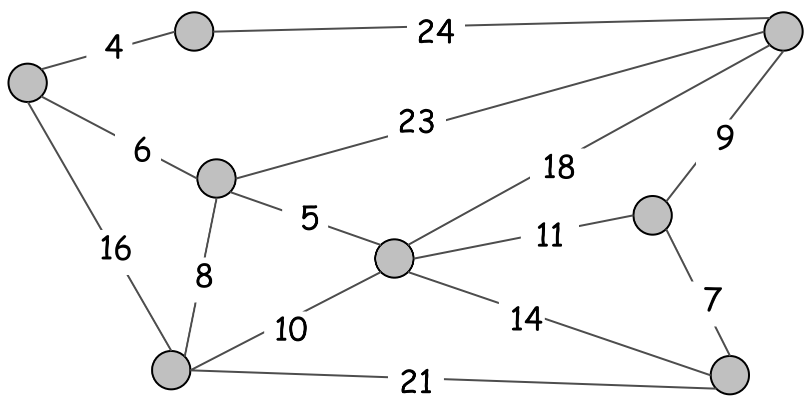
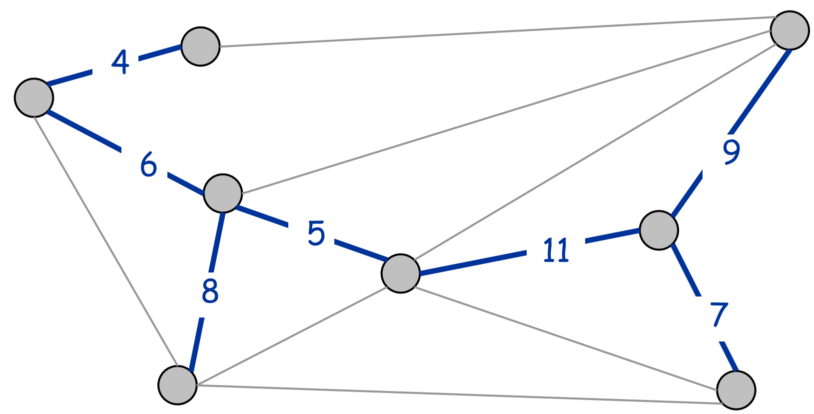
Minimum Spanning Tree

Minimum spanning tree. Given a connected graph $G = (V, E)$ with real-valued edge weights c_e, an MST is a subset of the edges $T \subseteq E$ such that T is a spanning tree whose sum of edge weights is minimized.



$G = (V, E)$



T, $\sum_{e \in T} c_e = 50$

Applications

MST is fundamental problem with diverse applications.

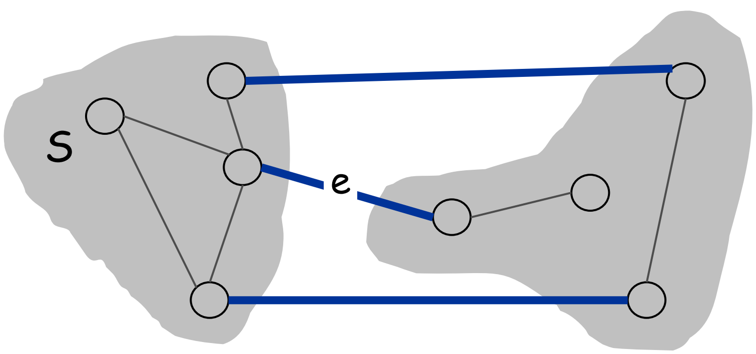
- Network design.
  - telephone, electrical, hydraulic, TV cable, computer, road
- Approximation algorithms for NP-hard problems.
  - traveling salesperson problem, Steiner tree
- Indirect applications.
  - max bottleneck paths
  - LDPC codes for error correction
  - image registration with Renyi entropy
  - learning salient features for real-time face verification
  - reducing data storage in sequencing amino acids in a protein
  - model locality of particle interactions in turbulent fluid flows
  - autoconfig protocol for Ethernet bridging to avoid cycles in a network
- Cluster analysis.

