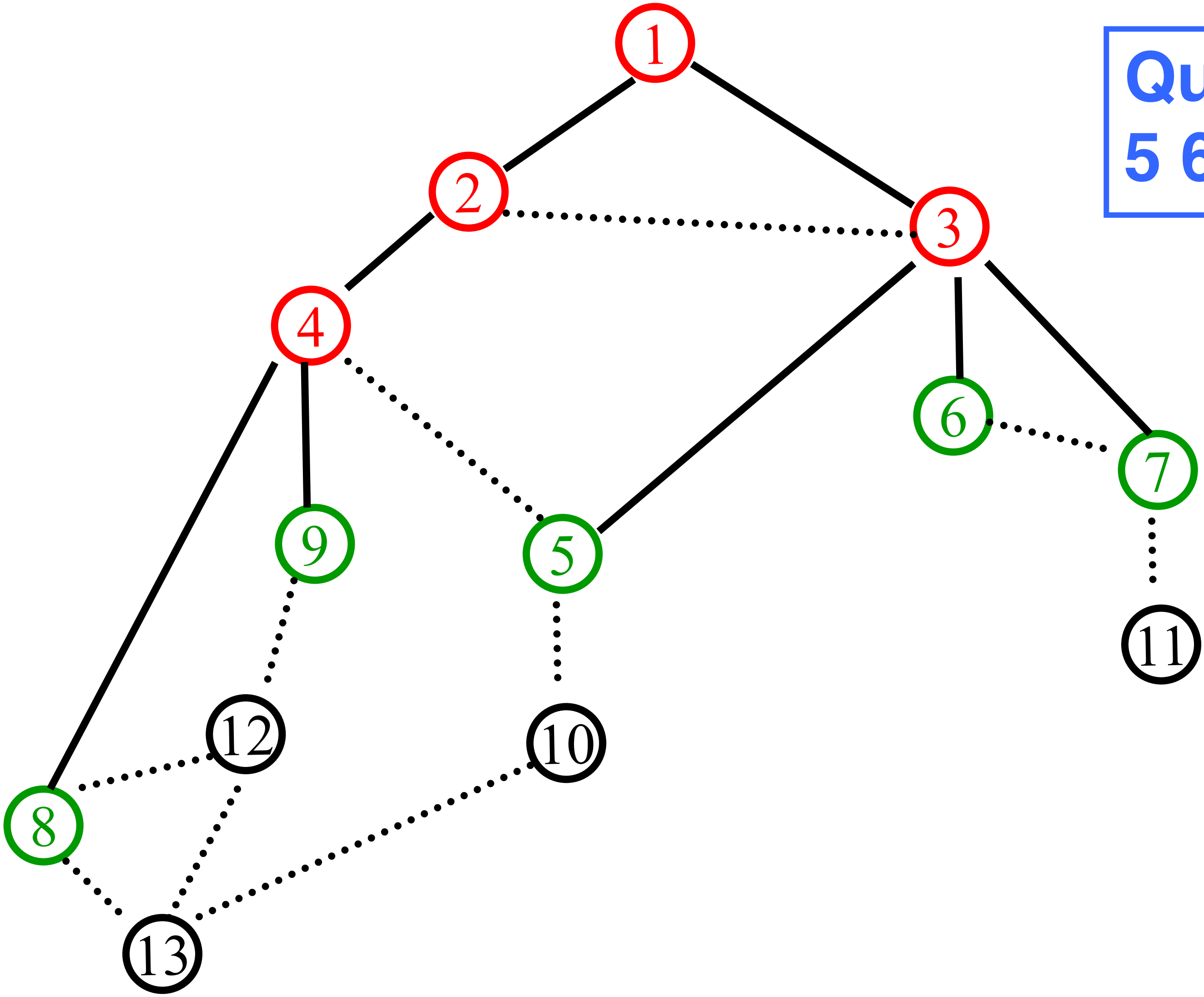
Queue:
3 4

BFS(v)



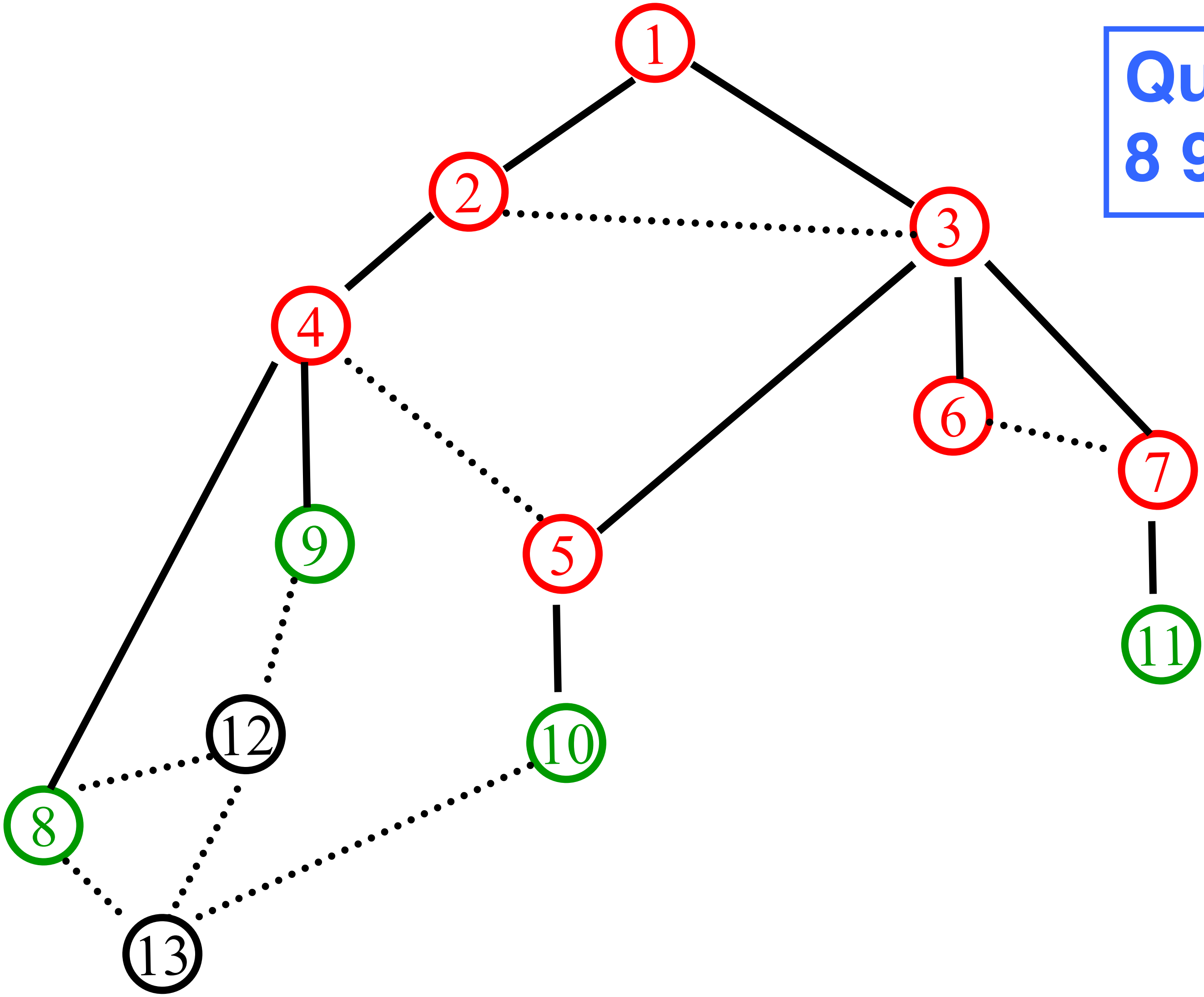
Queue:
4 5 6 7

BFS(v)



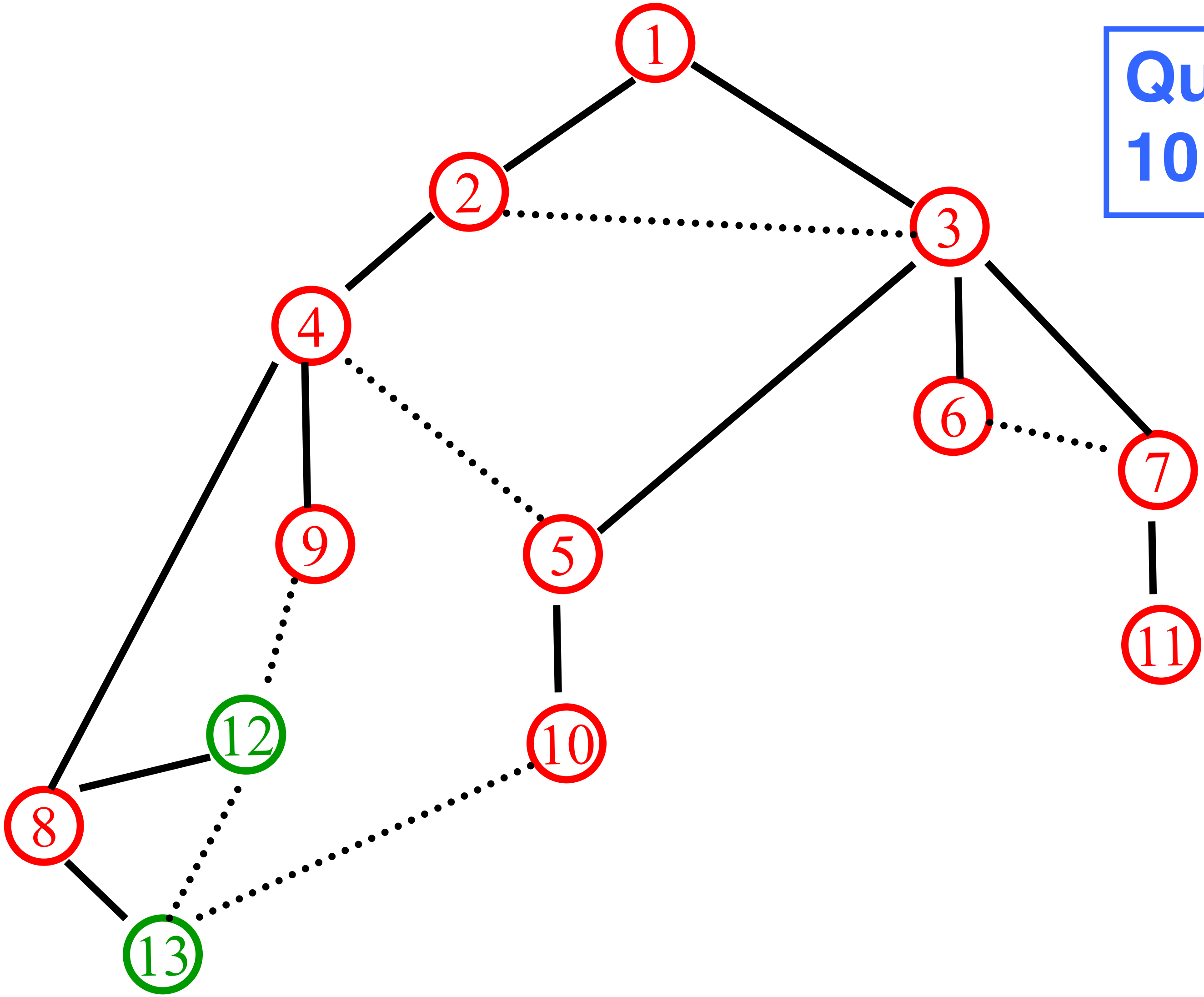
Queue:
5 6 7 8 9

BFS(v)



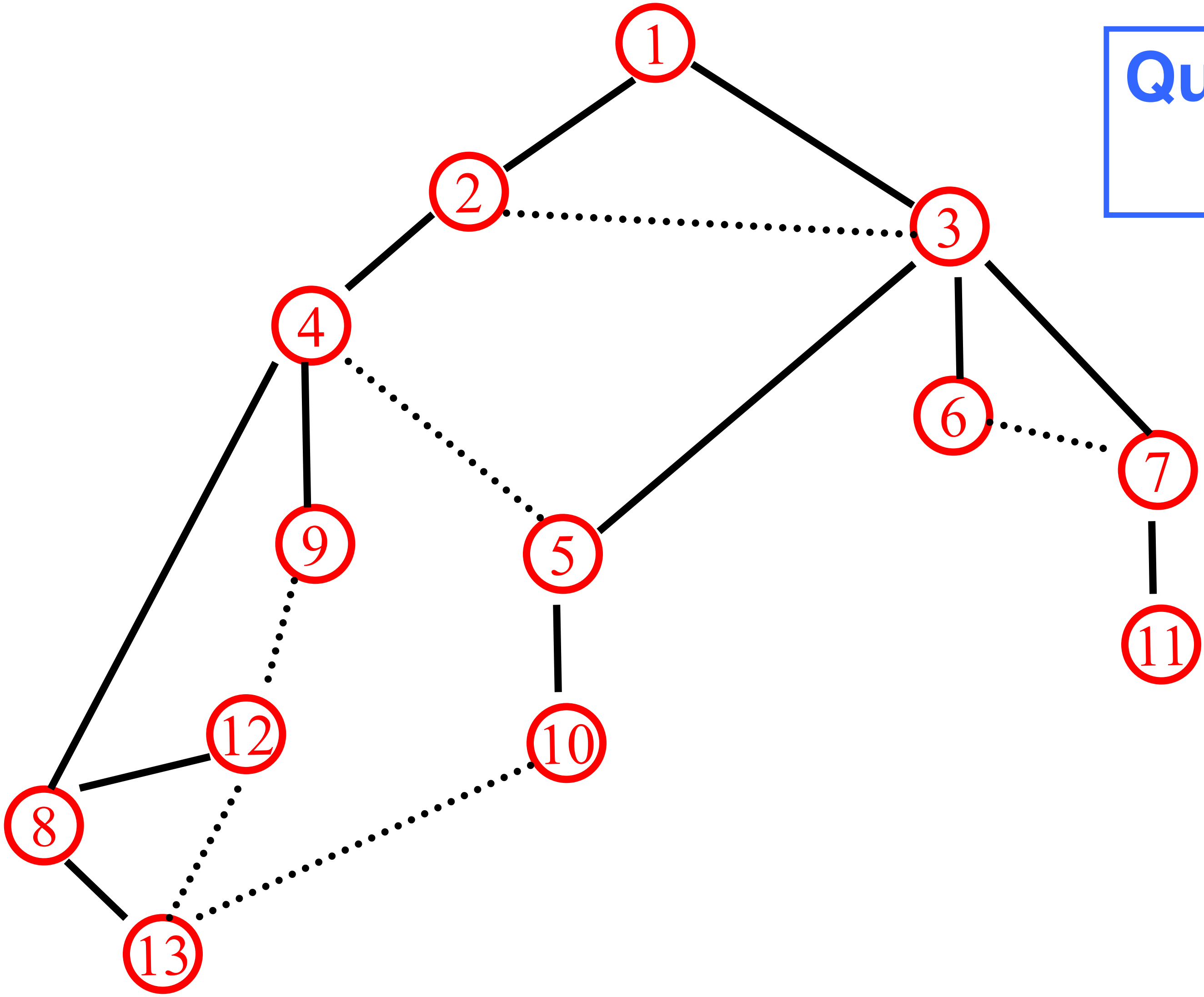
Queue:
8 9 10 11

BFS(v)



Queue:
10 11 12 13

BFS(v)



Queue:

BFS: Analysis, I

```
$O(n)$   Global initialization: mark all vertices "undiscovered"
  +      BFS(s)
$O(1)$       mark  s "discovered"
  +          queue = { s }
$O(n)$       while queue not empty
  x              u = remove_first(queue)
$O(n)$           for each edge {u,x}
                     if (x is undiscovered)
                         mark x discovered
                         append x on queue
  =              mark u fully explored
$O(n^2)$
```

Simple analysis:
2 nested loops.
Get worst-case number of iterations of each; multiply.

BFS: Analysis, II

Above analysis correct, but pessimistic (can't have $\Omega(n)$ edges incident to each of $\Omega(n)$ distinct "u" vertices if G is sparse). Alt, more global analysis:

Each edge is explored once from each end-point, so *total* runtime of inner loop is O(m).

Exercise: extend algorithm and analysis to non-connected graphs

Total O(n+m), n = # nodes, m = # edges

Properties of (Undirected) BFS(v)

BFS(v) visits x if and only if there is a path in G from v to x.

Edges into then-undiscovered vertices define a ***tree*** – the "breadth first spanning tree" of G

Level i in this tree are exactly those vertices *u* such that the shortest path (in G, not just the tree) from the root v is of length i.

All non-tree edges join vertices on the same or adjacent levels

not true of every spanning tree!

Proof of correctness

Lemma 1: Every vertex at level i is explored after every vertex at level i-1.

Proof is by induction on i.

Base case: i = 1. True.

Induction step: Let u be at level i, and v be at level i-1. Since we use a queue, it is enough to prove that u is added to the queue after v. But u was added when a vertex at level i-1 was explored, and v is added when a vertex of level i-2 was explored. So u is added after v by induction.

Proof of correctness

Lemma 2: Level i in this tree are exactly those vertices *u* such that the shortest path (in G, not just the tree) from the root is of length i.

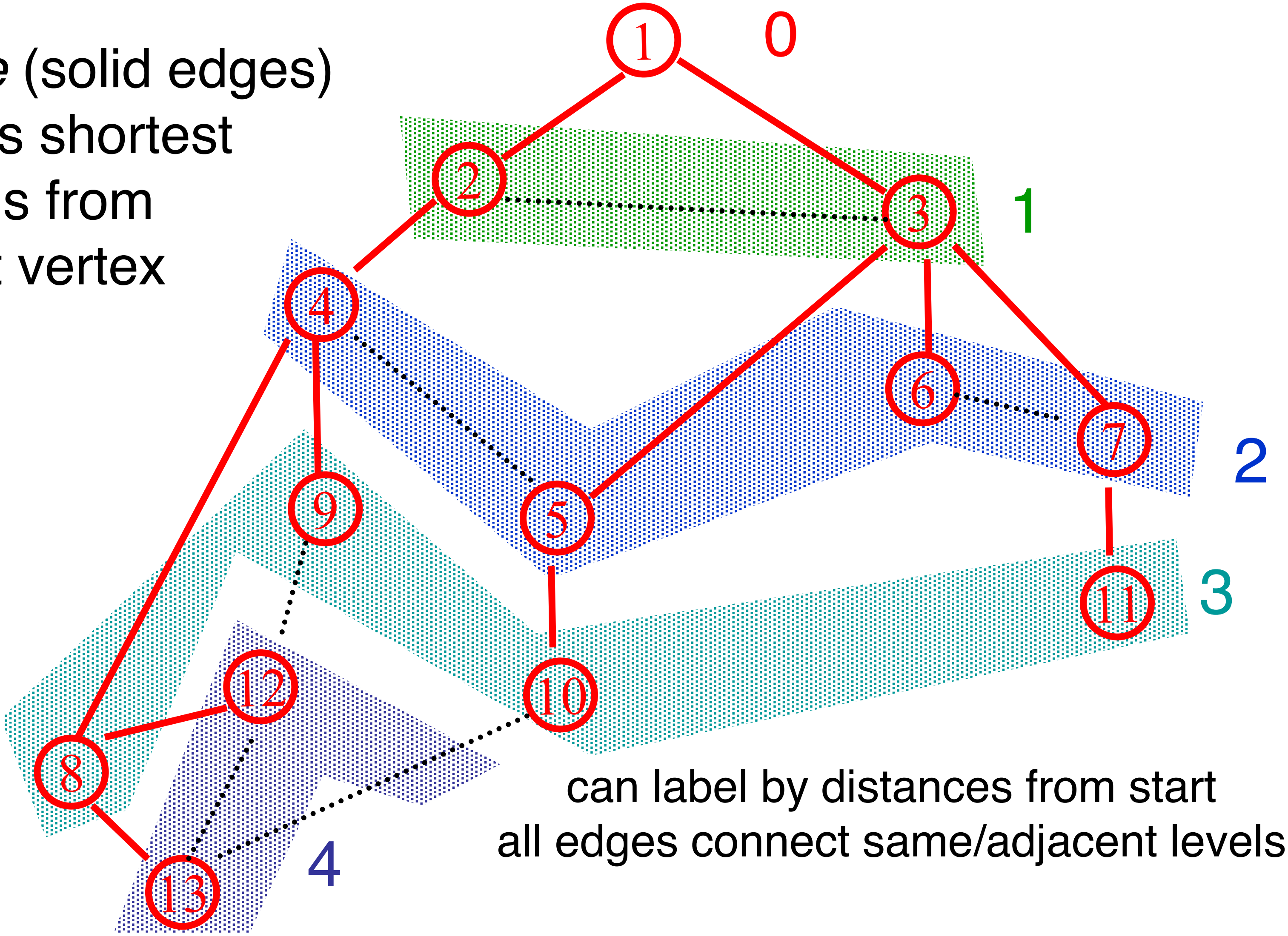
Proof is by induction on i.

Base case: i = 0. True.

Induction step: Every vertex u at level i certainly has distance at most i, because we discover a path of length i from u to v. If the distance from the root is less than i, and u was discovered when exploring v (at level i-1), then u is a neighbor of a vertex b at distance (and level) < i-1. But then, by Lemma 1, b would have been explored before v, and u would have been added in level i-1.

BFS Application: Shortest Paths

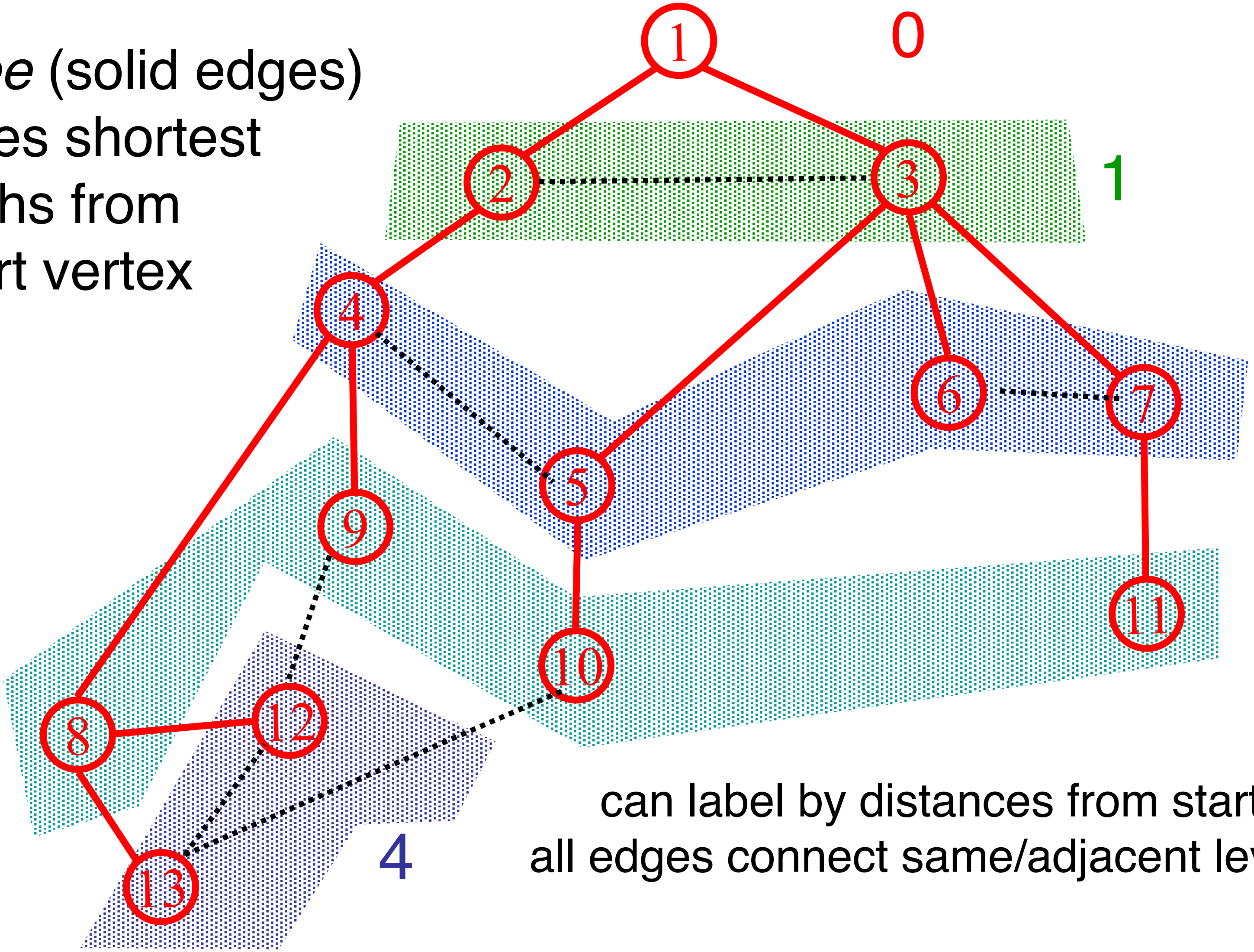
Tree (solid edges)
gives shortest
paths from
start vertex



can label by distances from start
all edges connect same/adjacent levels

BFS Application: Shortest Paths

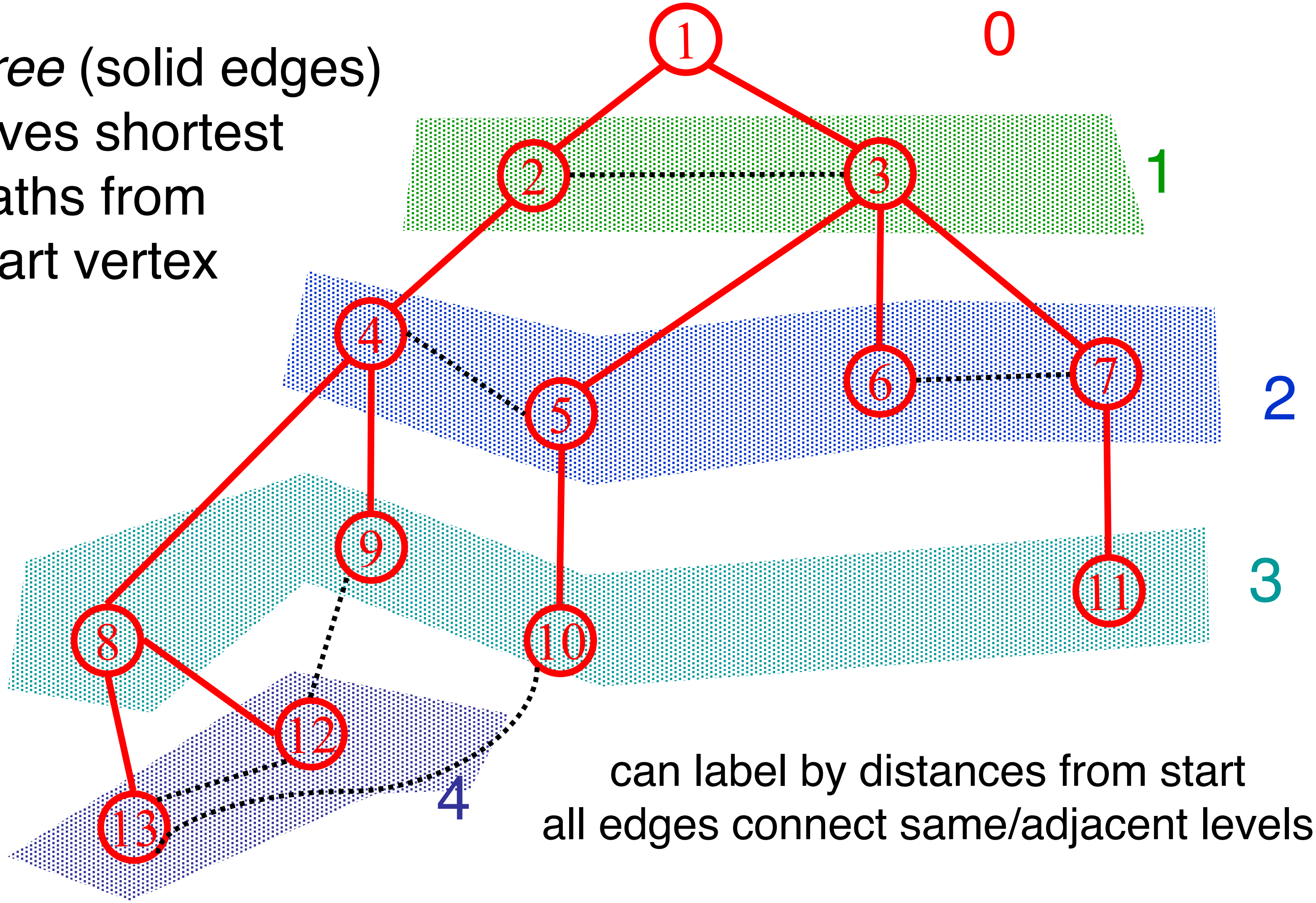
Tree (solid edges) gives shortest paths from start vertex



can label by distances from start
all edges connect same/adjacent levels

BFS Application: Shortest Paths

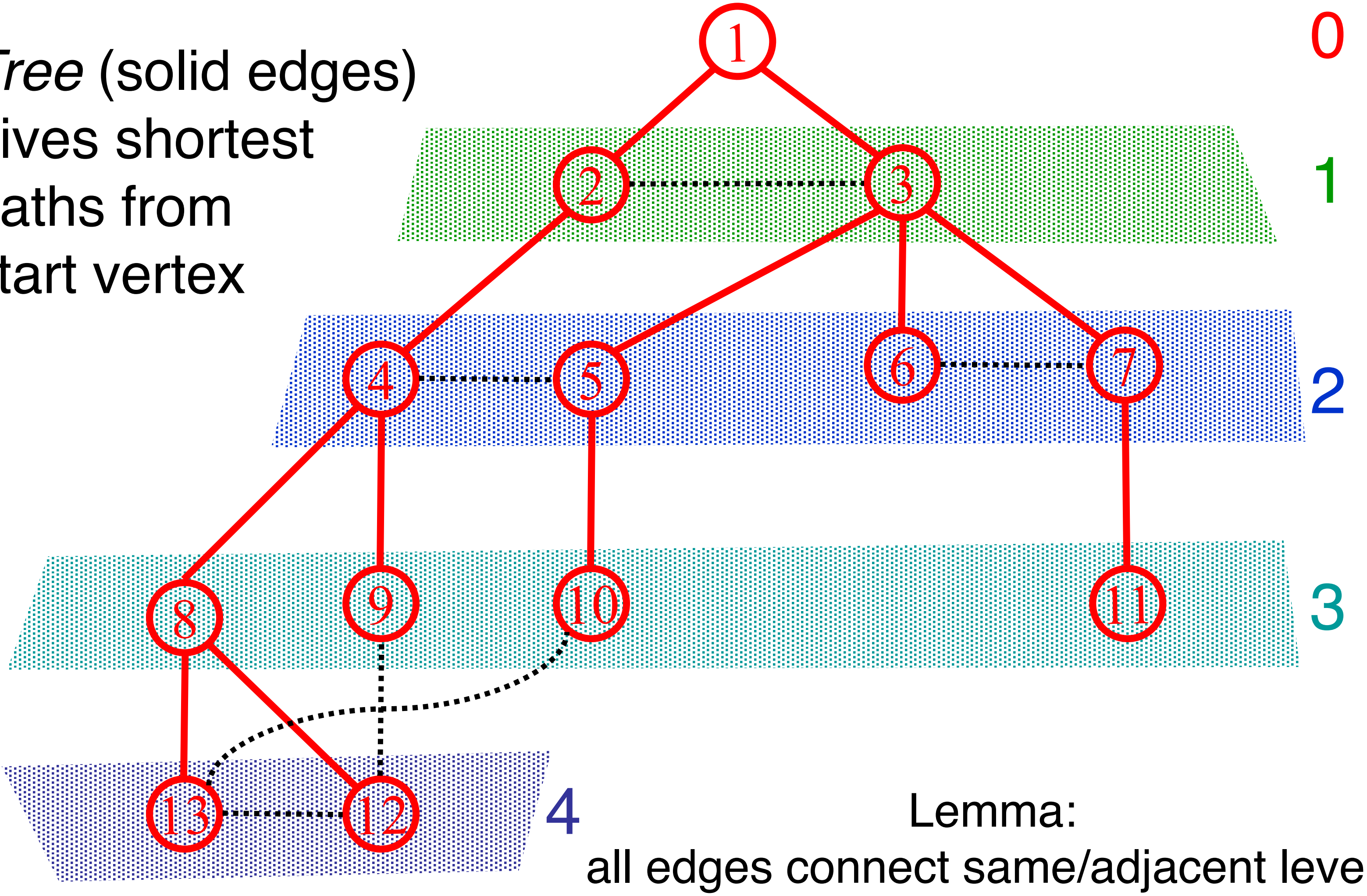
Tree (solid edges) gives shortest paths from start vertex



can label by distances from start
all edges connect same/adjacent levels

BFS Application: Shortest Paths

Tree (solid edges) gives shortest paths from start vertex



Lemma:
all edges connect same/adjacent levels

Why fuss about trees?

Trees are simpler than graphs

Ditto for algorithms on trees vs algs on graphs

So, this is often a good way to approach a graph problem: find a "nice" tree in the graph, i.e., one such that non-tree edges have some simplifying structure

E.g., BFS finds a tree s.t. level-jumps are minimized

DFS (below) finds a different tree, but it also has interesting structure…

Graph Search Application: Connected Components

Want to answer questions of the form:

given vertices u and v, is there a
path from u to v?

Set up one-time data structure to answer such questions efficiently.

Graph Search Application: Connected Components

Want to answer questions of the form:

given vertices u and v, is there a path from u to v?

Idea: create array A such that

A[u] = smallest numbered vertex that is connected to u. Question reduces to whether A[u]=A[v]?

Graph Search Application: Connected Components

```
initial state: all v undiscovered
for v = 1 to n do
    if state(v) != fully-explored then
        BFS(v): setting A[u] = v for each u found
        (and marking u discovered/fully-explored)
    endif
endfor
```

Total cost: O(n+m)

- each edge is touched a constant number of times (twice)
- works also with DFS

3.4 Testing Bipartiteness

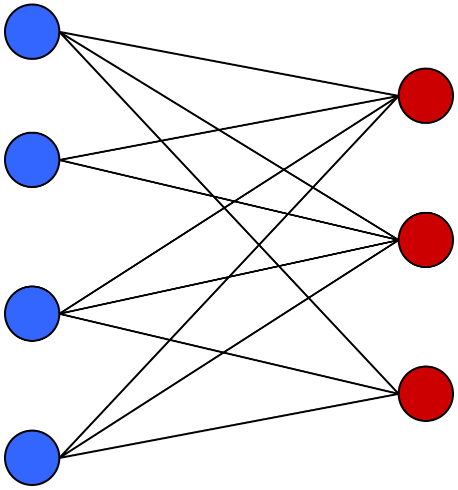
Bipartite Graphs

Def. An undirected graph G = (V, E) is *bipartite (2-colorable)* if the nodes can be colored red or blue such that no edge has both ends the same color.

Applications.

Stable marriage: men = red, women = blue
Scheduling: machines = red, jobs = blue

"bi-partite" means "two parts." An equivalent definition: G is bipartite if you can partition the node set into 2 parts (say, blue/red or left/right) so that all edges join nodes in different parts/no edge has both ends in the same part.

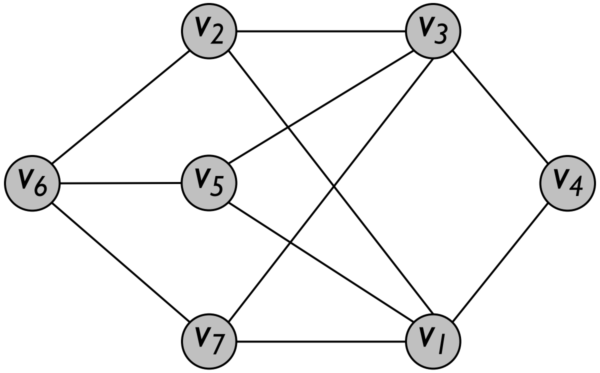


a bipartite graph

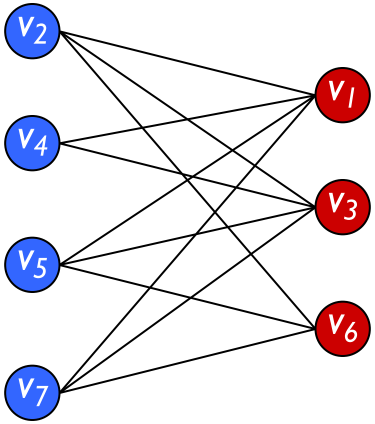
Testing Bipartiteness

Testing bipartiteness. Given a graph G, is it bipartite?

- Many graph problems become:
  - easier if the underlying graph is bipartite (matching)
  - tractable if the underlying graph is bipartite (independent set)
- Before attempting to design an algorithm, we need to understand structure of bipartite graphs.



a bipartite graph G

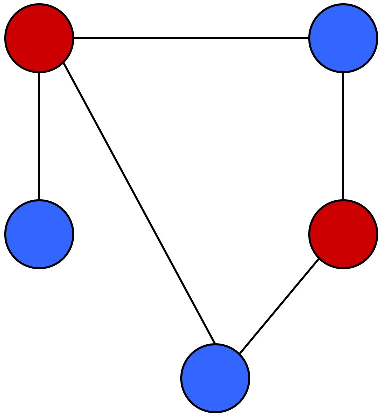


another drawing of G

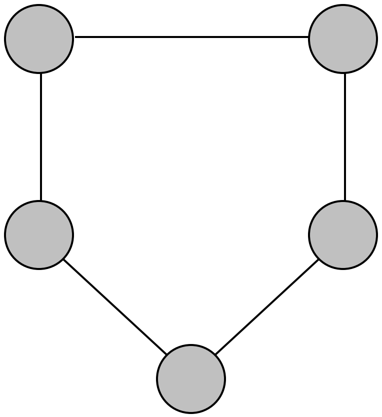
An Obstruction to Bipartiteness

Lemma. If a graph G is bipartite, it cannot contain an odd length cycle.

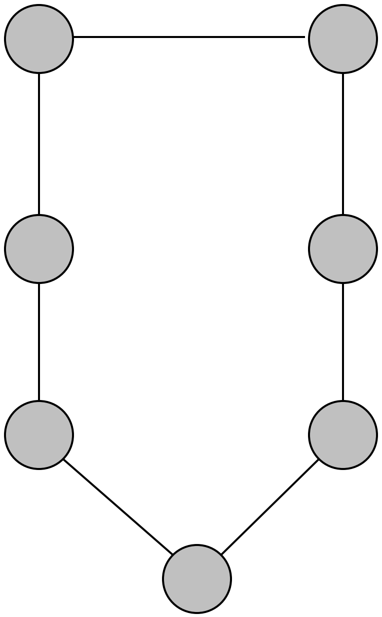
Pf. Impossible to 2-color the odd cycle, let alone G.



bipartite
(2-colorable)



not bipartite
(not 2-colorable)



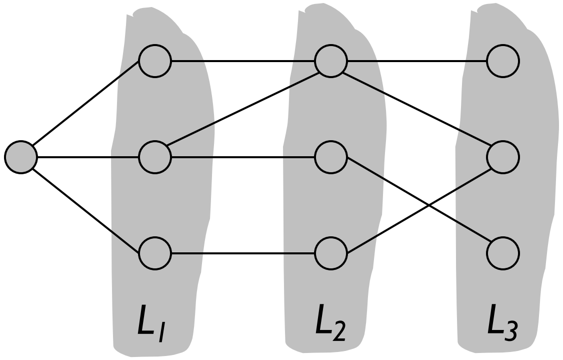
not bipartite
(not 2-colorable)

Bipartite Graphs

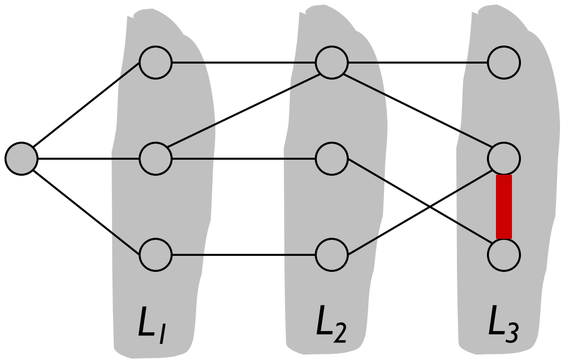
Lemma. Let G be a connected graph, and let L_0, …, L_k be the layers produced by BFS starting at node s. Exactly one of the following holds.

(i) No edge of G joins two nodes of the same layer, and G is bipartite.

(ii) An edge of G joins two nodes of the same layer, and G contains an odd-length cycle (and hence is not bipartite).



Case (i)



Case (ii)

Bipartite Graphs

Lemma. Let G be a connected graph, and let L_0, …, L_k be the layers produced by BFS starting at node s. Exactly one of the following holds.

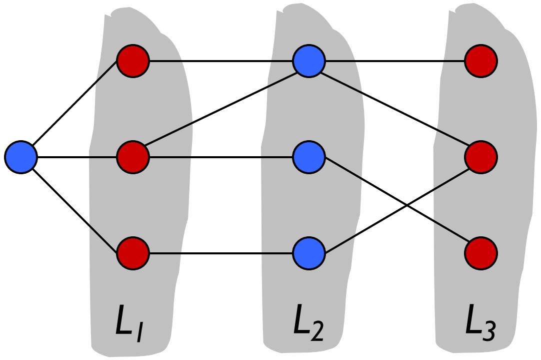
(i) No edge of G joins two nodes of the same layer, and G is bipartite.

(ii) An edge of G joins two nodes of the same layer, and G contains an odd-length cycle (and hence is not bipartite).

Pf. (i)

Suppose no edge joins two nodes in the same layer.

By previous lemma, all edges join nodes on adjacent levels.



Case (i)

Bipartition:

red = nodes on odd levels,

blue = nodes on even levels.

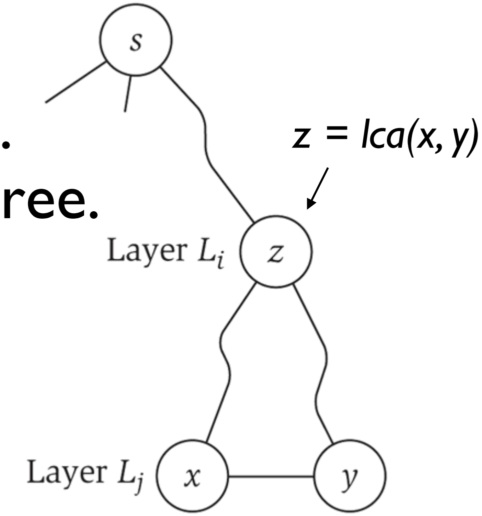
Bipartite Graphs

Lemma. Let G be a connected graph, and let $L_0, \ldots, L_k$ be the layers produced by BFS starting at node s. Exactly one of the following holds.

(i) No edge of G joins two nodes of the same layer, and G is bipartite.

(ii) An edge of G joins two nodes of the same layer, and G contains an odd-length cycle (and hence is not bipartite).

Pf. (ii)

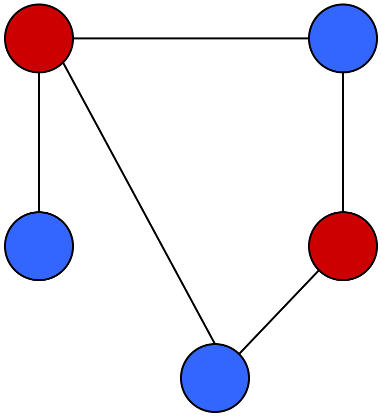


Suppose (x, y) is an edge & x, y in same level Lj.
Let z = their lowest common ancestor in BFS tree.
Let Li be level containing z.
Consider cycle that takes edge from x to y,
then tree from y to z, then tree from z to x.
Its length is $\underbrace{1}_{(x,y)} + \underbrace{(j-i)}_{\text{path from } y \text{ to } z} + \underbrace{(j-i)}_{\text{path from } z \text{ to } x}$, which is odd.

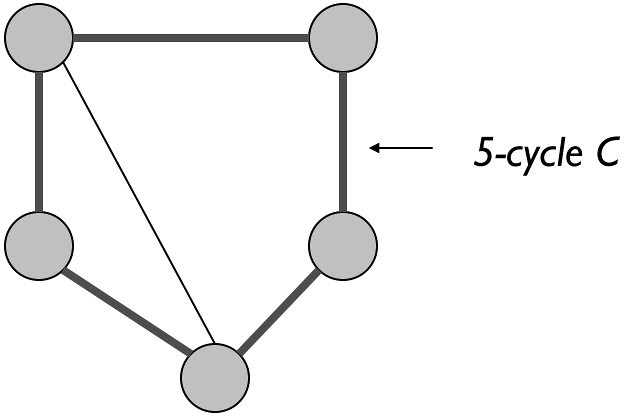
Obstruction to Bipartiteness

Cor: A graph G is bipartite iff it contains no odd length cycle.

NB: the proof is algorithmic–it *finds* a coloring or odd cycle.



bipartite
(2-colorable)



not bipartite
(not 2-colorable)

3.6 DAGs and Topological Ordering

Precedence Constraints

Precedence constraints. Edge (v_i, v_j) means task v_i must occur before v_j.

Applications

Course prerequisites: course v_i must be taken before v_j

Compilation: must compile module v_i before v_j

Computing workflow: output of job v_i is input to job v_j

Manufacturing or assembly: sand it before you paint it…

Spreadsheet evaluation order: if A7 is "=A6+A5+A4", evaluate them first

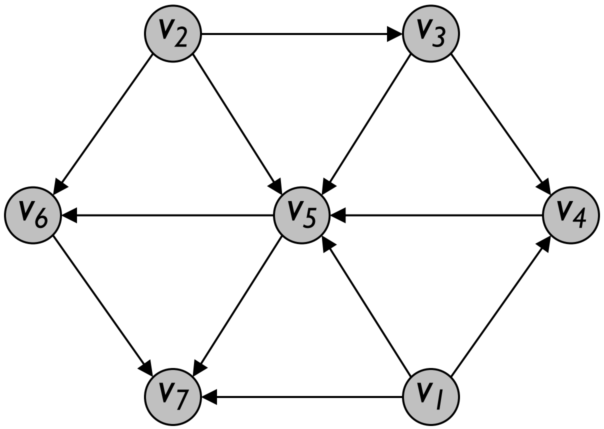
Directed Acyclic Graphs

Def. A DAG is a directed acyclic graph, i.e., one that contains no directed cycles.

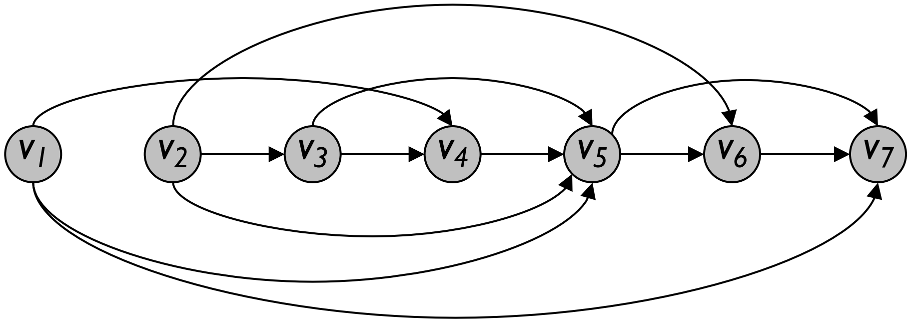
Ex. Precedence constraints: edge (v_i, v_j) means v_i must precede v_j.

Def. A topological order of a directed graph $G = (V, E)$ is an ordering of its nodes as $v_1, v_2, \ldots, v_n$ so that for every edge (v_i, v_j) we have $i < j$.

E.g., $\forall$edge (v_i, v_j), finish v_i before starting v_j



a DAG



a topological ordering of that DAG–all edges left-to-right

Directed Acyclic Graphs

Lemma. If G has a topological order, then G is a DAG.

if all edges go L→R, you can't loop back to close a cycle

Pf. (by contradiction)

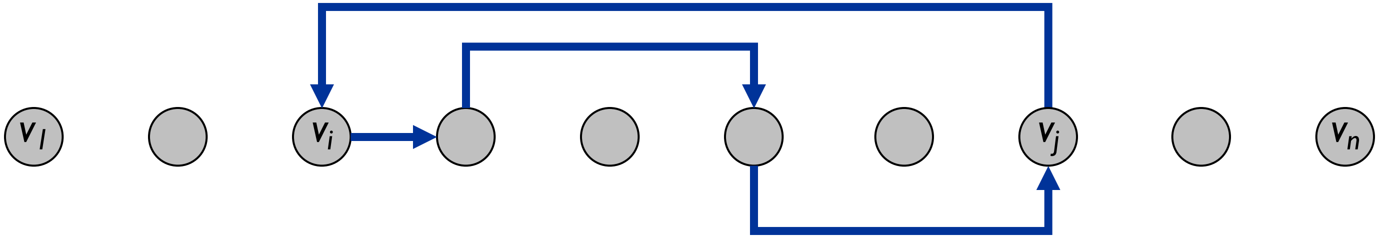
Suppose that G has a topological order $v_1, \ldots, v_n$ and that G also has a directed cycle C.

Let v_i be the lowest-indexed node in C, and let v_j be the node just before v_i; thus (v_j, v_i) is an edge.

By our choice of i, we have $i < j$.

On the other hand, since (v_j, v_i) is an edge and $v_1, \ldots, v_n$ is a topological order, we must have $j < i$, a contradiction.

the directed cycle C



the supposed topological order: $v_1, \ldots, v_n$

Directed Acyclic Graphs

Lemma.

If G has a topological order, then G is a DAG.

Q. Does every DAG have a topological ordering?

Q. If so, how do we compute one?

Directed Acyclic Graphs

Lemma. If G is a DAG, then G has a node with no incoming edges.

Pf. (by contradiction)

Suppose that G is a DAG and every node has at least one incoming edge. Let's see what happens.

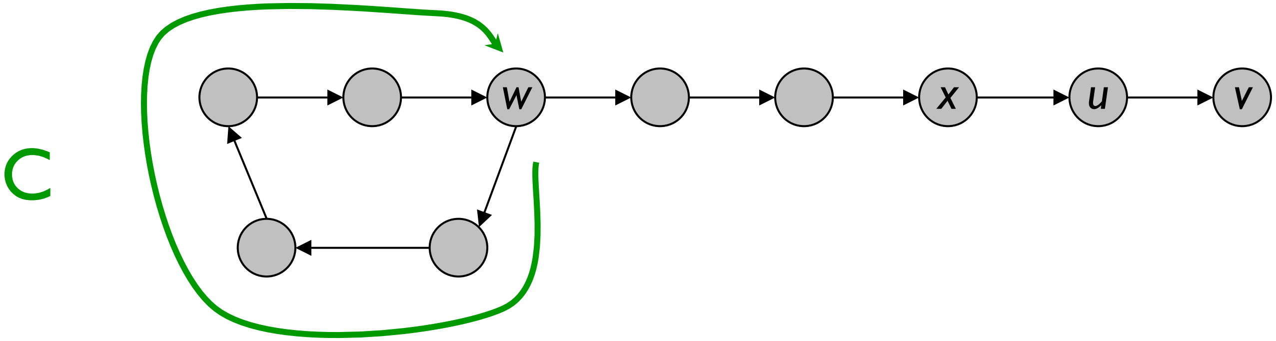
Pick any node v, and begin following edges backward from v. Since v has at least one incoming edge (u, v) we can walk backward to u.

Then, since u has at least one incoming edge (x, u), we can walk backward to x.

Repeat until we visit a node, say w, twice.

Why must this happen?

Let C be the sequence of nodes encountered between successive visits to w. C is a cycle.



Directed Acyclic Graphs

Lemma. If G is a DAG, then G has a topological ordering.

Pf. (by induction on n)

Base case: true if n = 1.

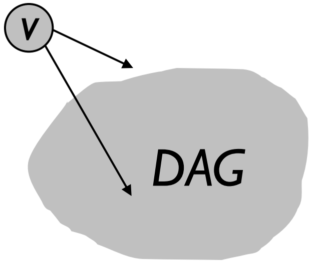
Given DAG on n > 1 nodes, find a node v with no incoming edges.

G - { v } is a DAG, since deleting v cannot create cycles.

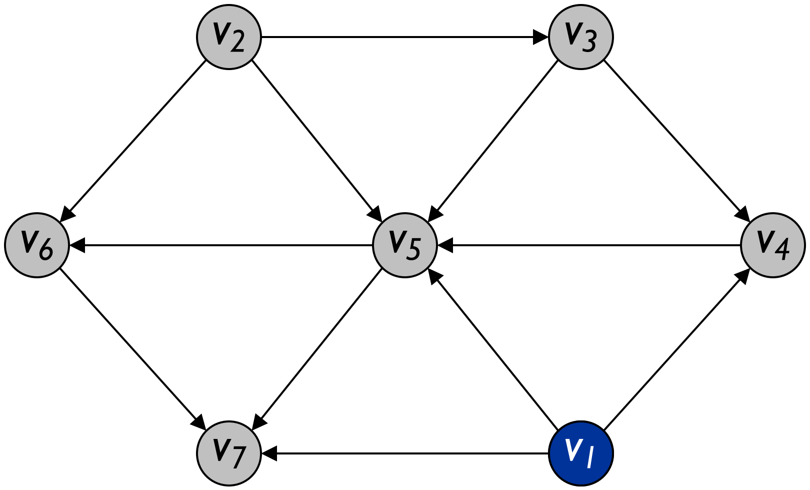
By inductive hypothesis, G - { v } has a topological ordering.

Place v first in topological ordering; then append nodes of G - { v } in topological order. This is valid since v has no incoming edges. ▪

```
To compute a topological ordering of $G$:
Find a node $v$ with no incoming edges and order it first
Delete $v$ from $G$
Recursively compute a topological ordering of $G-\{v\}$
  and append this order after $v$
```

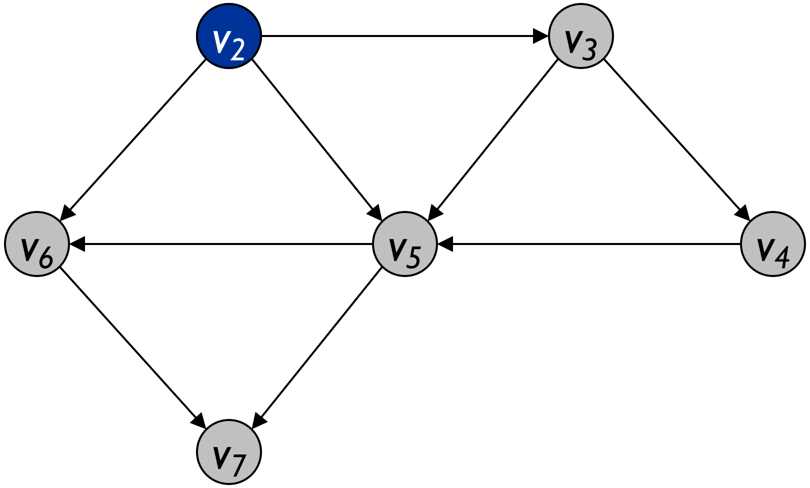


Topological Ordering Algorithm: Example



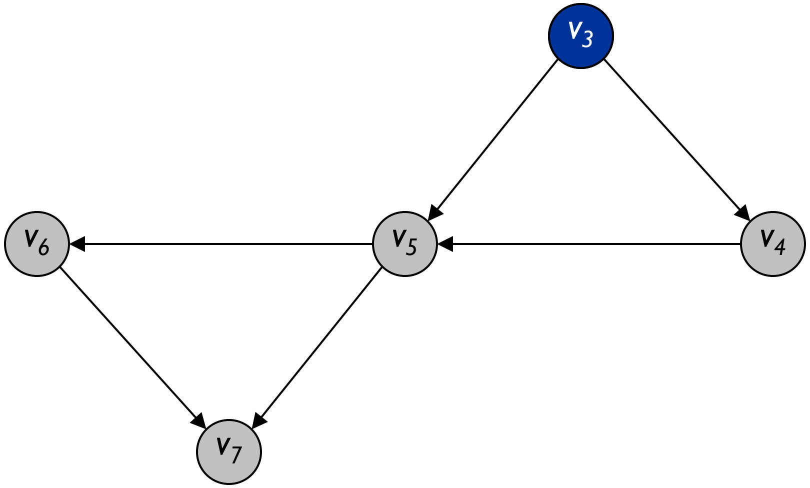
Topological order:

Topological Ordering Algorithm: Example



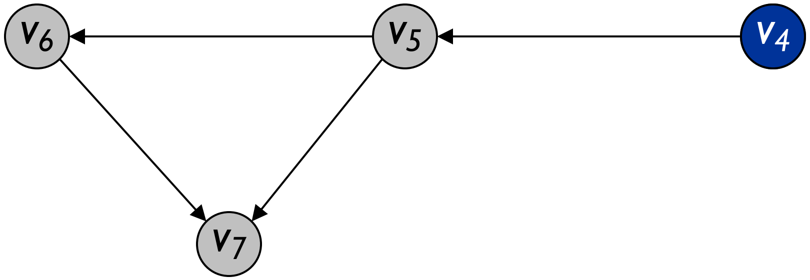
Topological order: v_1

Topological Ordering Algorithm: Example



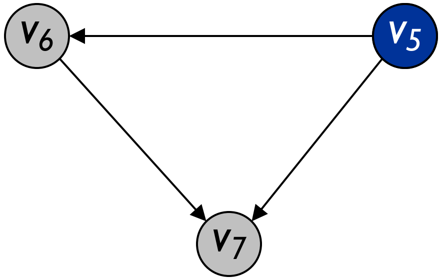
Topological order: v_1, v_2

Topological Ordering Algorithm: Example



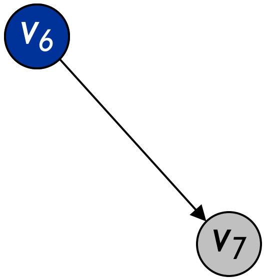
Topological order: v_1, v_2, v_3

Topological Ordering Algorithm: Example



Topological order: v_1, v_2, v_3, v_4

Topological Ordering Algorithm: Example

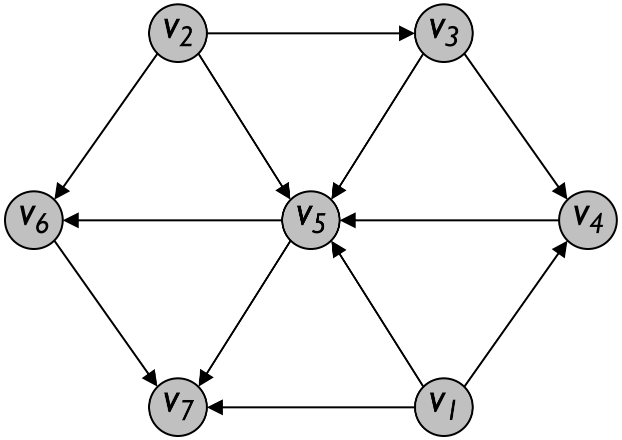


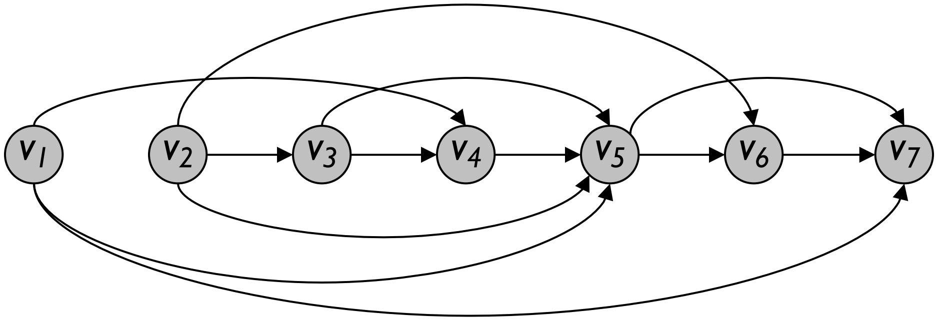
Topological order: v_1, v_2, v_3, v_4, v_5

Topological Ordering Algorithm: Example

Topological order: $v_1, v_2, v_3, v_4, v_5, v_6$

Topological Ordering Algorithm: Example





Topological order: $v_1, v_2, v_3, v_4, v_5, v_6, v_7$.

Topological Sorting Algorithm

Maintain the following:

```
count[w] = (remaining) number of incoming edges to node w
S = set of (remaining) nodes with no incoming edges
```

Initialization:

```
count[w] = 0 for all w
count[w]++ for all edges (v,w)              O(m + n)
S = S ∪ {w} for all w with count[w]==0
```

Main loop:

```
while S not empty
        remove some v from S
        make v next in topo order           O(1) per node
        for all edges from v to some w      O(1) per edge
        decrement count[w]
        add w to S if count[w] hits 0
```

Correctness: clear, I hope

Time: $O(m + n)$ (assuming edge-list representation of graph)

Depth-First Search

Follow the first path you find as far as you can go

Back up to last unexplored edge when you reach a dead end, then go as far you can

Naturally implemented using recursive calls or a stack

DFS(v) – Recursive version

```
Global Initialization:
    for all nodes v, v.dfs# = -1    // mark v "undiscovered"
    dfscounter = 0

DFS(v)
    v.dfs# = dfscounter++           // v "discovered", number it
    for each edge (v,x)
        if (x.dfs# = -1)            // tree edge (x previously  undiscovered)
            DFS(x)
        else …                      // code for back-, fwd-, parent,
                                    // edges, if needed
                                    // mark v "completed," if needed
```

Why fuss about trees (again)?

BFS tree ≠ **DFS tree, but, as with** BFS, DFS has found a tree in the graph s.t. non-tree edges are "simple”

DFS(A)

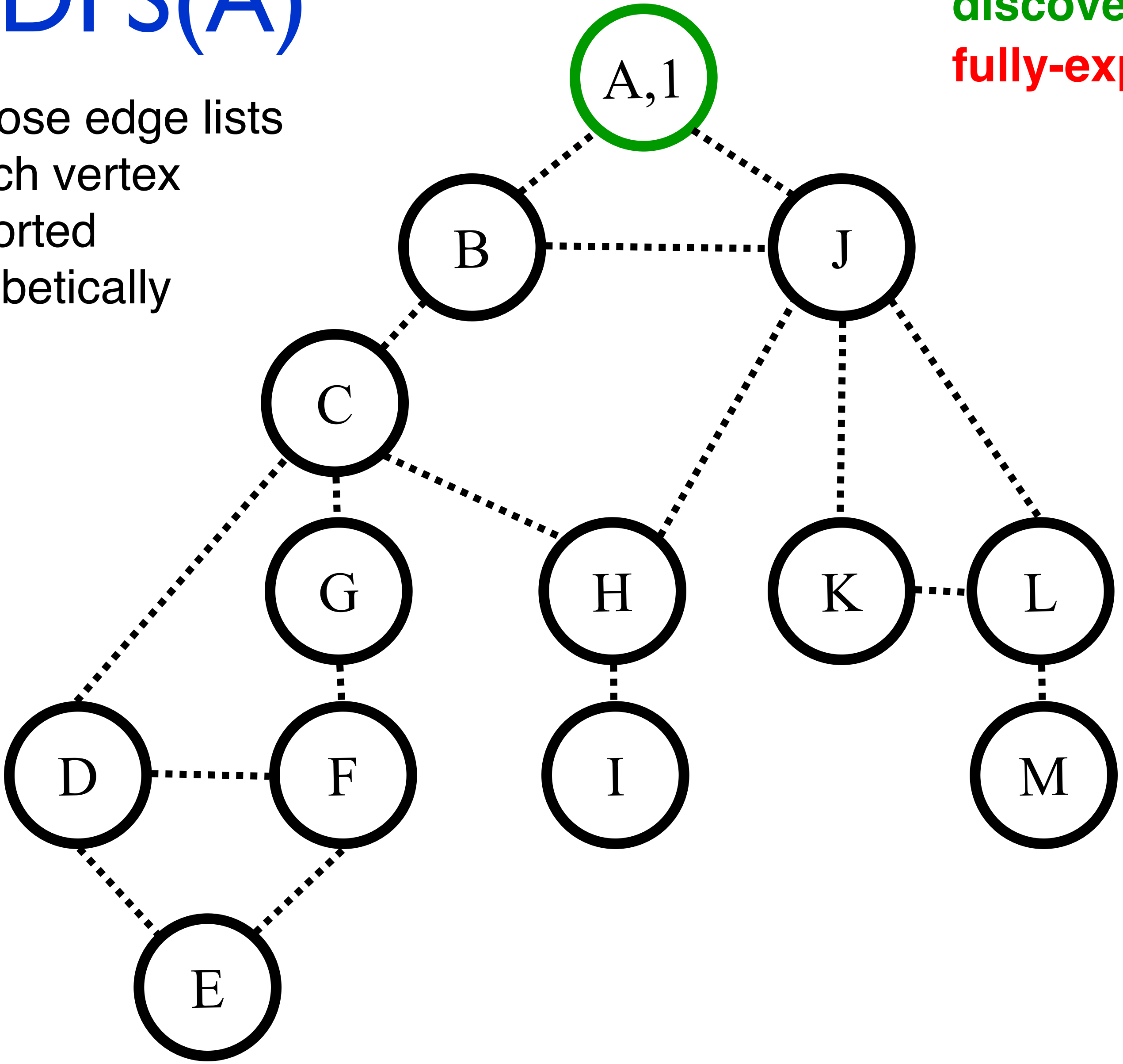
Suppose edge lists at each vertex are sorted alphabetically

Color code:

undiscovered

discovered

fully-explored



Call Stack
(Edge list):

A (B,J)

DFS(A)

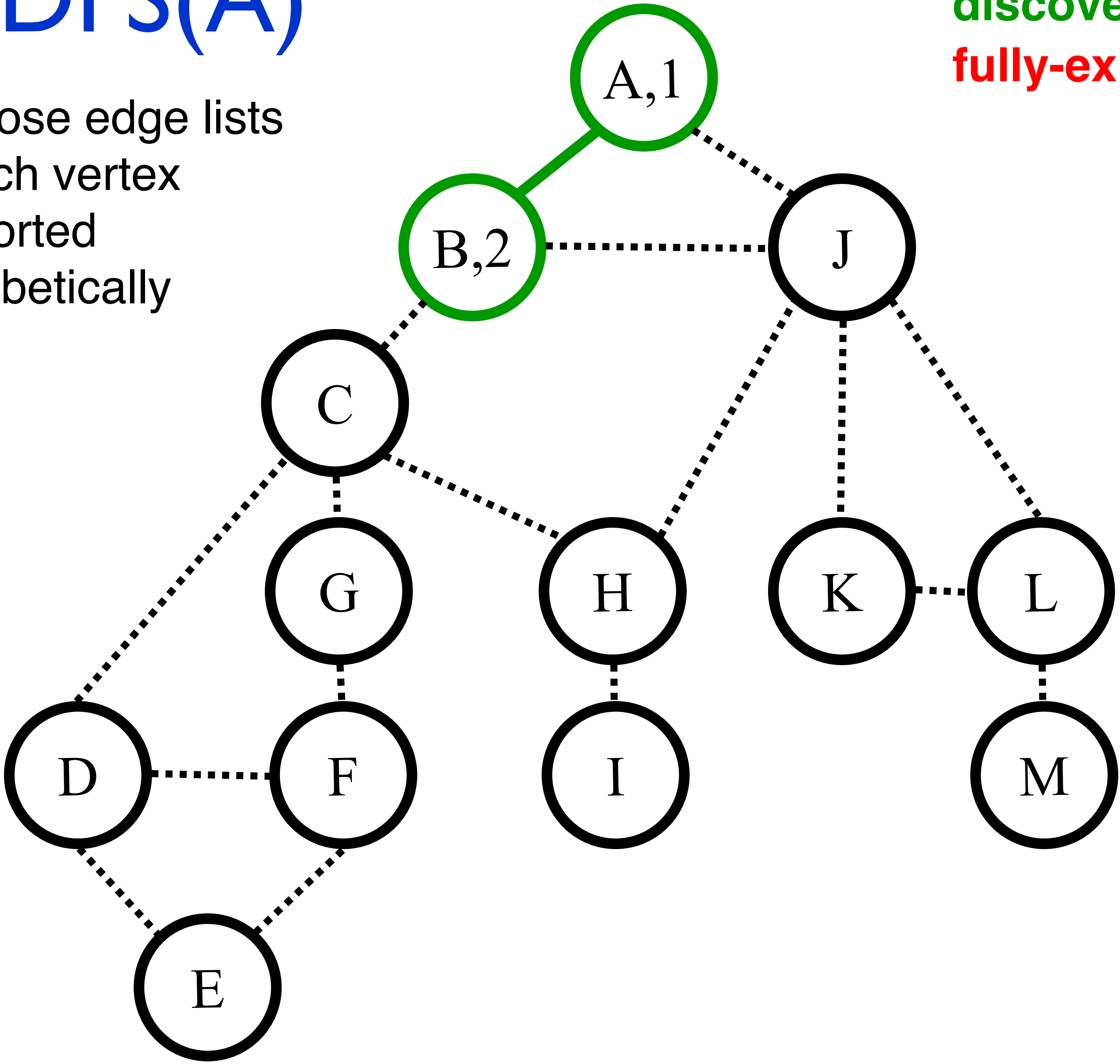
Suppose edge lists at each vertex are sorted alphabetically

Color code:

undiscovered

discovered

fully-explored



Call Stack:
(Edge list)

A (~~B~~,J)
B (A,C,J)

DFS(A)

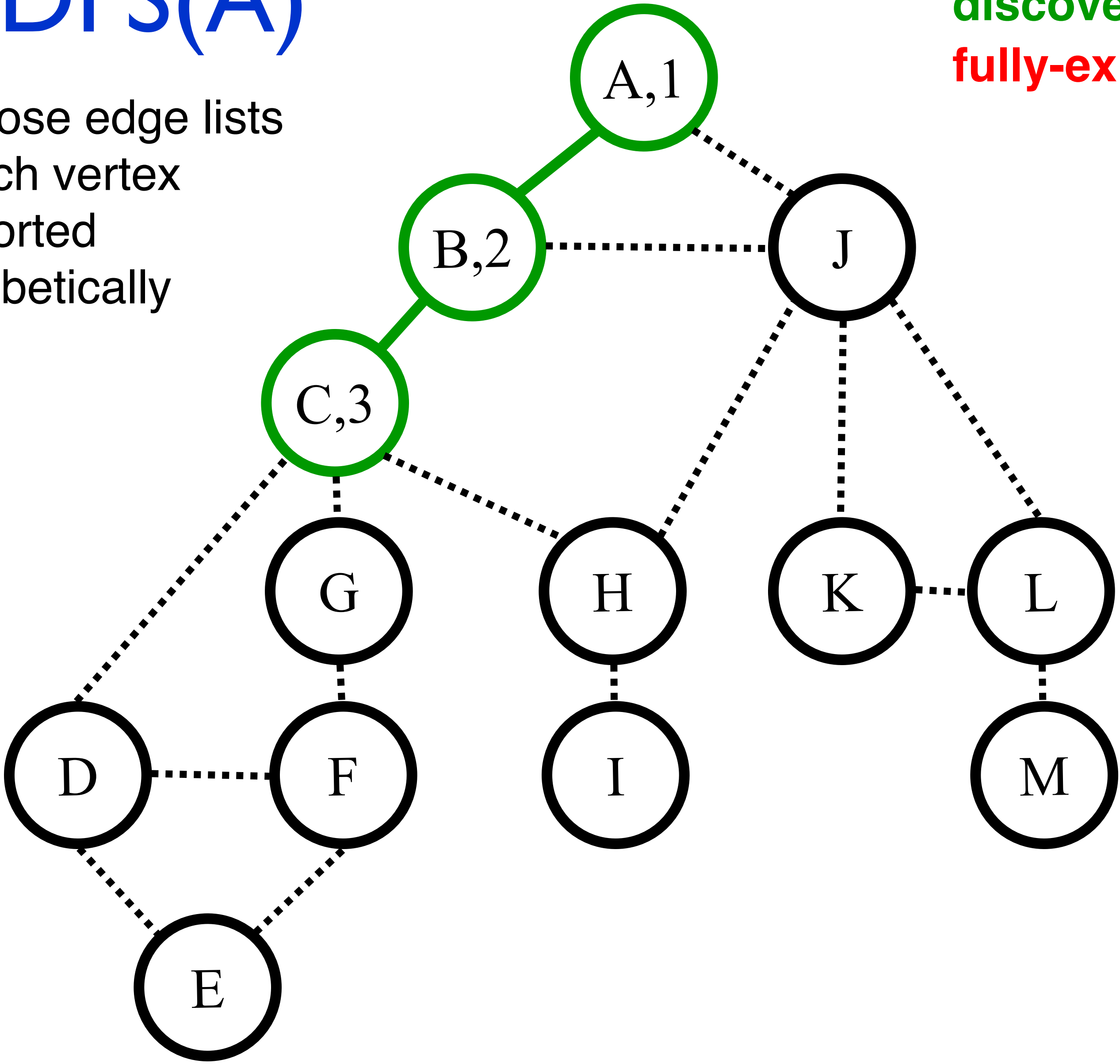
Suppose edge lists at each vertex are sorted alphabetically

Color code:

undiscovered

discovered

fully-explored



Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (B,D,G,H)

DFS(A)

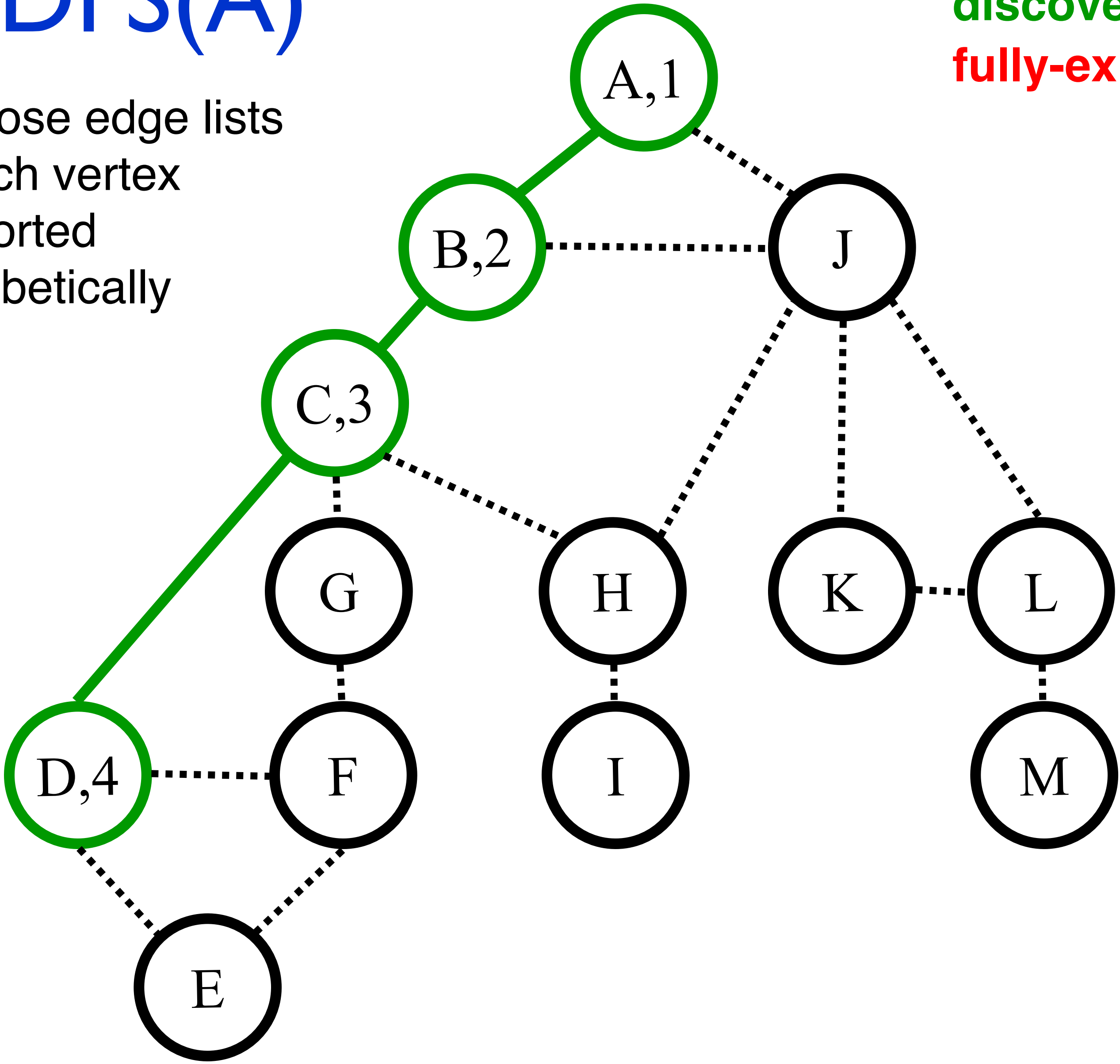
Suppose edge lists at each vertex are sorted alphabetically

Color code:

undiscovered

discovered

fully-explored



Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,G,H)
D (C,E,F)

DFS(A)

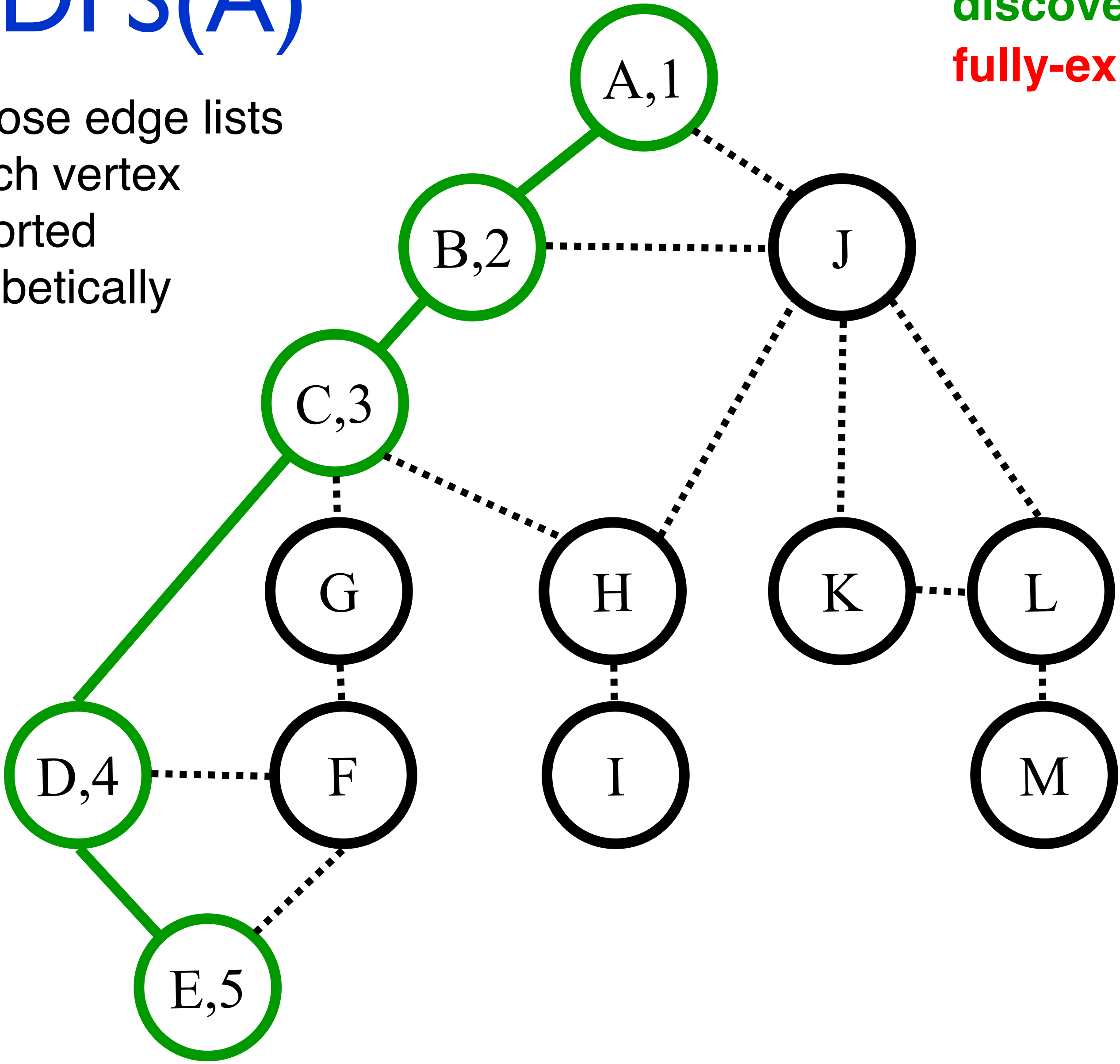
Suppose edge lists at each vertex are sorted alphabetically

Color code:

undiscovered

discovered

fully-explored



Call Stack:
(Edge list)

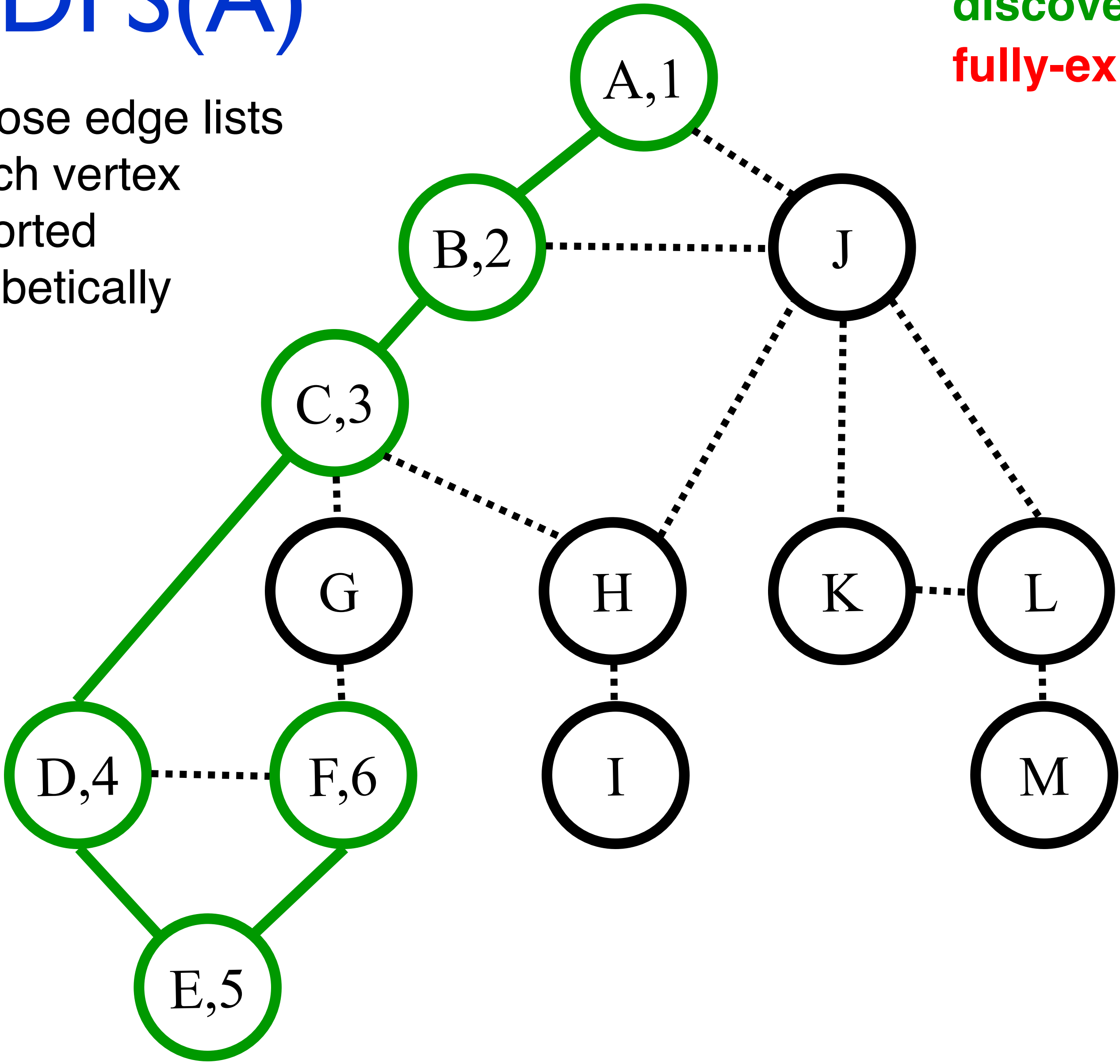
A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,G,H)
D (~~C~~,~~E~~,F)
E (D,F)

DFS(A)

Suppose edge lists at each vertex are sorted alphabetically

Color code:

undiscovered
discovered
fully-explored



Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,G,H)
D (~~C~~,~~E~~,F)
E (~~D~~,~~F~~)
F (D,E,G)

DFS(A)

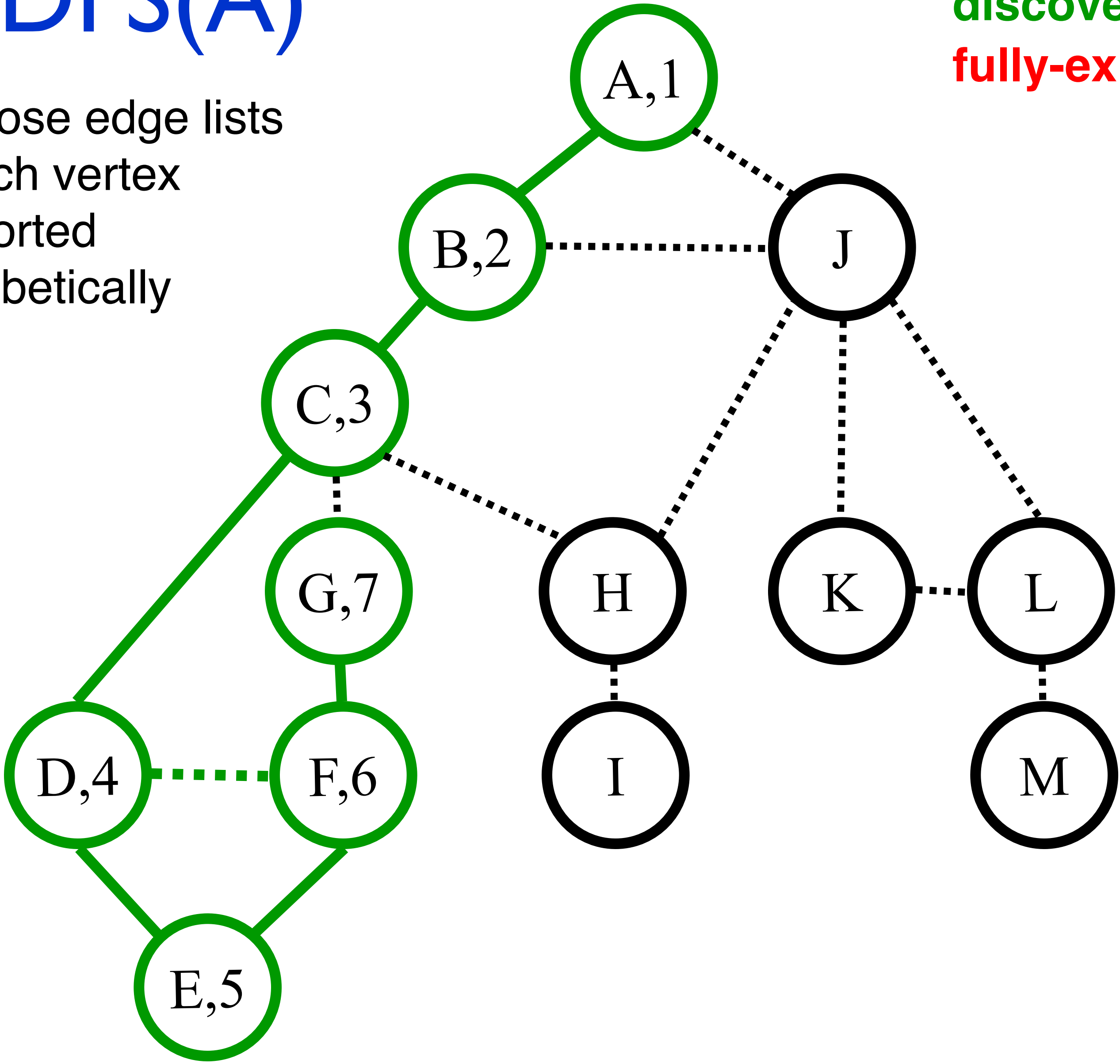
Suppose edge lists at each vertex are sorted alphabetically

Color code:

undiscovered

discovered

fully-explored



Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,G,H)
D (~~C~~,~~E~~,F)
E (~~D~~,~~F~~)
F (~~D~~,~~E~~,~~G~~)
G (C,F)

DFS(A)

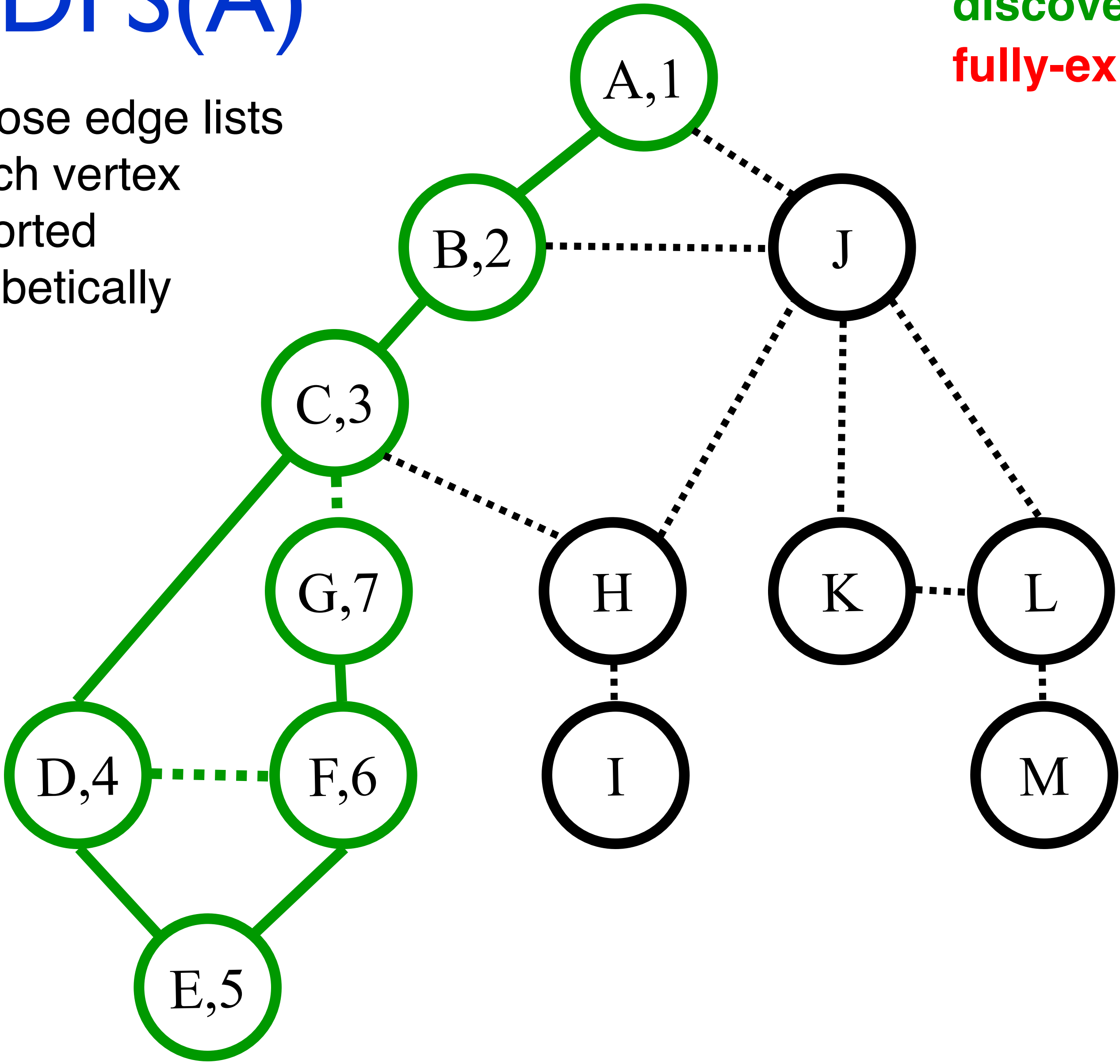
Suppose edge lists at each vertex are sorted alphabetically

Color code:

undiscovered

discovered

fully-explored

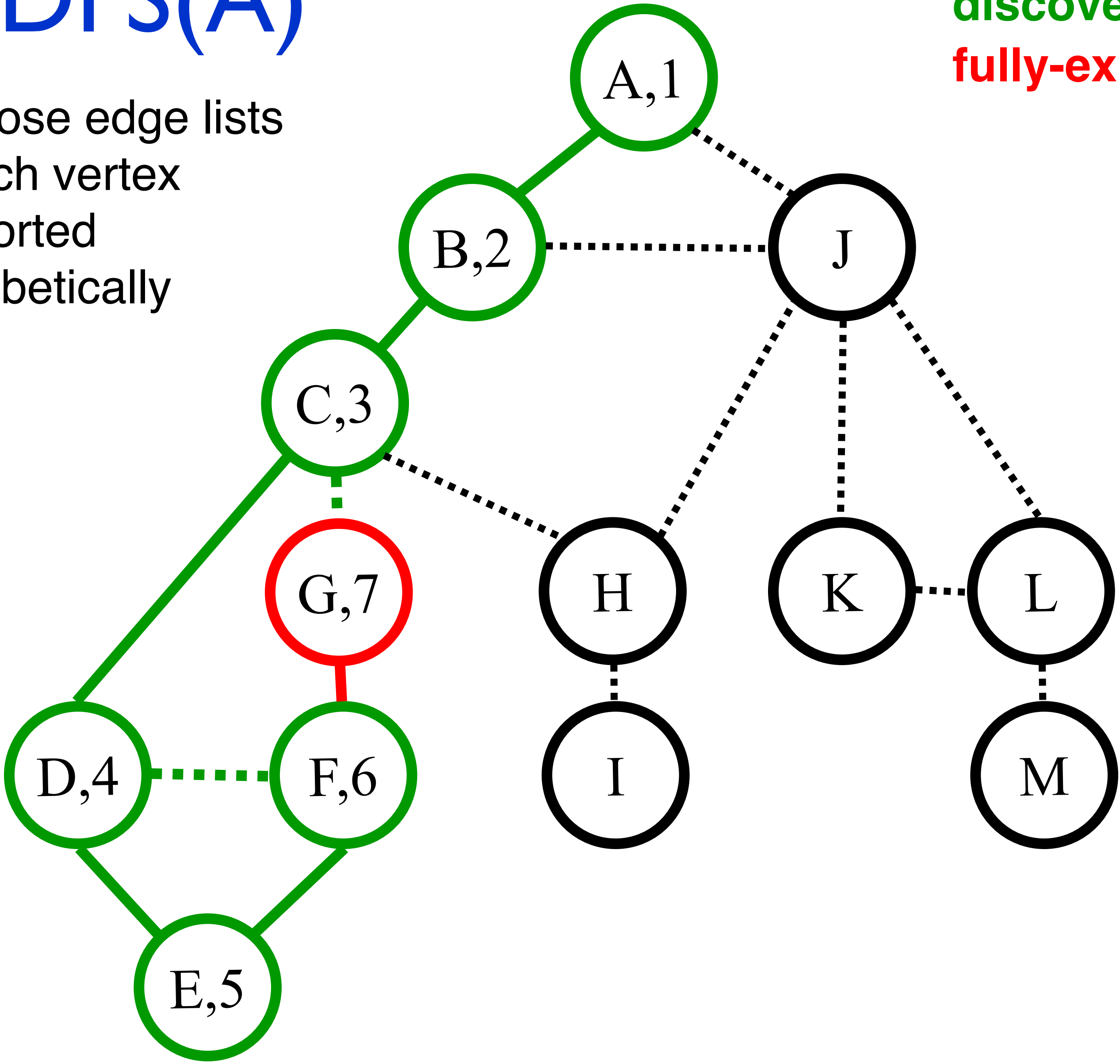


Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,G,H)
D (~~C~~,~~E~~,F)
E (~~D~~,~~F~~)
F (~~D~~,~~E~~,~~G~~)
G (~~C~~,~~F~~)

DFS(A)

Suppose edge lists at each vertex are sorted alphabetically



Color code:

undiscovered

discovered

fully-explored

Call Stack:
(Edge list)

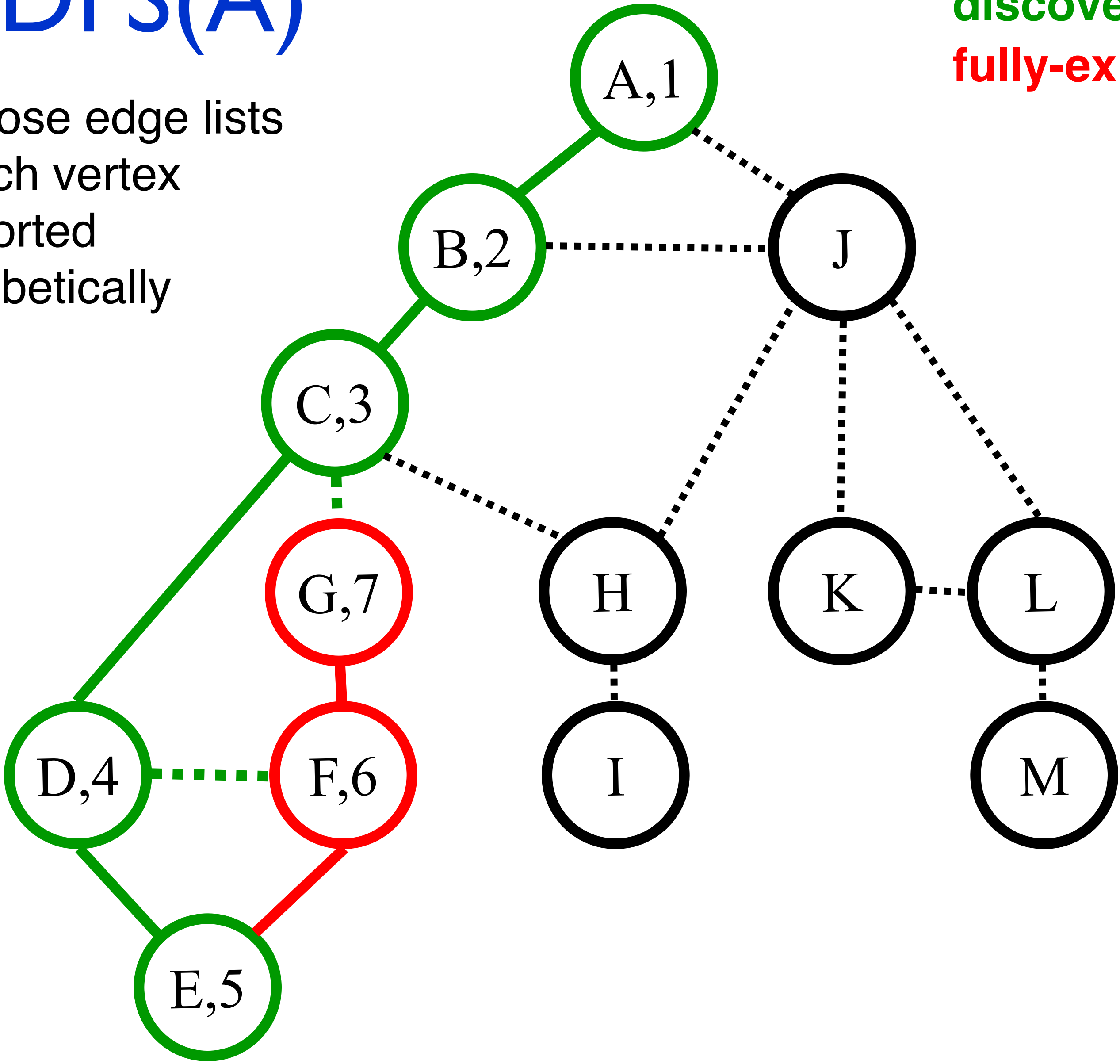
A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,G,H)
D (~~C~~,~~E~~,F)
E (~~D~~,~~F~~)
F (~~D~~,~~E~~,~~G~~)

DFS(A)

Suppose edge lists at each vertex are sorted alphabetically

Color code:

undiscovered
discovered
fully-explored



Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,G,H)
D (~~C~~,~~E~~,F)
E (~~D~~,~~F~~)

DFS(A)

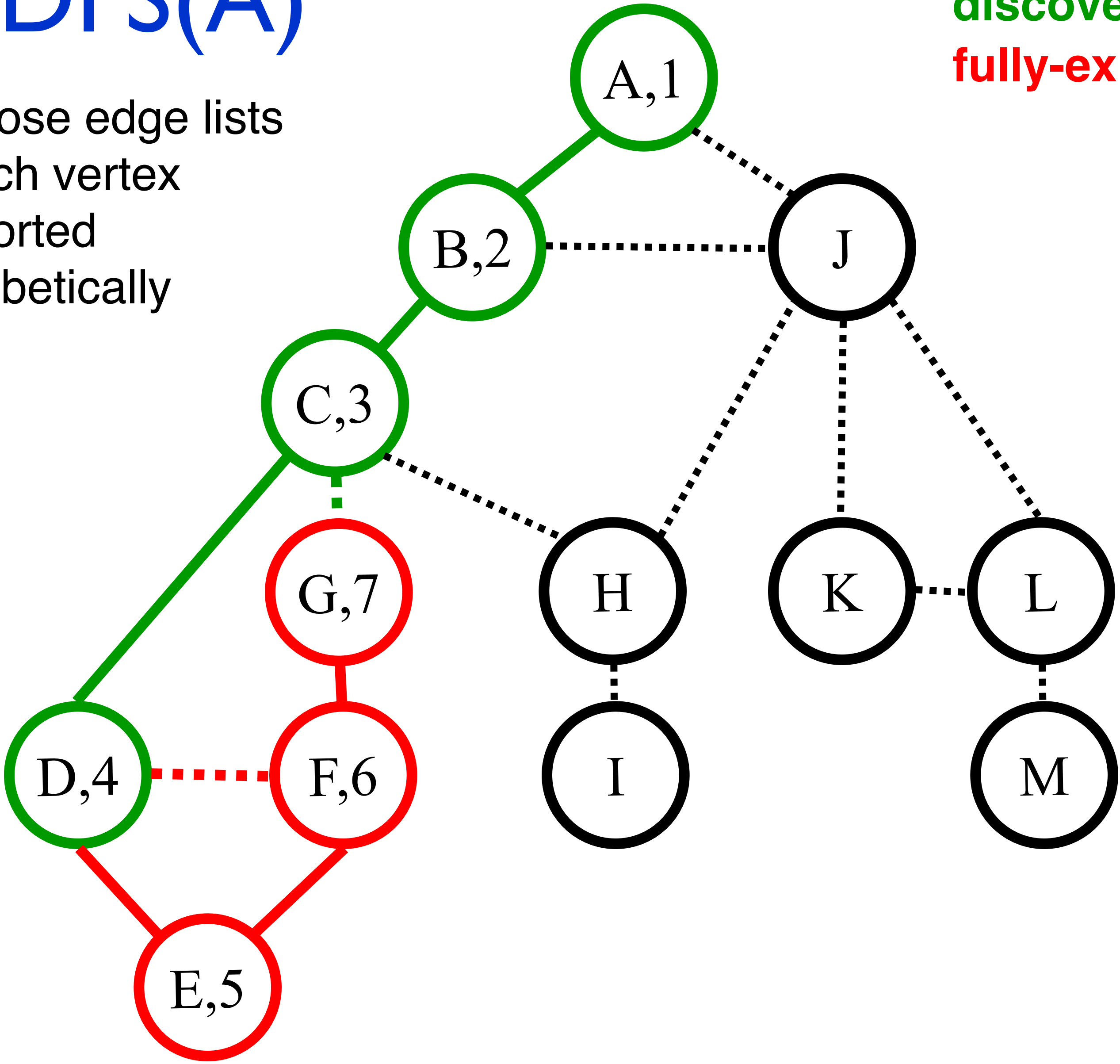
Suppose edge lists at each vertex are sorted alphabetically

Color code:

undiscovered

discovered

fully-explored



Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,G,H)
D (~~C~~,~~E~~,~~F~~)

DFS(A)

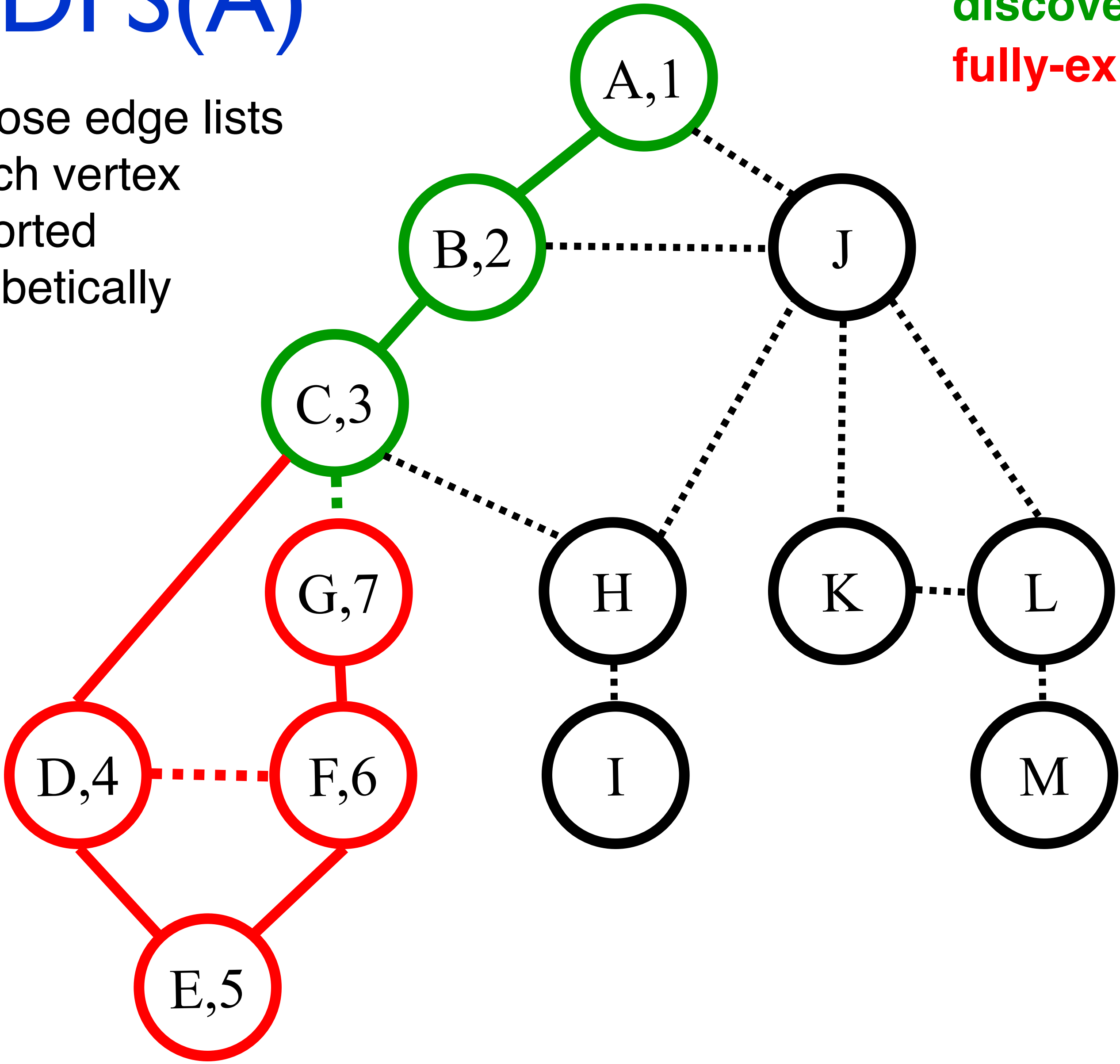
Suppose edge lists at each vertex are sorted alphabetically

Color code:

undiscovered

discovered

fully-explored



Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,G,H)

DFS(A)

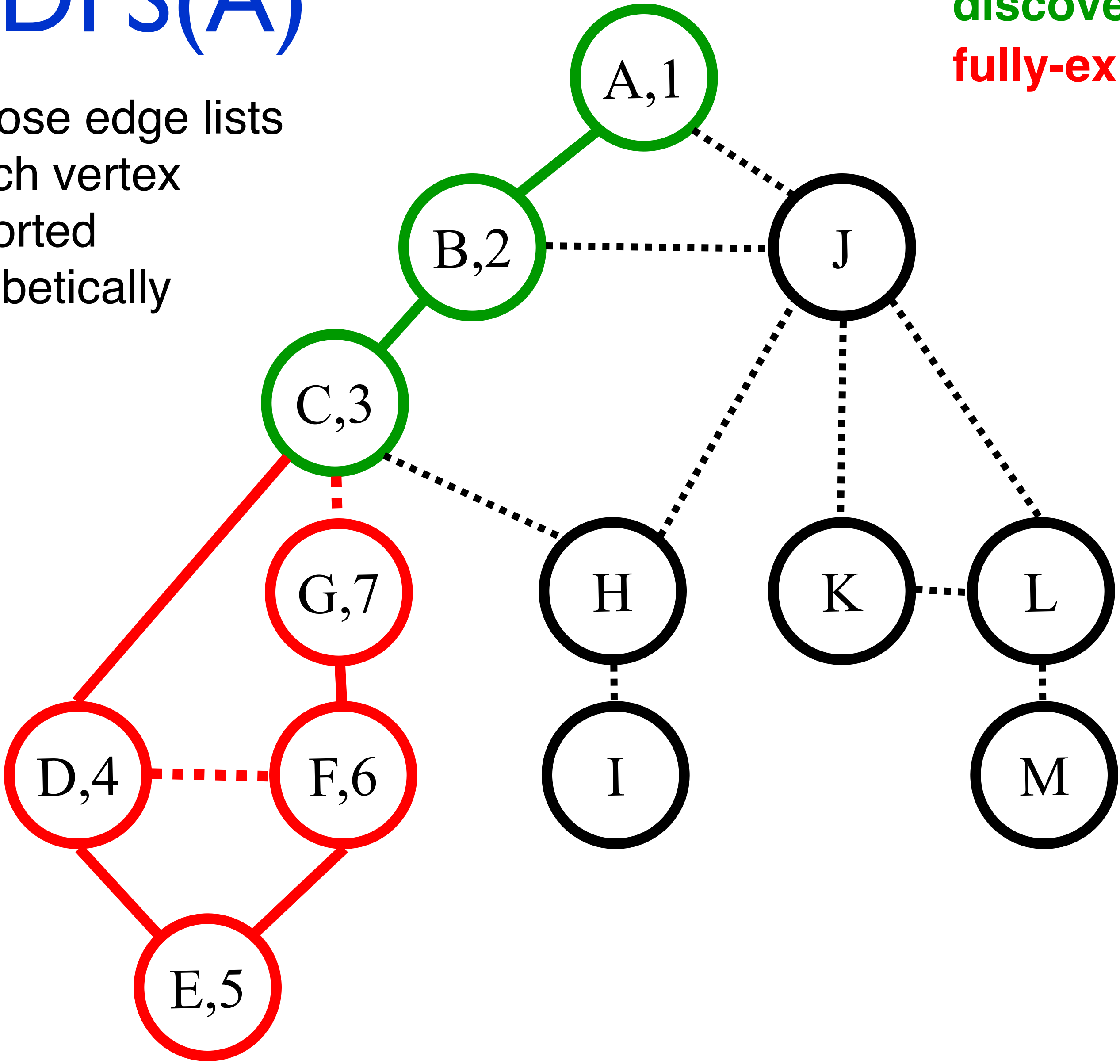
Suppose edge lists at each vertex are sorted alphabetically

Color code:

undiscovered

discovered

fully-explored



Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,~~G~~,H)

DFS(A)

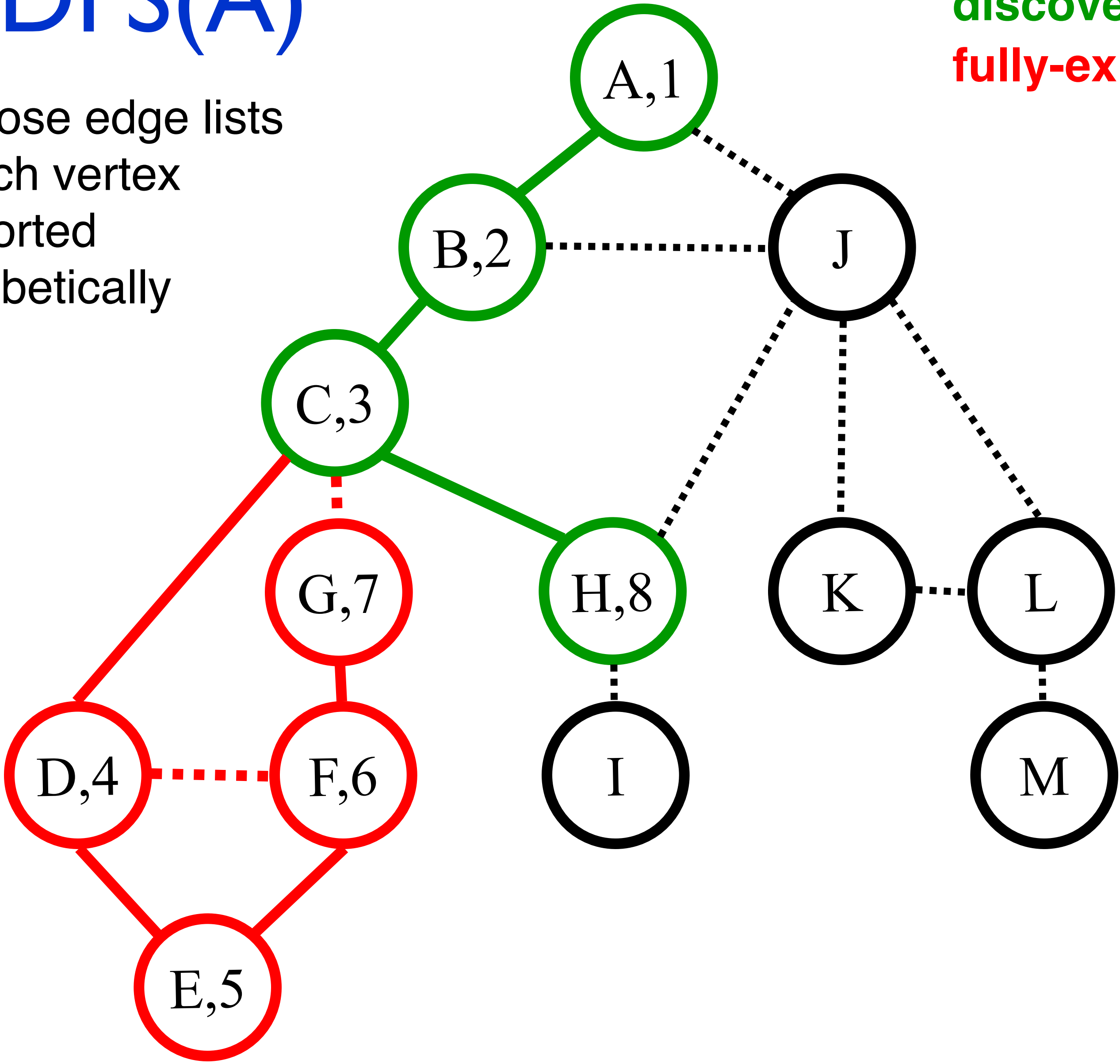
Suppose edge lists at each vertex are sorted alphabetically

Color code:

undiscovered

discovered

fully-explored



Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,~~G~~,~~H~~)
H (C,I,J)

DFS(A)

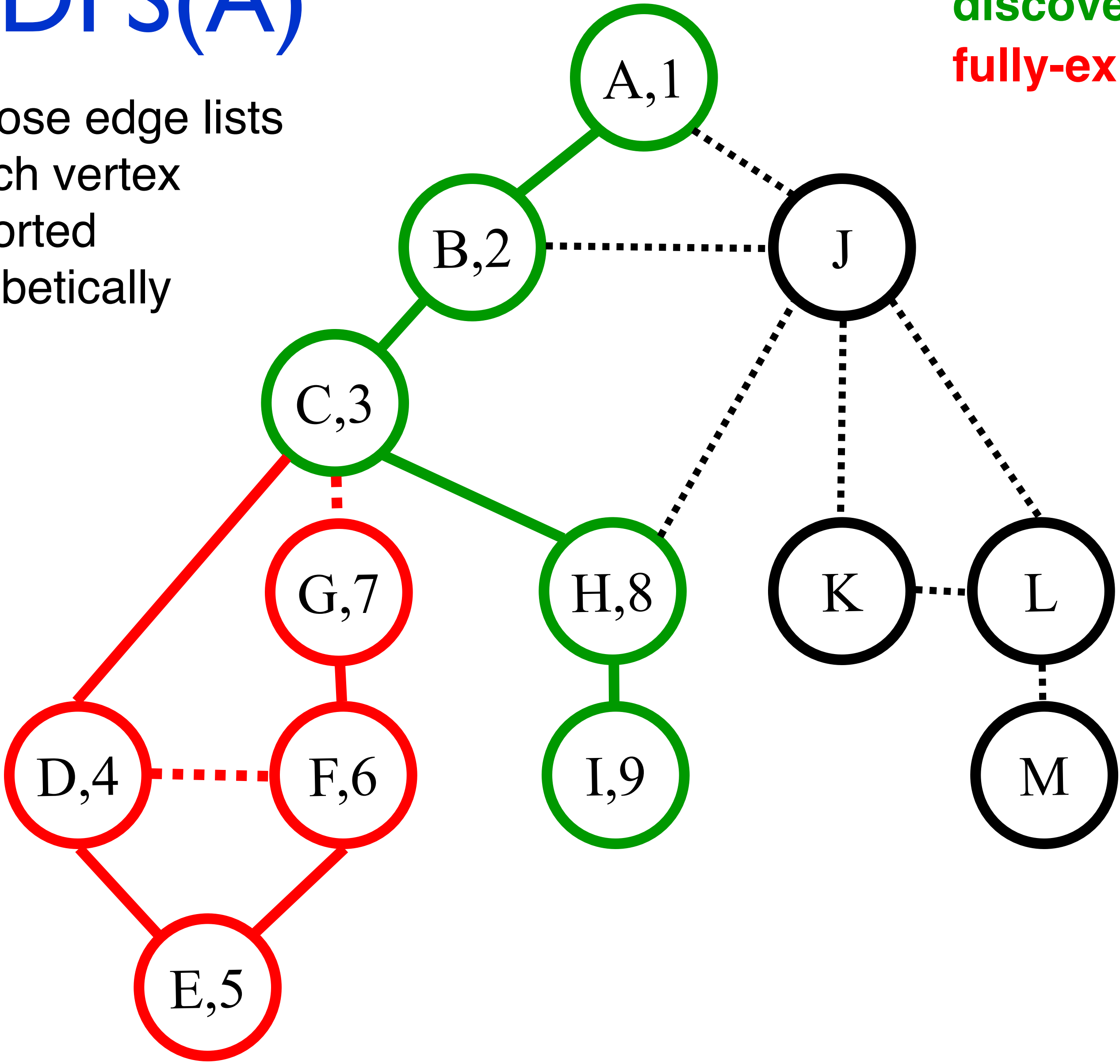
Suppose edge lists at each vertex are sorted alphabetically

Color code:

undiscovered

discovered

fully-explored



Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,~~G~~,~~H~~)
H (~~C~~,~~I~~,J)
I (H)

DFS(A)

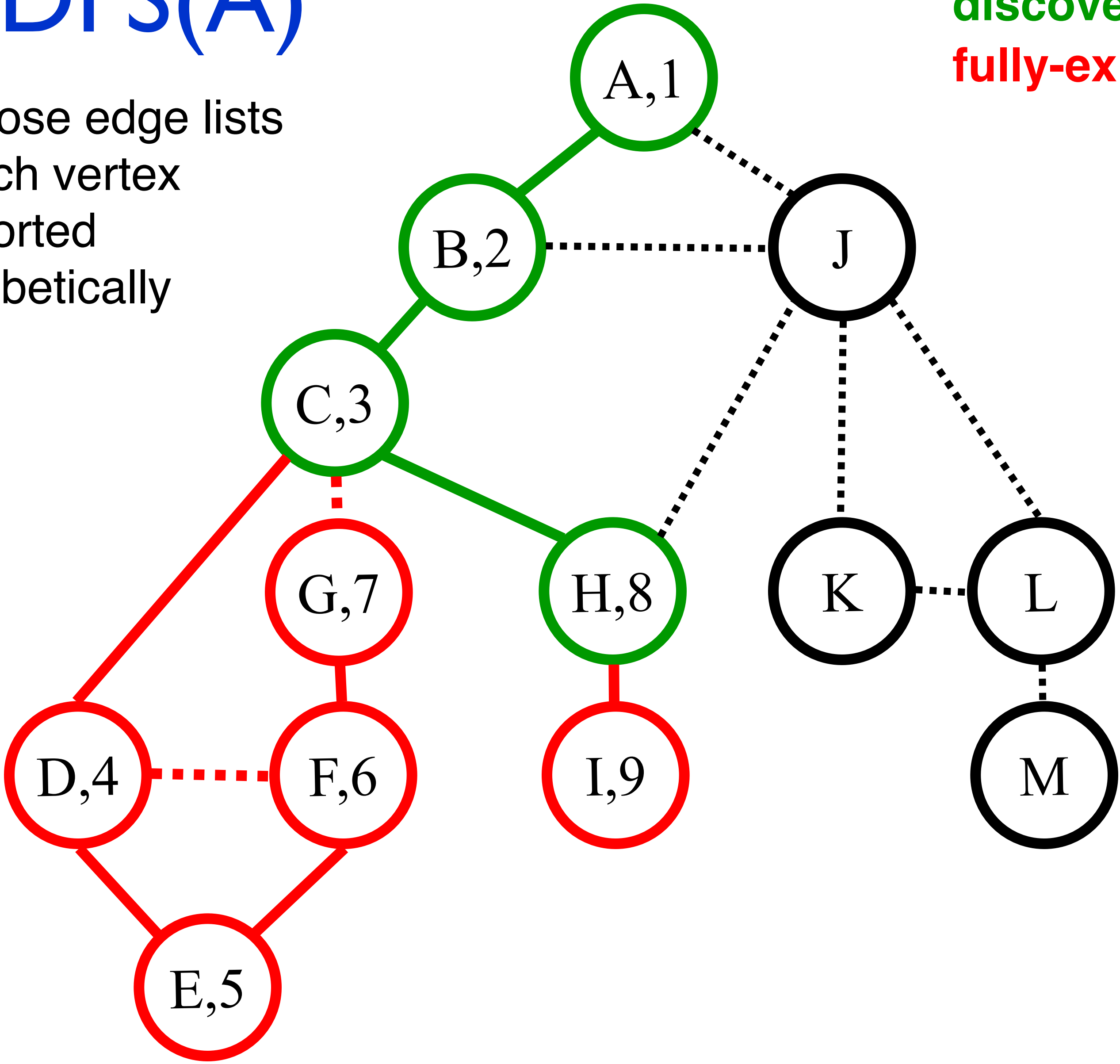
Suppose edge lists at each vertex are sorted alphabetically

Color code:

undiscovered

discovered

fully-explored



Call Stack:
(Edge list)

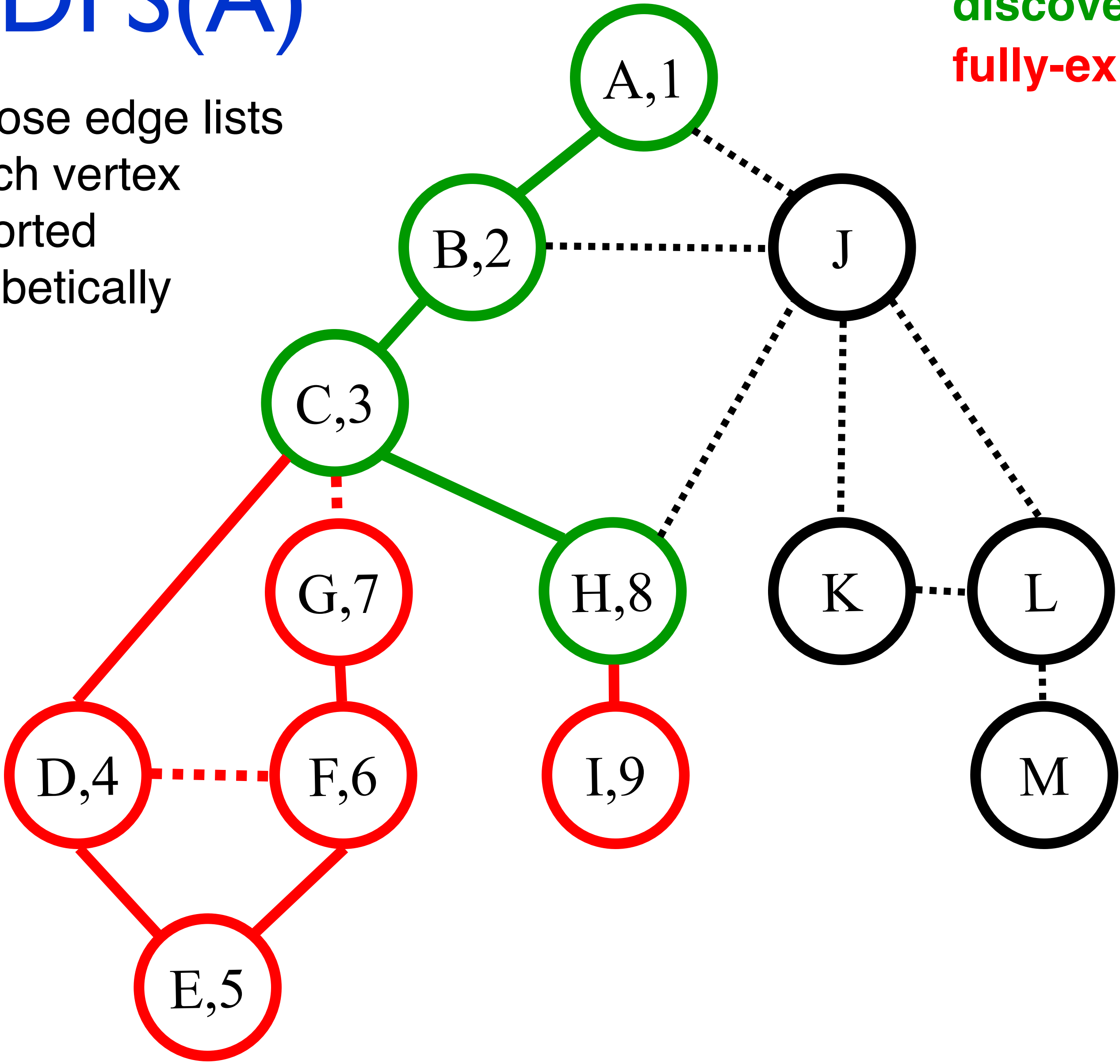
A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,~~G~~,~~H~~)
H (~~C~~,~~I~~,J)
I (~~H~~)

DFS(A)

Suppose edge lists at each vertex are sorted alphabetically

Color code:

undiscovered
discovered
fully-explored



Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,~~G~~,~~H~~)
H (~~C~~,~~I~~,J)

DFS(A)

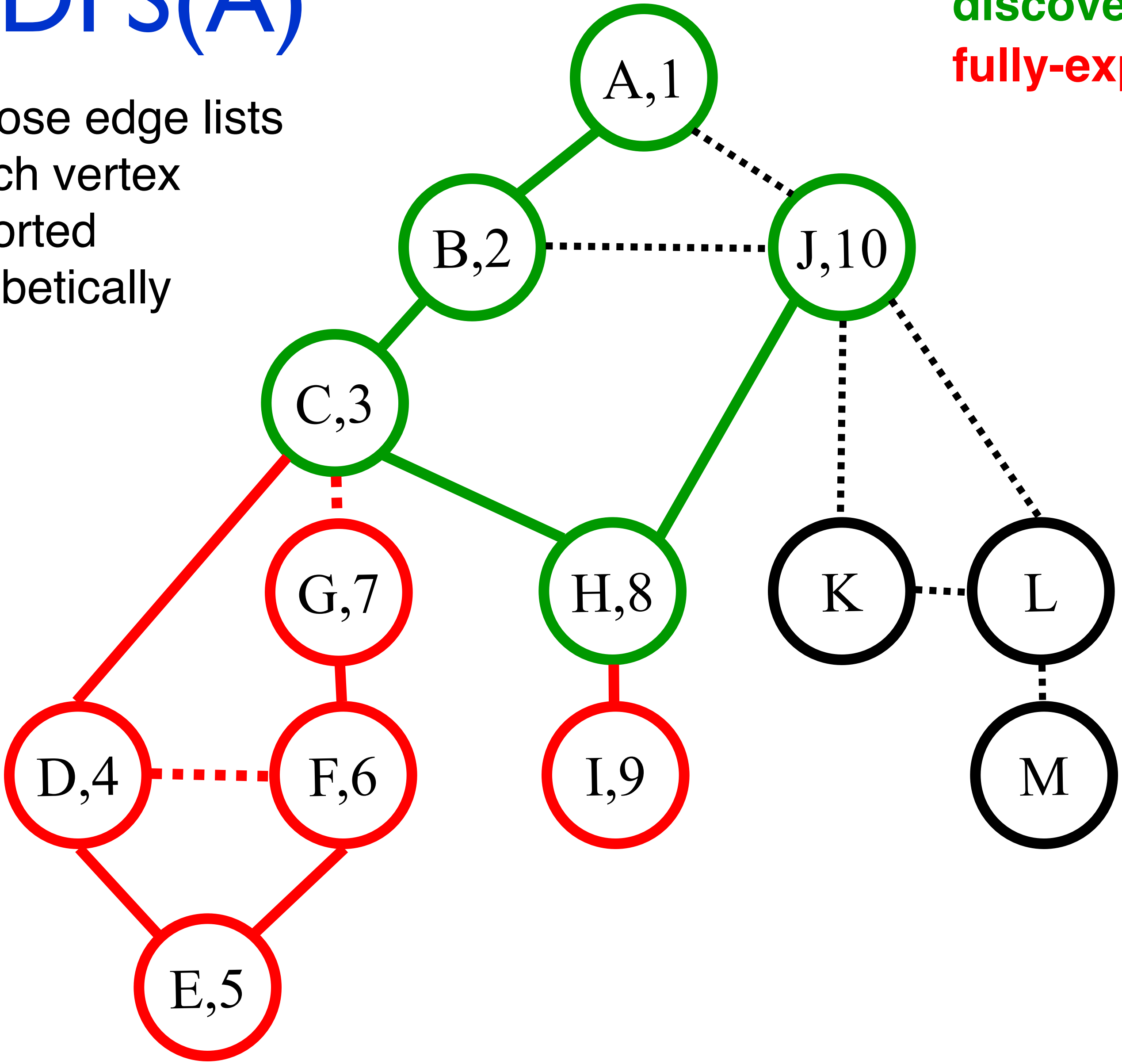
Suppose edge lists at each vertex are sorted alphabetically

Color code:

undiscovered

discovered

fully-explored



Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,~~G~~,~~H~~)
H (~~C~~,~~I~~,~~J~~)
J (A,B,H,K,L)

DFS(A)

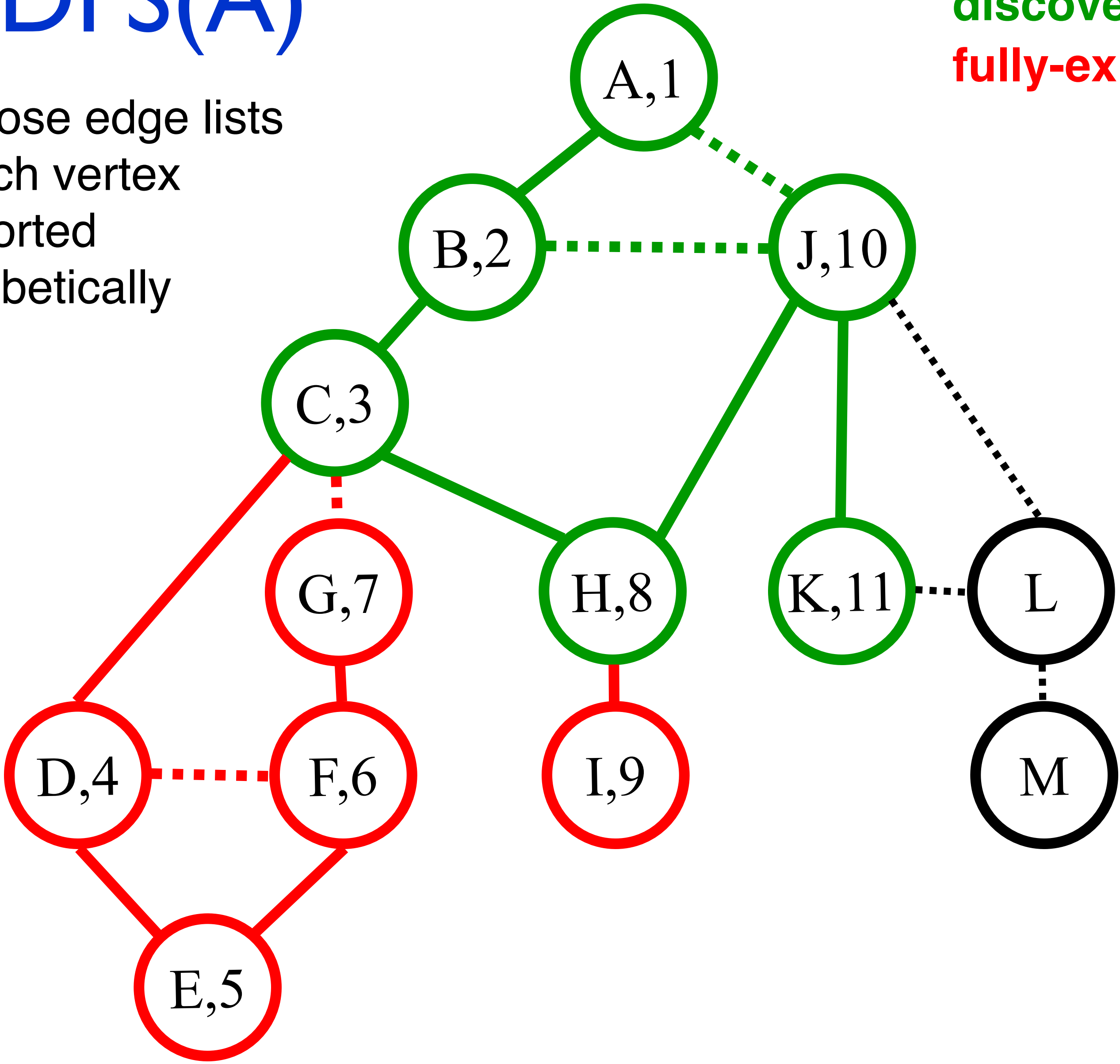
Suppose edge lists at each vertex are sorted alphabetically

Color code:

undiscovered

discovered

fully-explored



Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,~~G~~,~~H~~)
H (~~C~~,~~I~~,~~J~~)
J (~~A~~,~~B~~,~~H~~,~~K~~,L)
K (J,L)

DFS(A)

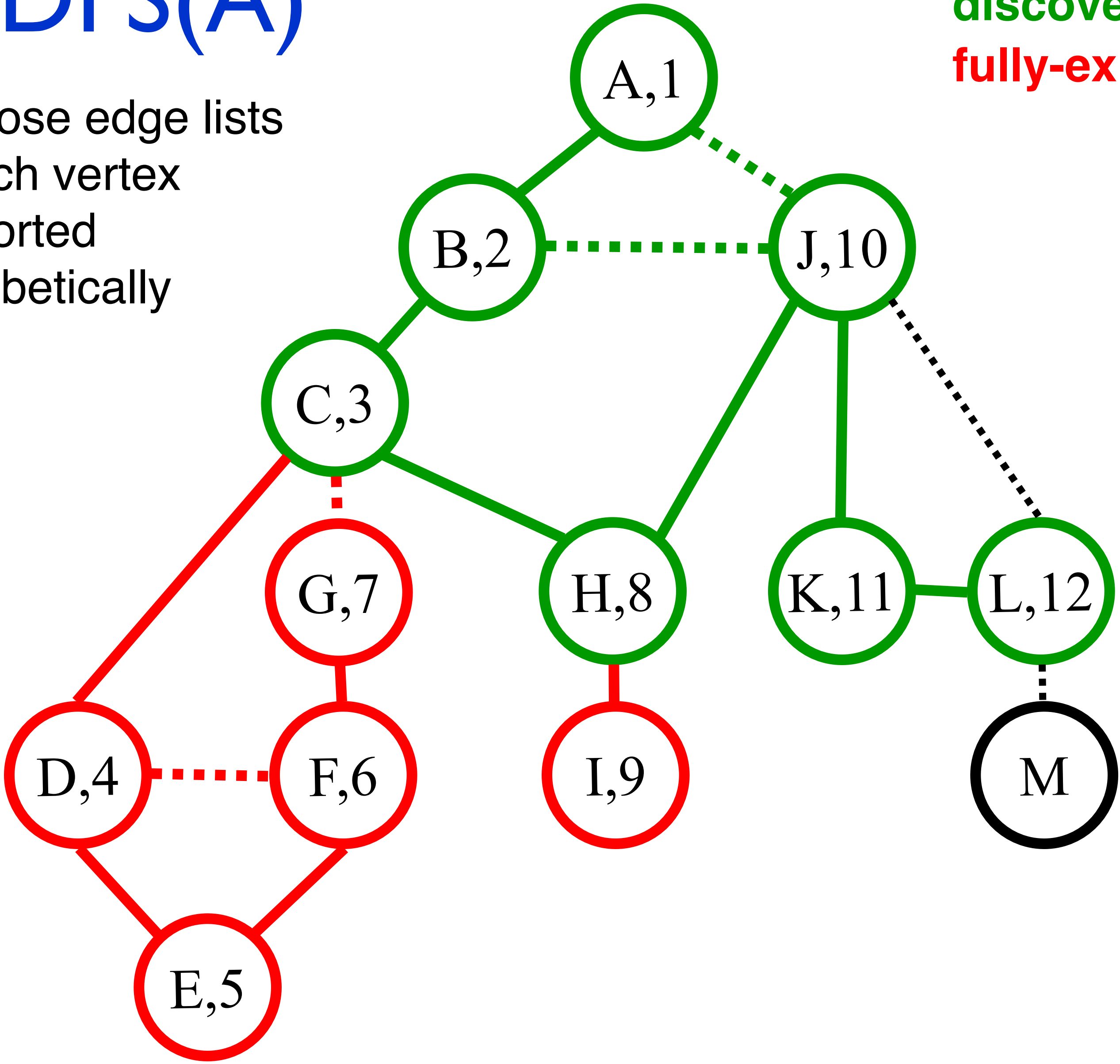
Suppose edge lists at each vertex are sorted alphabetically

Color code:

undiscovered

discovered

fully-explored



Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,~~G~~,~~H~~)
H (~~C~~,~~I~~,~~J~~)
J (~~A~~,~~B~~,~~H~~,~~K~~,L)
K (~~J~~,~~L~~)
L (J,K,M)

DFS(A)

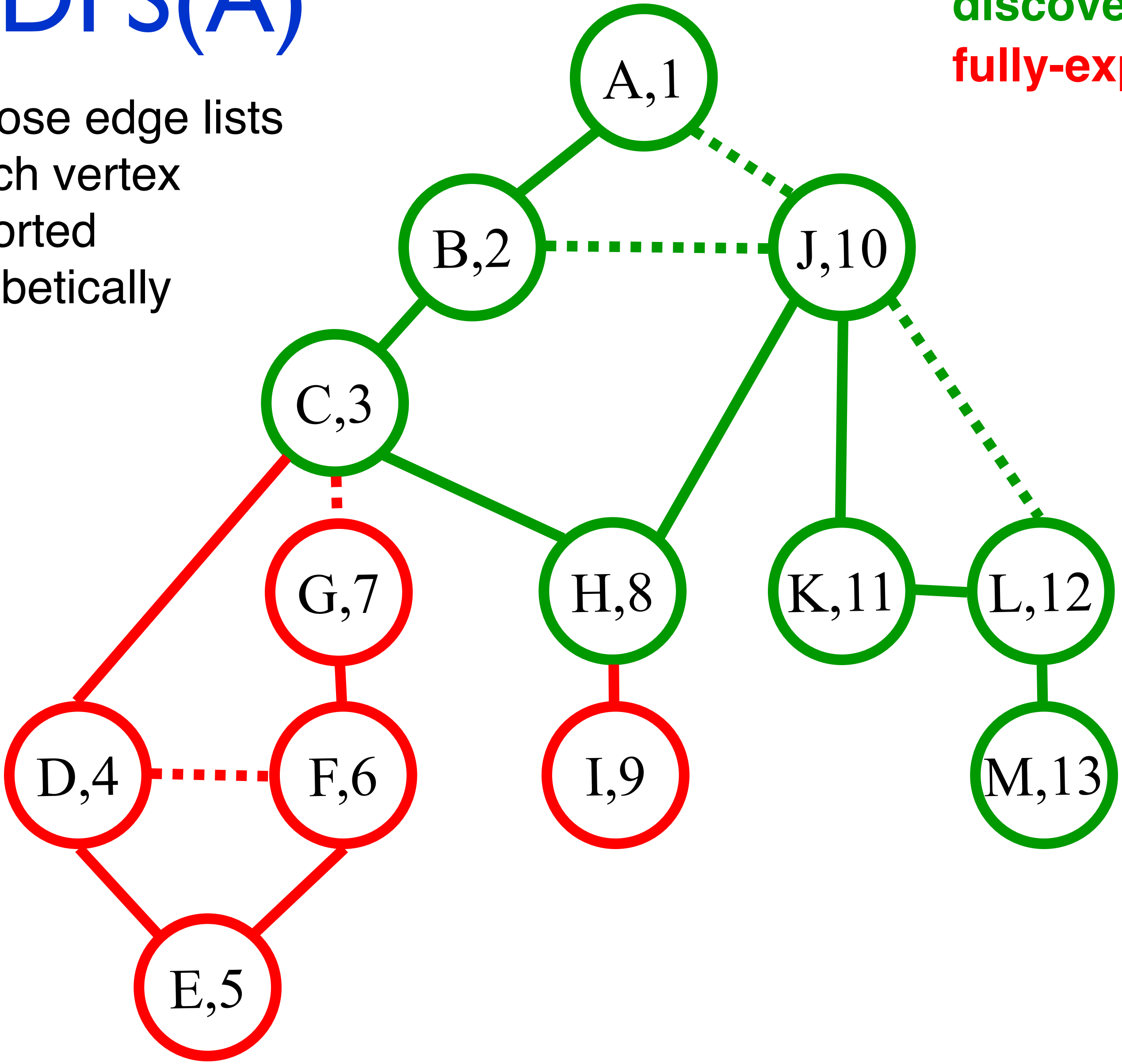
Suppose edge lists at each vertex are sorted alphabetically

Color code:

undiscovered

discovered

fully-explored



Call Stack:
(Edge list)

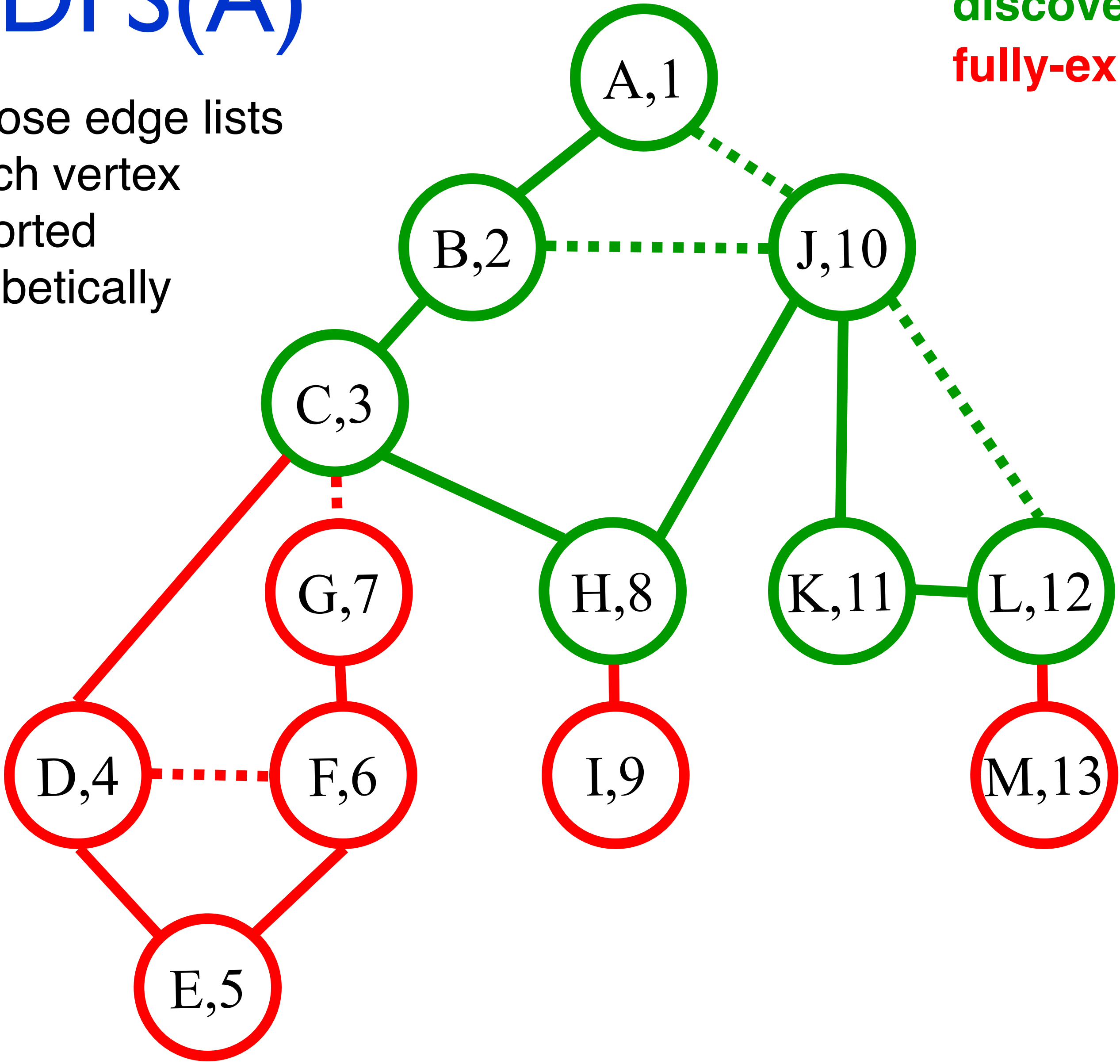
A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,~~G~~,~~H~~)
H (~~C~~,~~I~~,~~J~~)
J (~~A~~,~~B~~,~~H~~,~~K~~,L)
K (~~J~~,~~L~~)
L (~~J~~,~~K~~,~~M~~)
M (L)

DFS(A)

Suppose edge lists at each vertex are sorted alphabetically

Color code:

undiscovered
discovered
fully-explored



Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,~~G~~,~~H~~)
H (~~C~~,~~I~~,~~J~~)
J (~~A~~,~~B~~,~~H~~,~~K~~,L)
K (~~J~~,~~L~~)
L (~~J~~,~~K~~,~~M~~)

DFS(A)

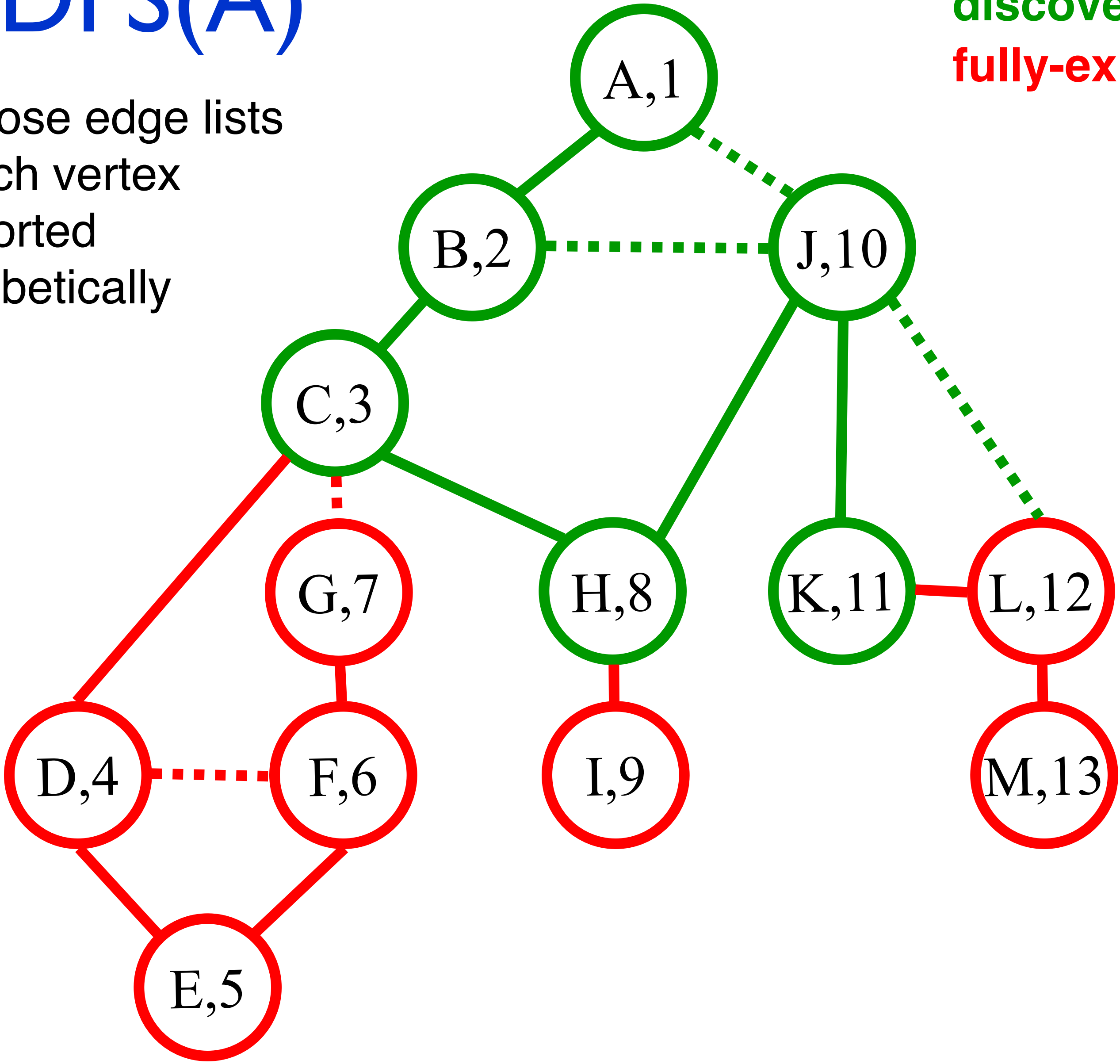
Suppose edge lists at each vertex are sorted alphabetically

Color code:

undiscovered

discovered

fully-explored



Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,~~G~~,~~H~~)
H (~~C~~,~~I~~,~~J~~)
J (~~A~~,~~B~~,~~H~~,~~K~~,L)
K (~~J~~,~~L~~)

DFS(A)

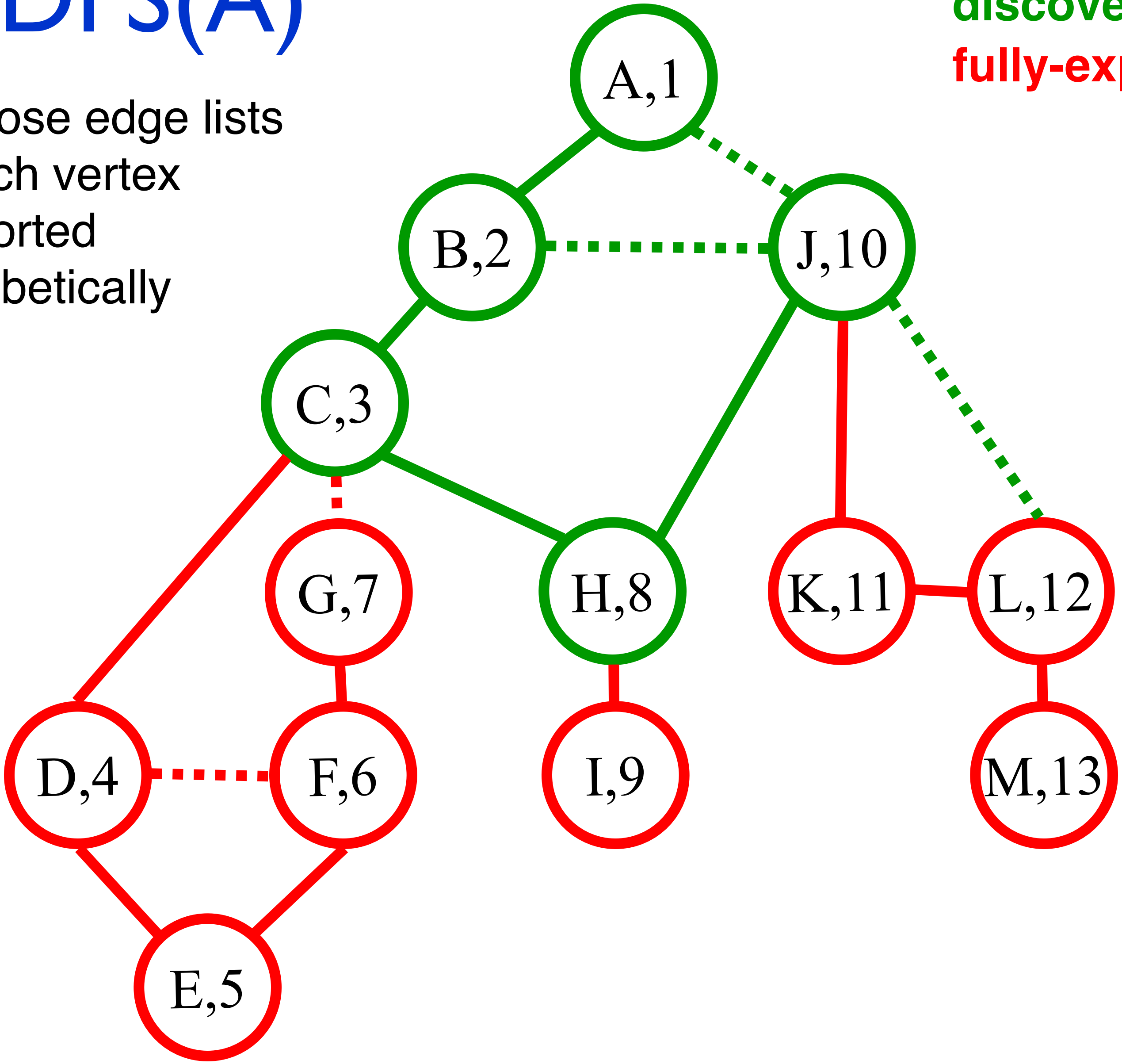
Suppose edge lists at each vertex are sorted alphabetically

Color code:

undiscovered

discovered

fully-explored



Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,~~G~~,~~H~~)
H (~~C~~,~~I~~,~~J~~)
J (~~A~~,~~B~~,~~H~~,~~K~~,L)

DFS(A)

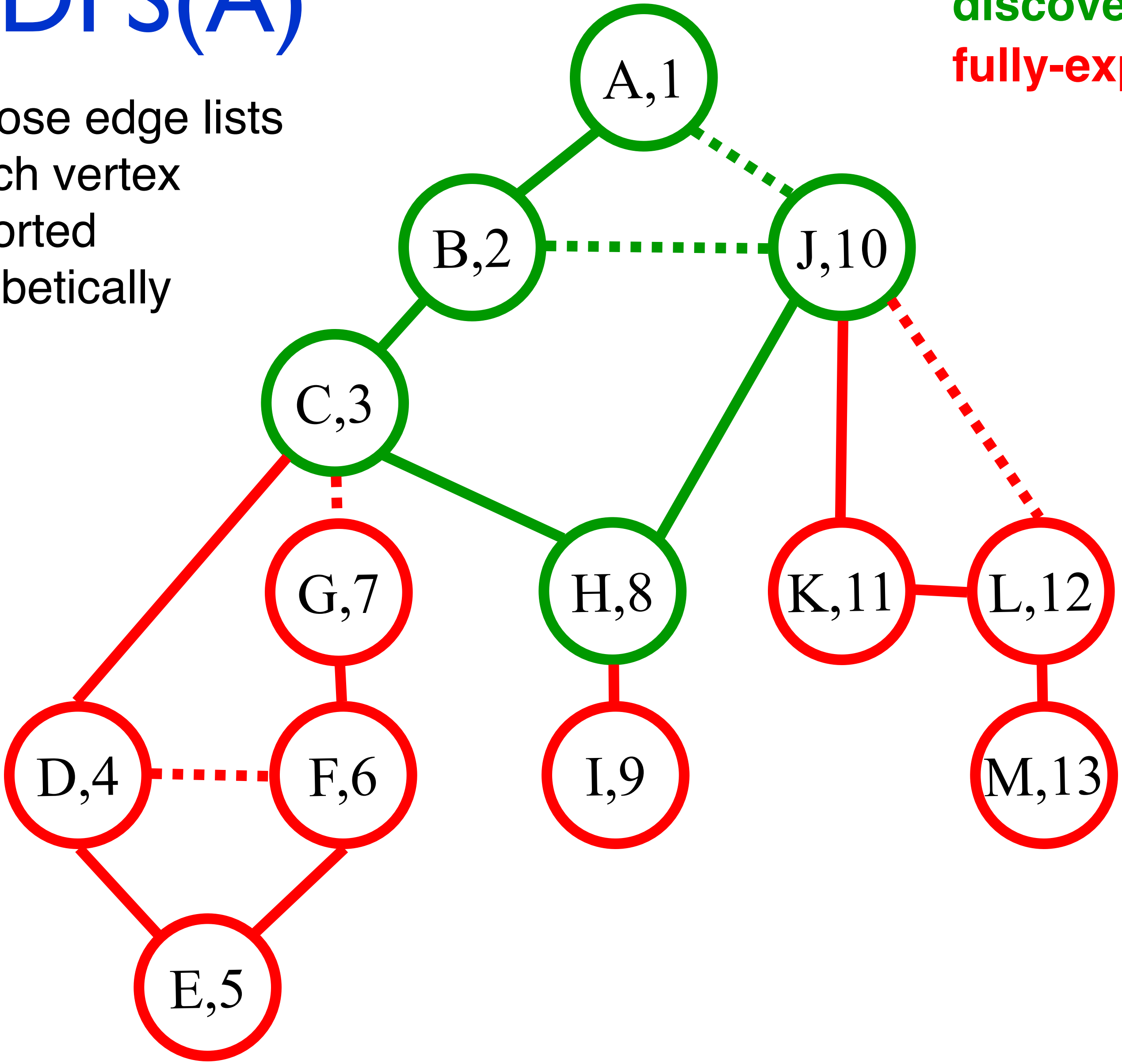
Suppose edge lists at each vertex are sorted alphabetically

Color code:

undiscovered

discovered

fully-explored



Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,~~G~~,~~H~~)
H (~~C~~,~~I~~,~~J~~)
J (~~A~~,~~B~~,~~H~~,~~K~~,~~L~~)

DFS(A)

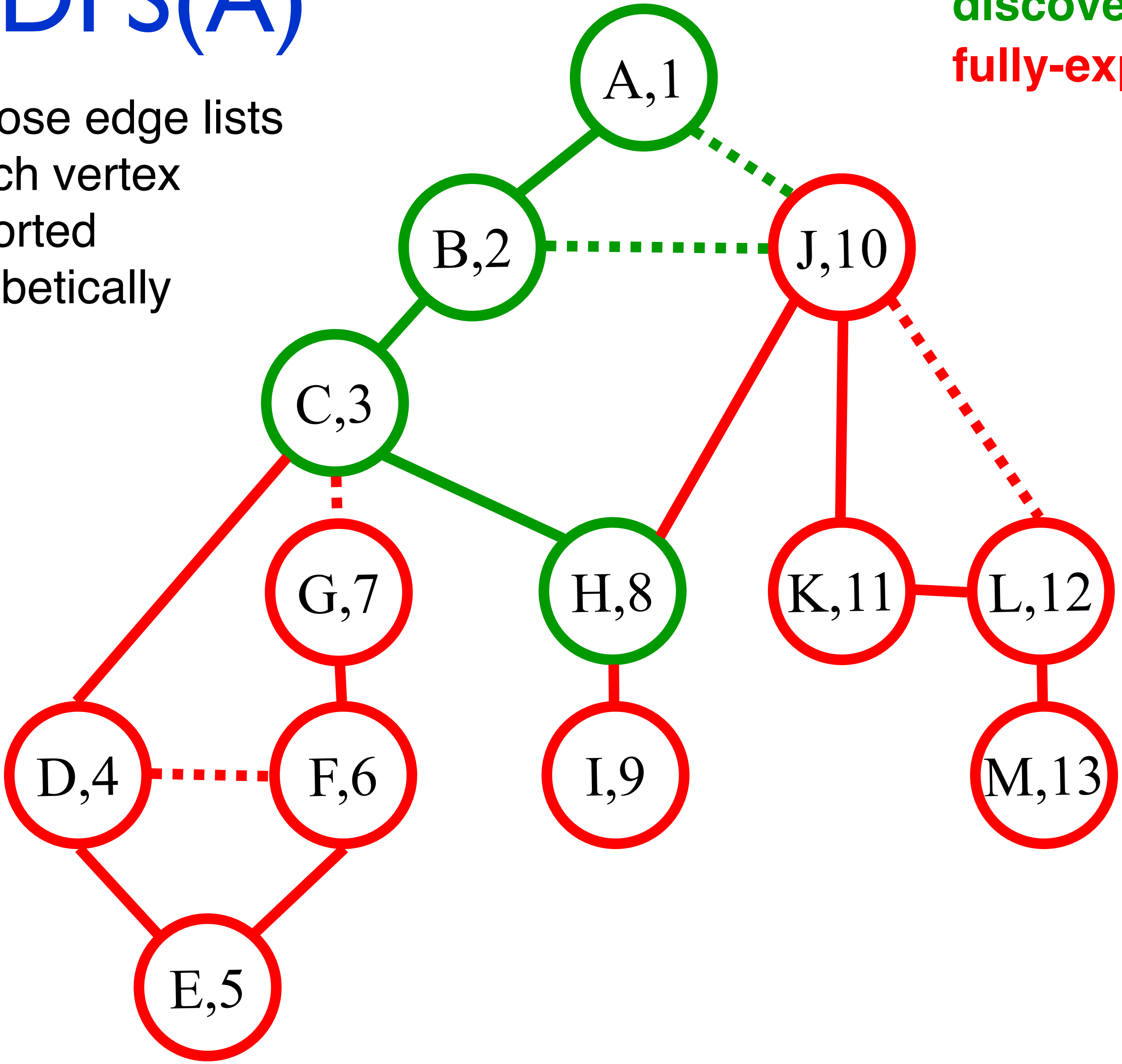
Suppose edge lists at each vertex are sorted alphabetically

Color code:

undiscovered

discovered

fully-explored



Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,~~G~~,~~H~~)
H (~~C~~,~~I~~,~~J~~)

DFS(A)

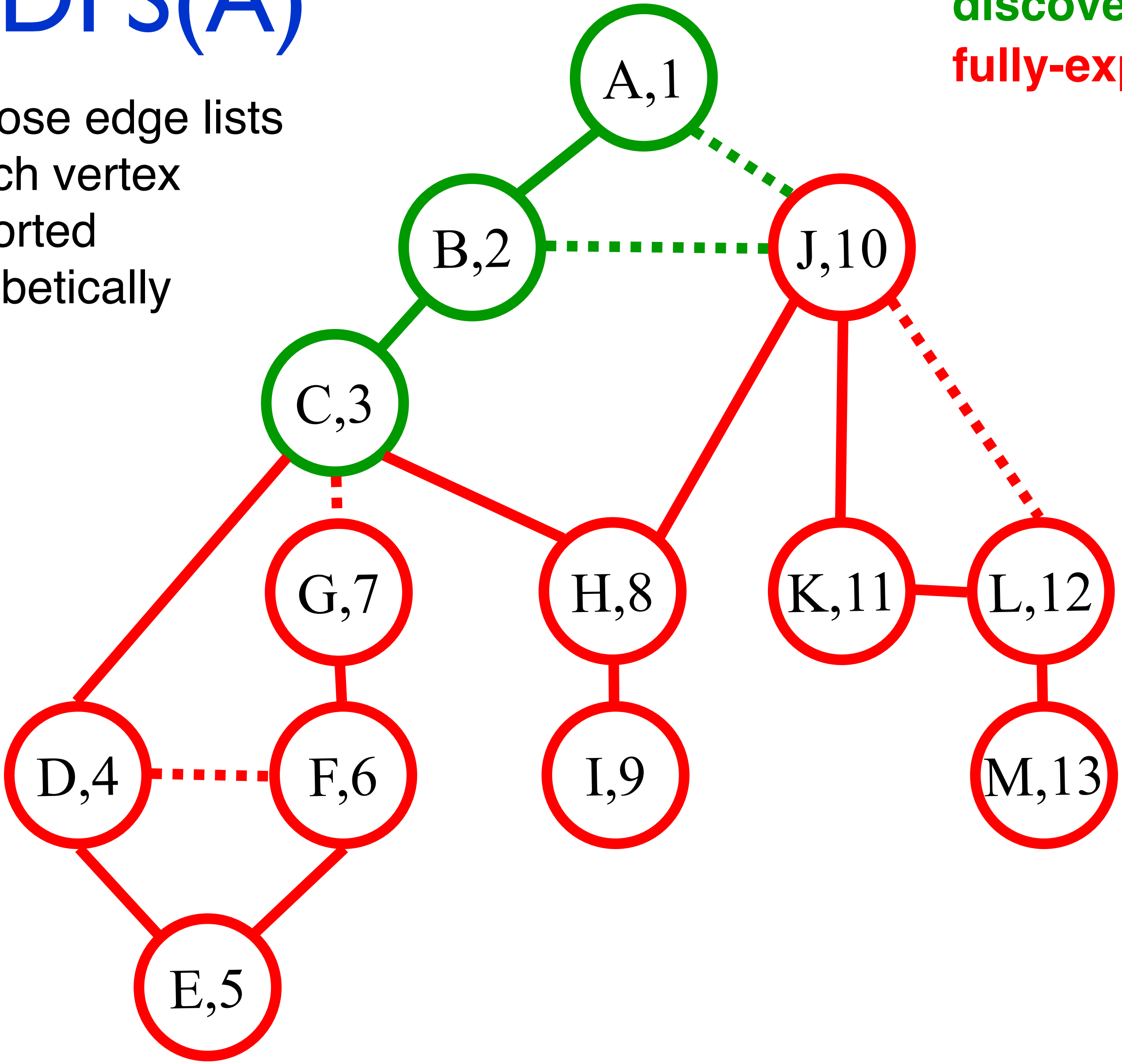
Suppose edge lists at each vertex are sorted alphabetically

Color code:

undiscovered

discovered

fully-explored



Call Stack:
(Edge list)

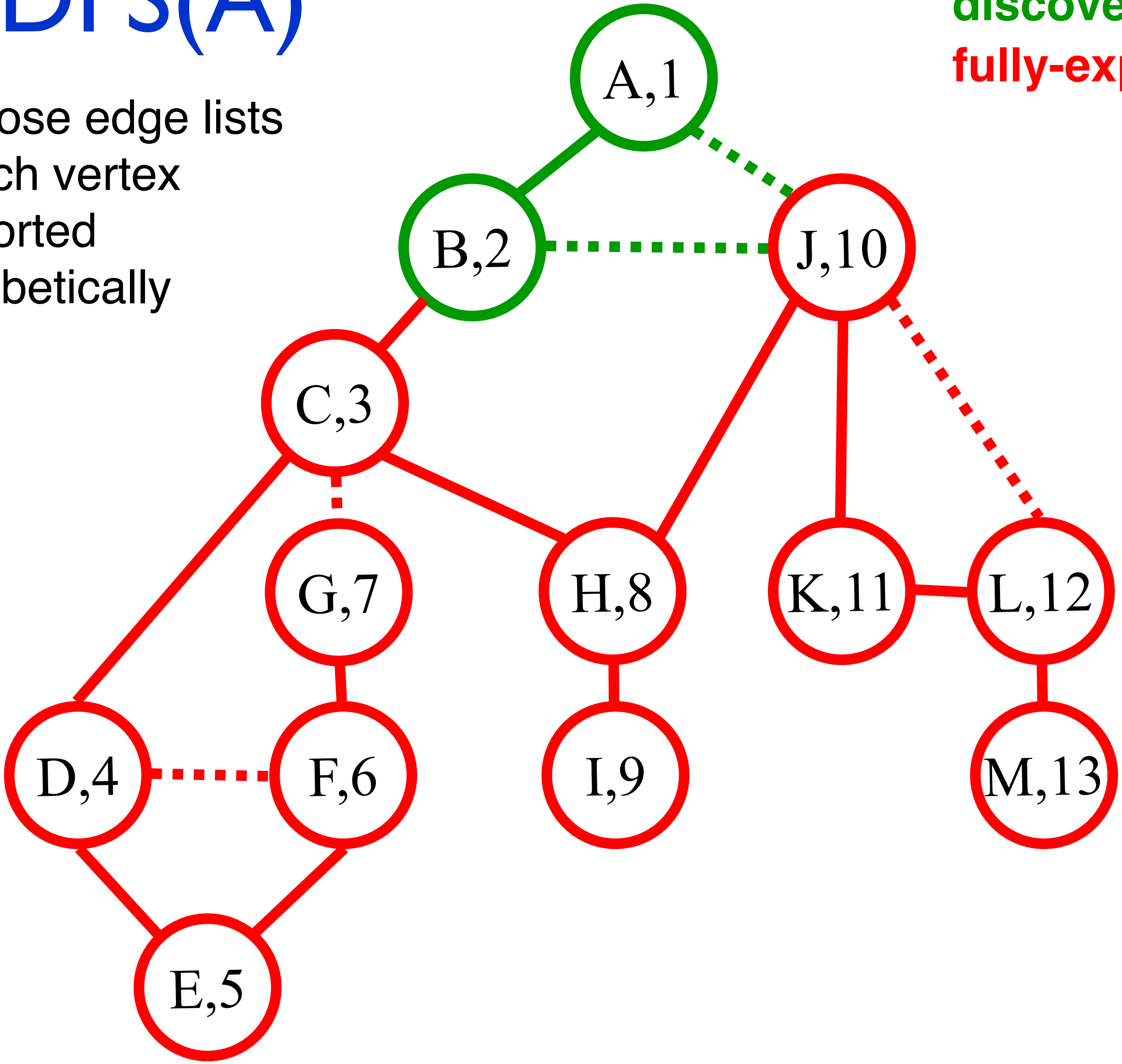
A (~~B~~,J)
B (~~A~~,~~C~~,J)
C (~~B~~,~~D~~,~~G~~,~~H~~)

DFS(A)

Suppose edge lists at each vertex are sorted alphabetically

Color code:

undiscovered
discovered
fully-explored

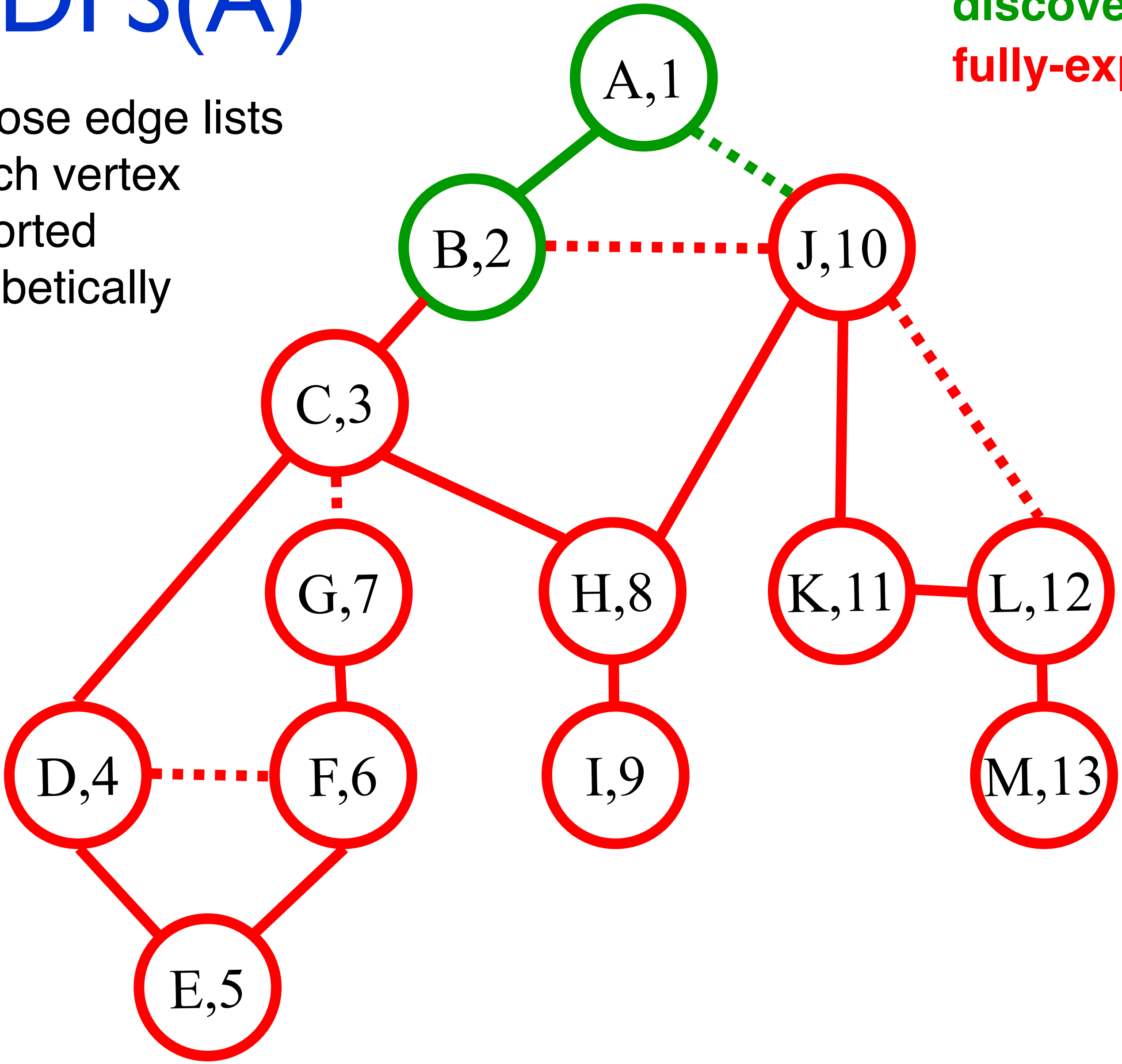


Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,J)

DFS(A)

Suppose edge lists at each vertex are sorted alphabetically



Color code:

undiscovered

discovered

fully-explored

Call Stack:
(Edge list)

A (~~B~~,J)
B (~~A~~,~~C~~,~~J~~)

DFS(A)

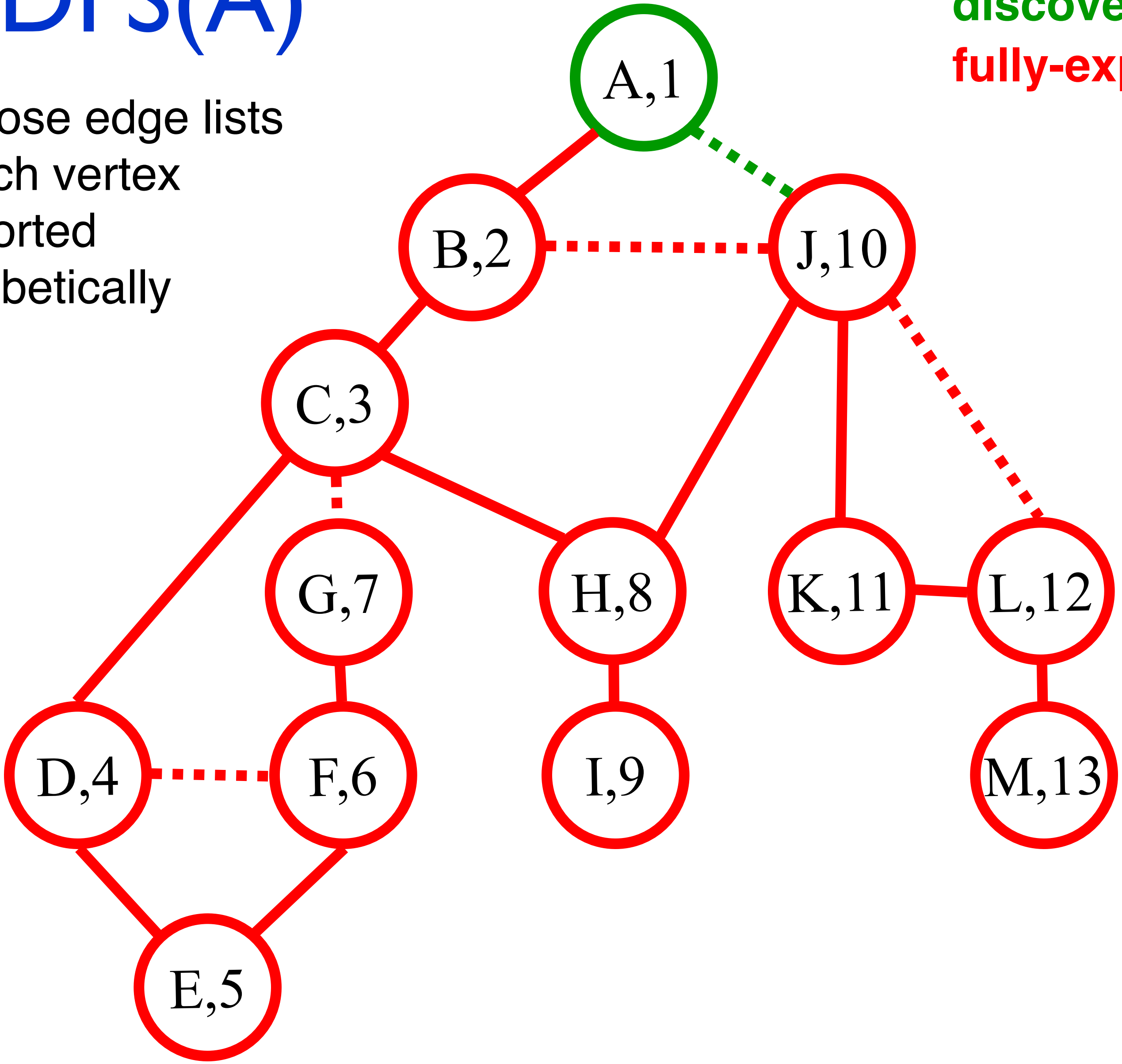
Suppose edge lists at each vertex are sorted alphabetically

Color code:

undiscovered

discovered

fully-explored



Call Stack:
(Edge list)

A (~~B~~,J)

DFS(A)

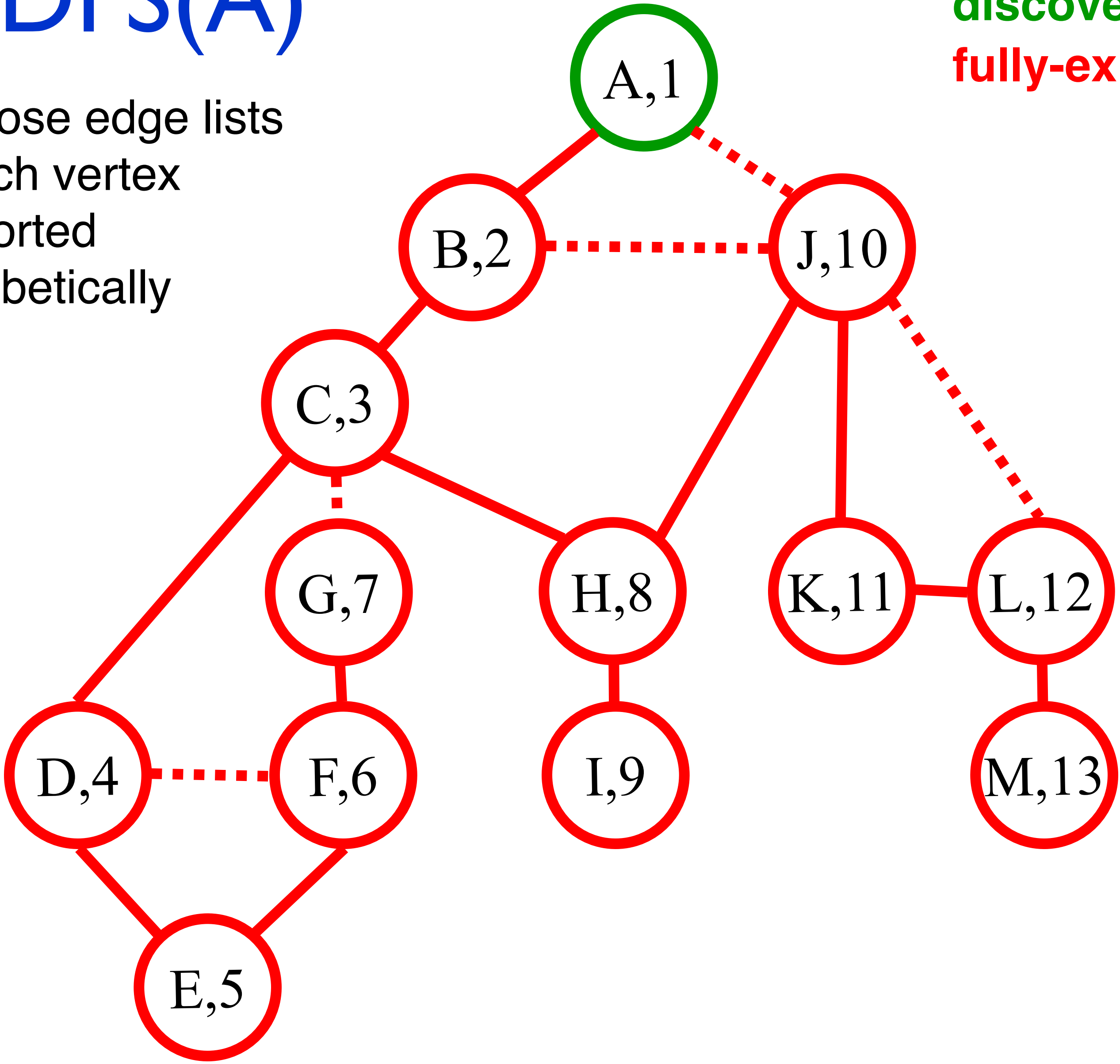
Suppose edge lists at each vertex are sorted alphabetically

Color code:

undiscovered

discovered

fully-explored



Call Stack:
(Edge list)

A (~~B~~,~~J~~)

DFS(A)

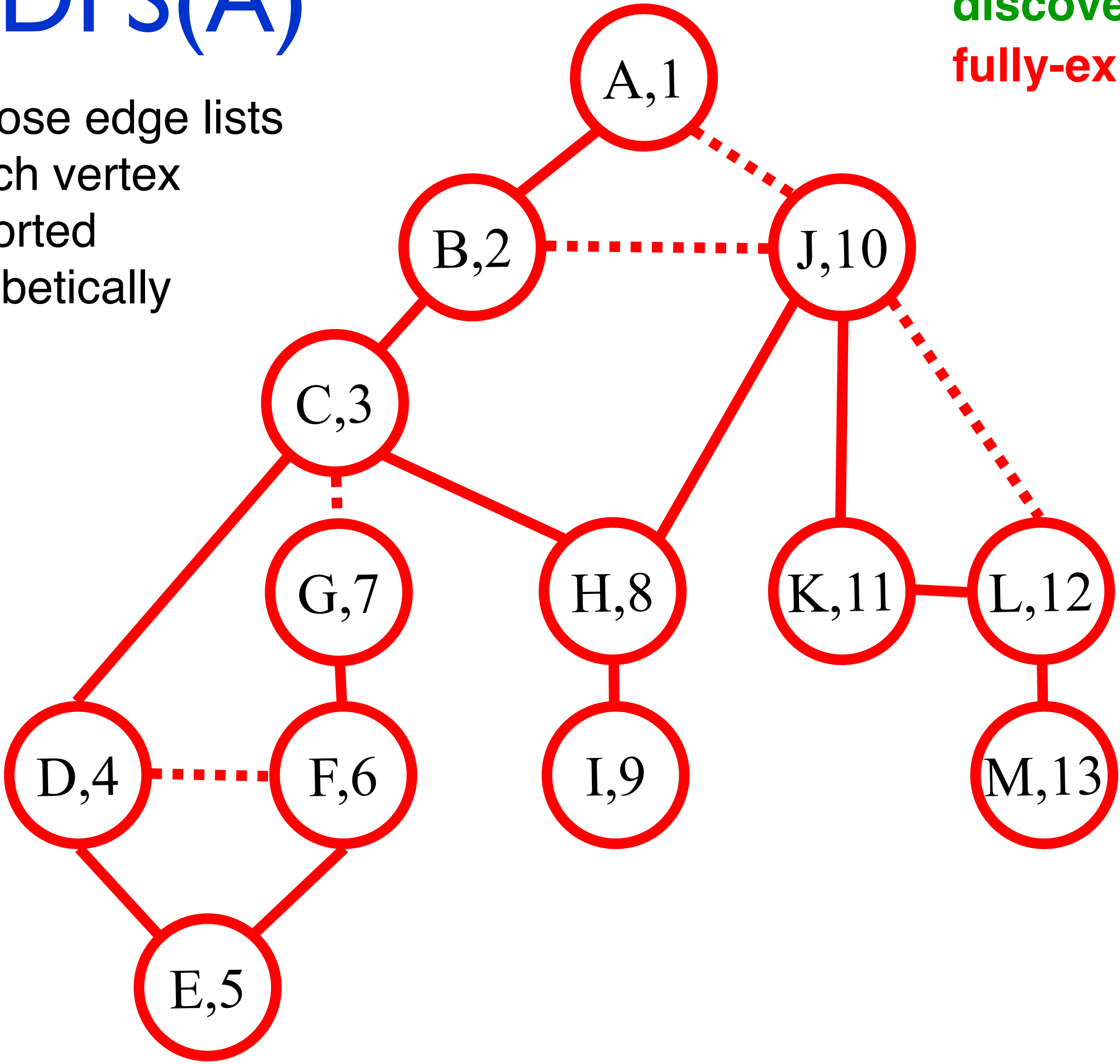
Suppose edge lists at each vertex are sorted alphabetically

Color code:

undiscovered

discovered

fully-explored



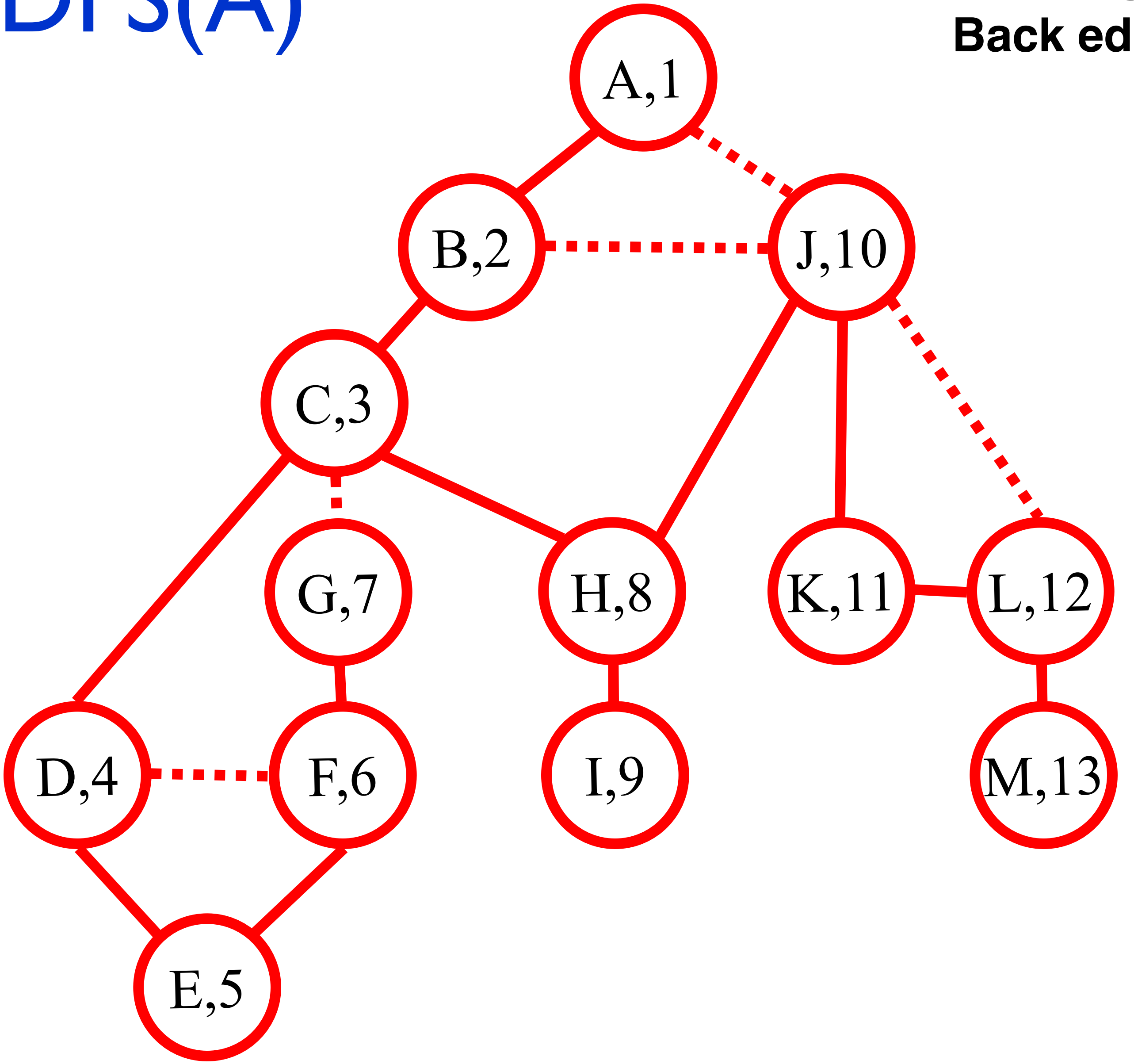
Call Stack: (Edge list)
TA-DA!!

DFS(A)

Edge code:

Tree edge —

Back edge ⋯

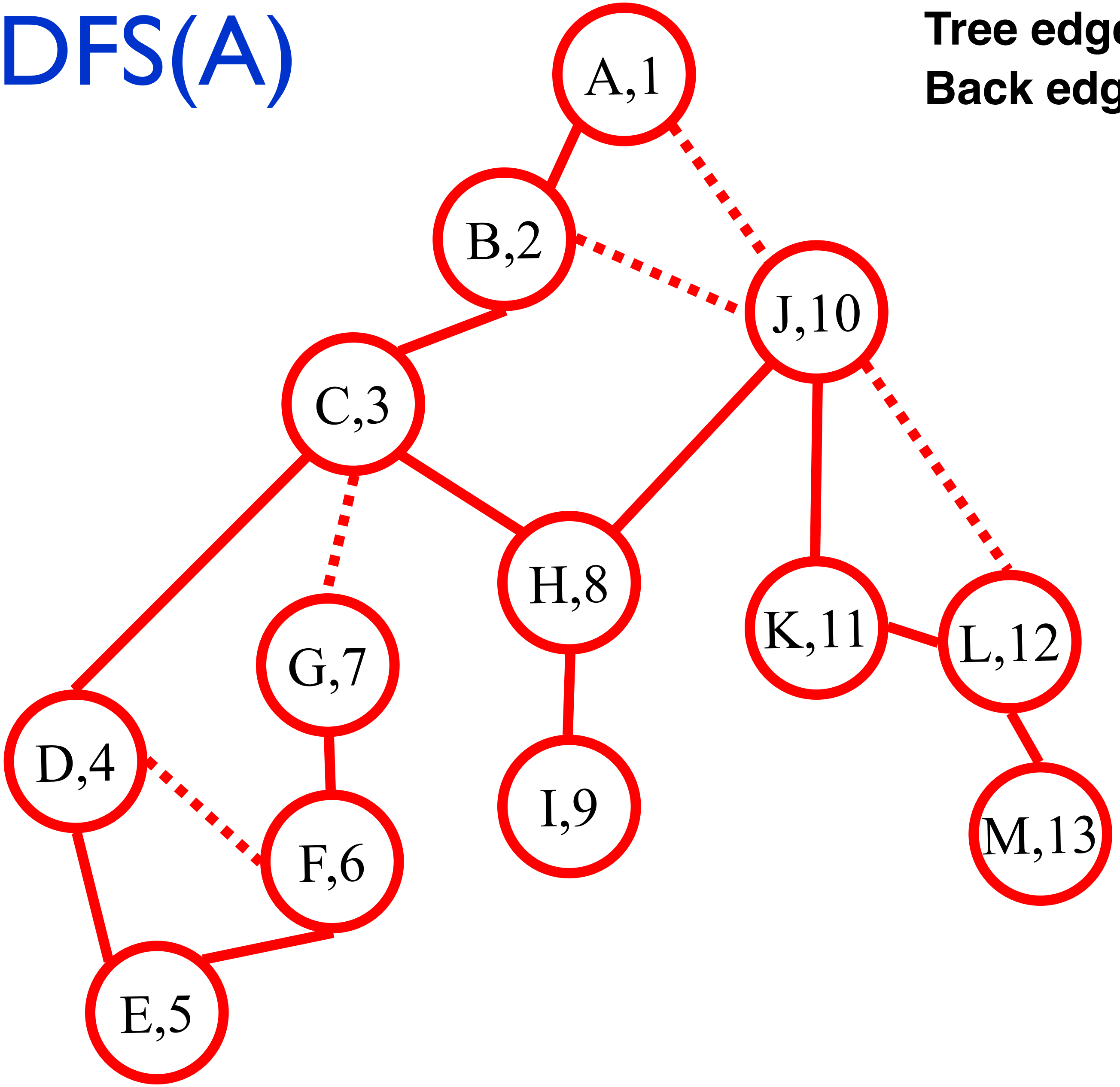


DFS(A)

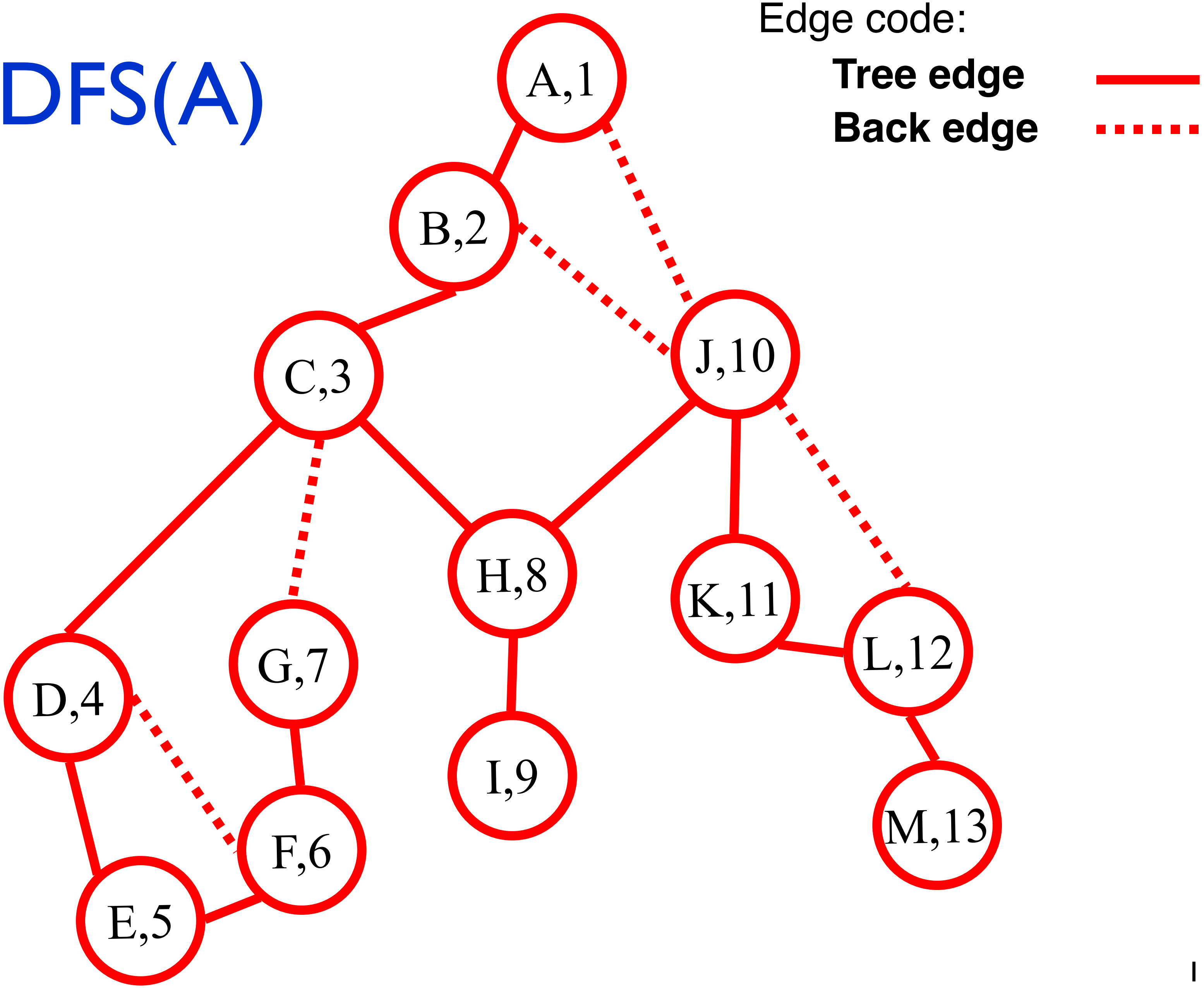
Edge code:

Tree edge ——

Back edge ······



DFS(A)

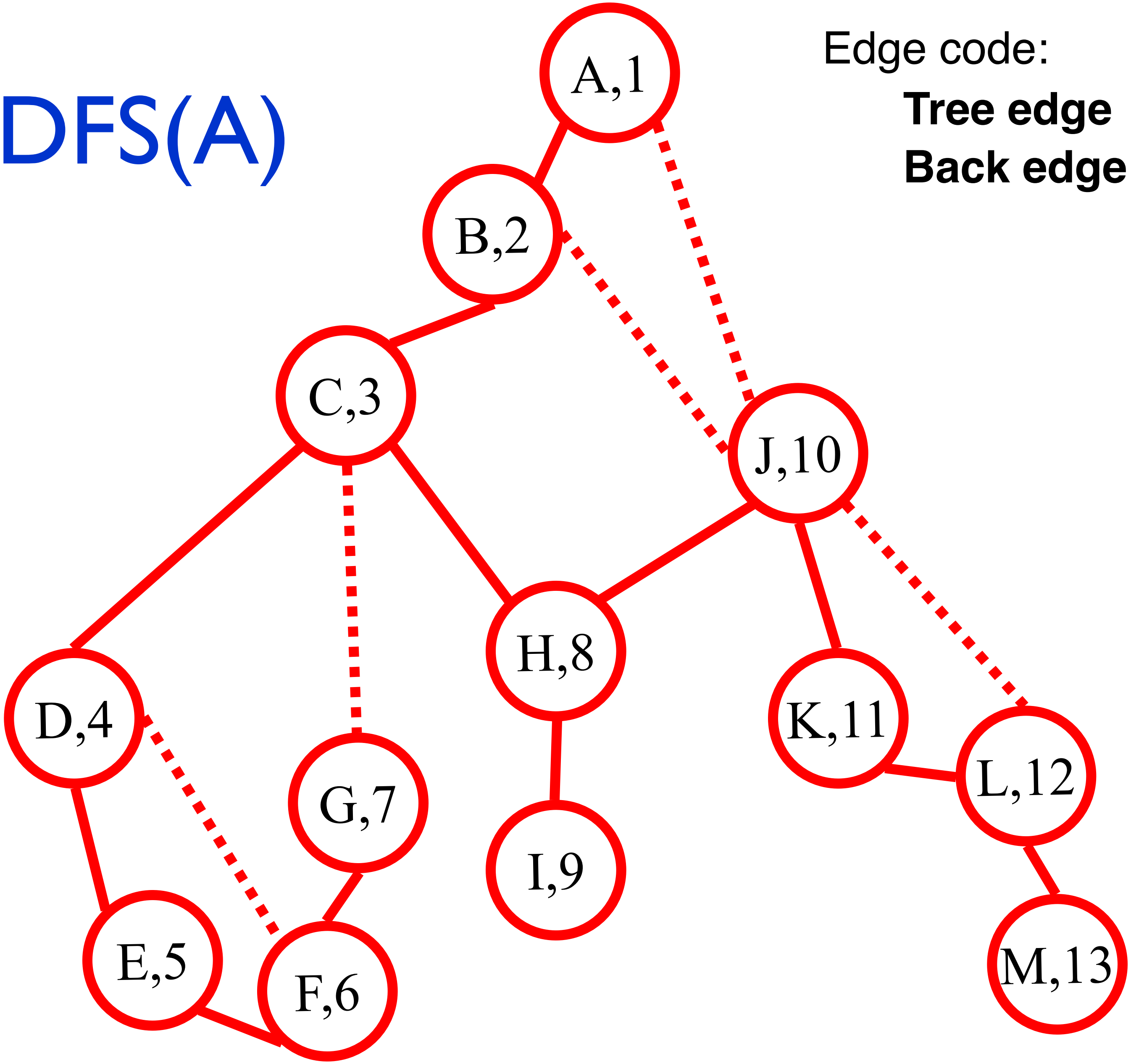


Edge code:

Tree edge ▬

Back edge ┅

DFS(A)

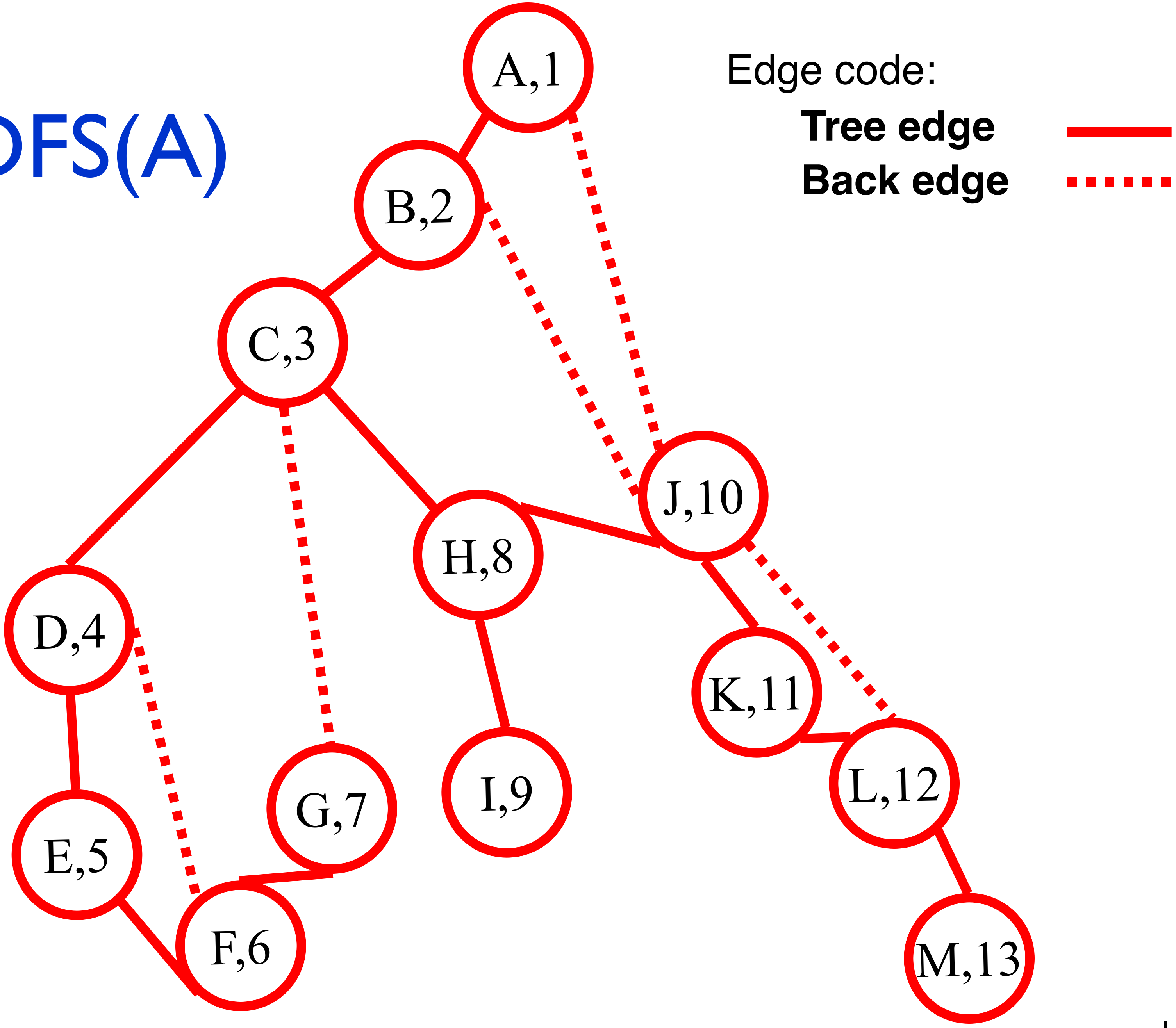


Edge code:

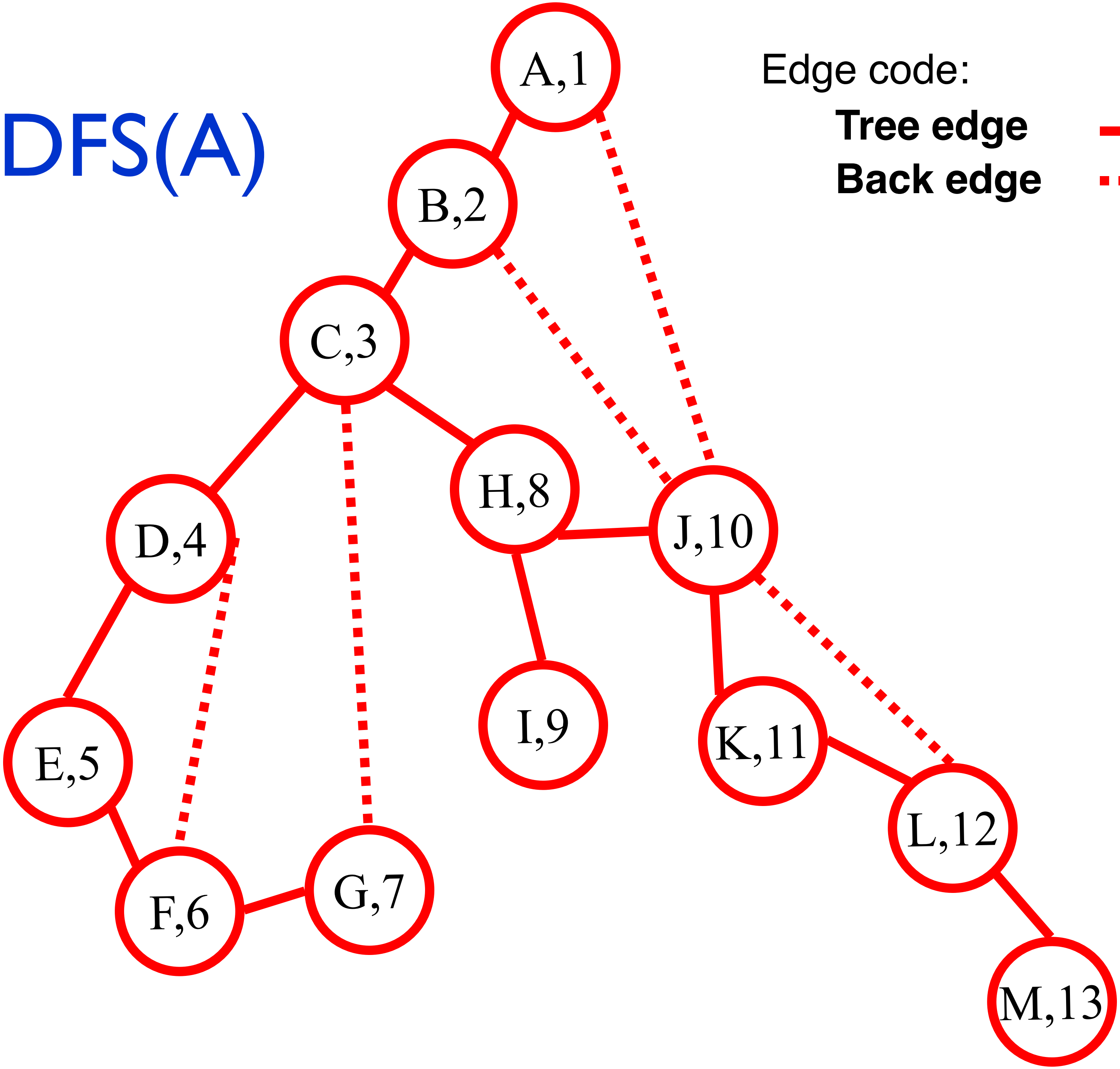
Tree edge ━━━

Back edge ┅┅┅

DFS(A)


A,1
B,2
C,3
D,4
E,5
F,6
G,7
H,8
I,9
J,10
K,11
L,12
M,13
Edge code:
Tree edge ——
Back edge ⋯⋯

DFS(A)

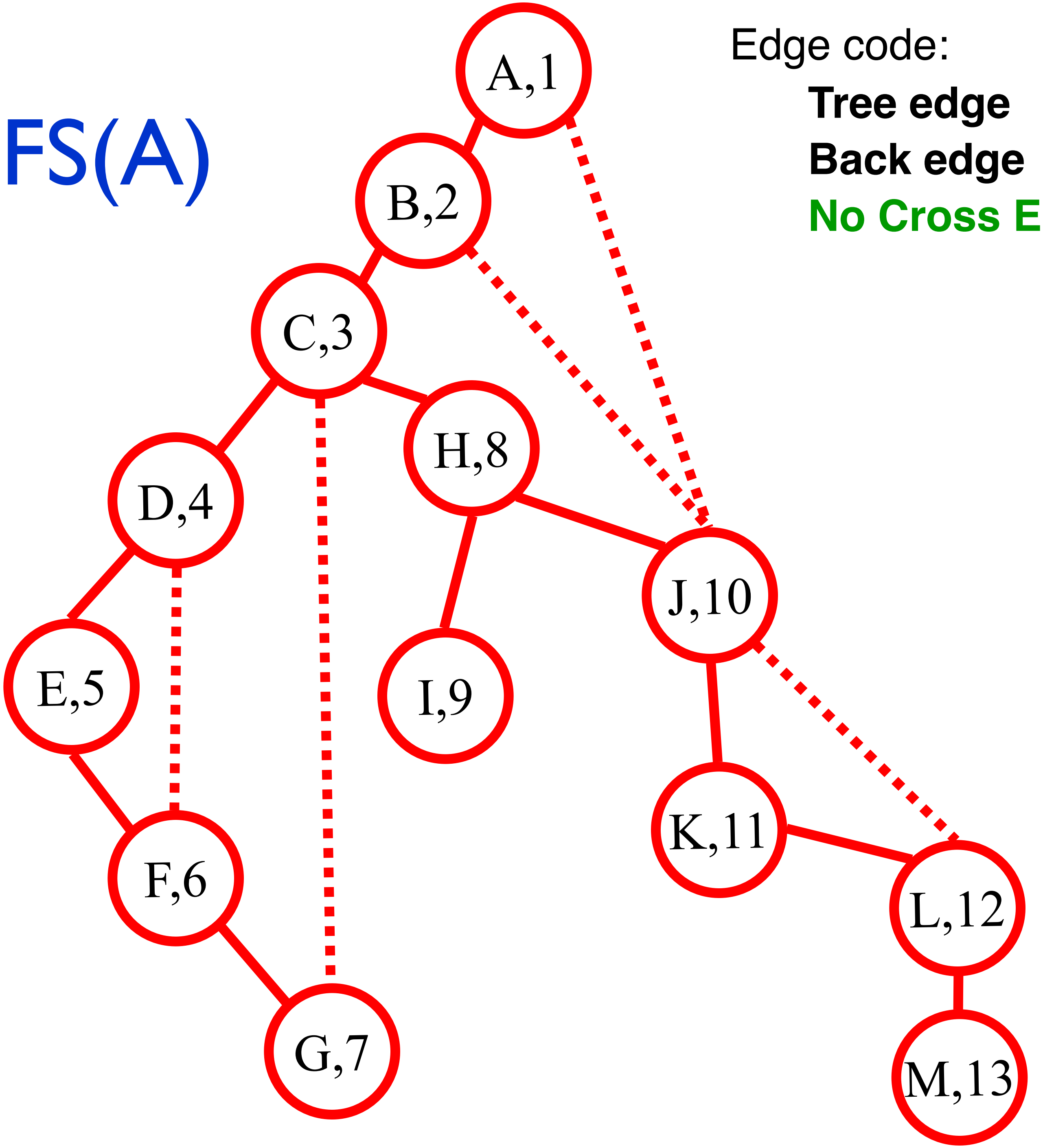


Edge code:

Tree edge —

Back edge ⋯

DFS(A)



Edge code:

Tree edge ———

Back edge ······

No Cross Edges!

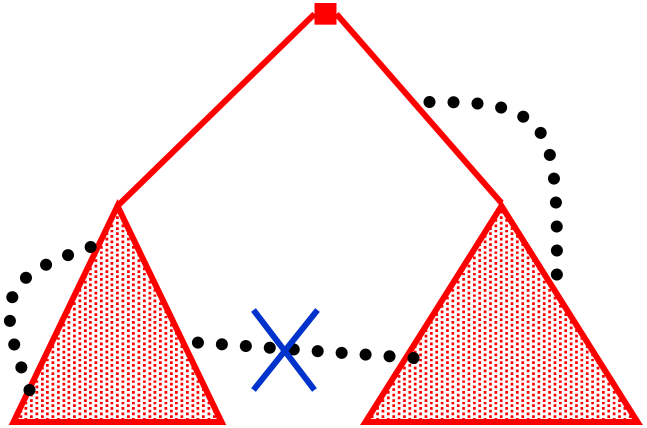
Properties of (Undirected) DFS(v)

- Like BFS(v):
  - DFS(v) visits x if and only if there is a path in G from v to x (through previously unvisited vertices)
  - Edges into then-undiscovered vertices define a ***tree*** – the "depth first spanning tree" of G
- Unlike the BFS tree:
  - the DF spanning tree isn't minimum depth
  - its levels don't reflect min distance from the root
  - non-tree edges never join vertices on the same or adjacent levels
- BUT…

Non-tree edges

All non-tree edges join a vertex and one of its descendents/ancestors in the DFS tree

No cross edges!



Why fuss about trees (again)?

As with BFS, DFS has found a tree in the graph s.t. non-tree edges are "simple"--only descendant/ancestor

A simple problem on trees

Given: tree T, a value L(v) defined for every vertex v in T

Goal: find M(v), the min value of L(v) anywhere in the subtree rooted at v (including v itself).

How?

DFS(v) – Recursive version

```
Global Initialization:
    for all nodes v, v.dfs# = -1   // mark v "undiscovered"
    dfscounter = 0

DFS(v)
    v.dfs# = dfscounter++          // v "discovered", number it
    for each edge (v,x)
        if (x.dfs# = -1)           // tree edge (x previously  undiscovered)
            DFS(x)
        else …                     // code for back-, fwd-, parent,
                                   // edges, if needed
                                   // mark v "completed," if needed
```