Greedy Algorithms

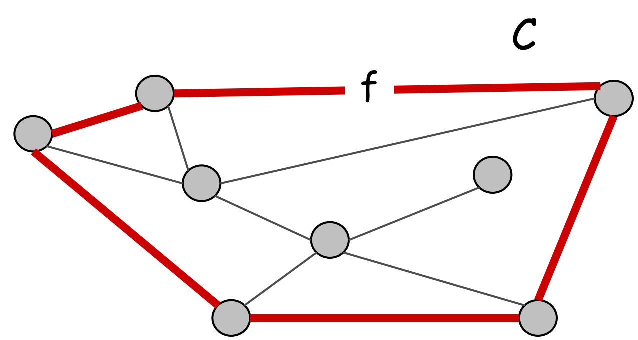
Simplifying assumption. All edge costs c_e are distinct.

Cut property. Let S be any subset of nodes (called a cut), and let e be the min cost edge with exactly one endpoint in S. Then every MST contains e.

Cycle property. Let C be any cycle, and let f be the max cost edge belonging to C. Then no MST contains f.



e is in the MST



f is not in the MST

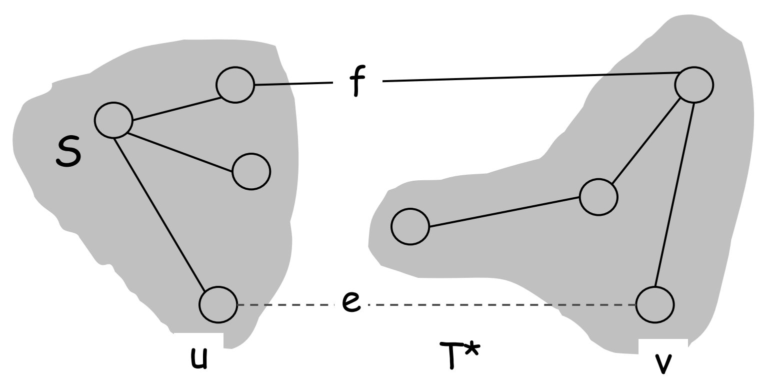
Greedy Algorithms

Simplifying assumption. All edge costs c_e are distinct.

Cut property. Let S be any subset of nodes, and let e be the min cost edge with exactly one endpoint in S. Then the MST T* contains e.

Pf. By contradiction

- Suppose e = {u,v} does not belong to T*.
- Adding e to T* creates a cycle C in T*.
- There is a path from u to v in T* $\Rightarrow$ there exists another edge, say f, that leaves S.
- T' = T* $\cup$ { e } - { f } is also a spanning tree.
- Since $c_e < c_f$, cost(T') < cost(T*).
- This is a contradiction. ▪

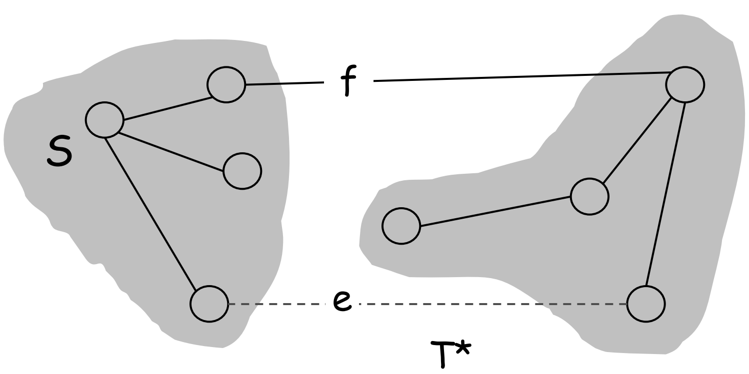


Simplifying assumption. All edge costs c_e are distinct.

Cycle property. Let C be any cycle in G, and let f be the max cost edge belonging to C. Then the MST T* does not contain f.

Pf. By contradiction

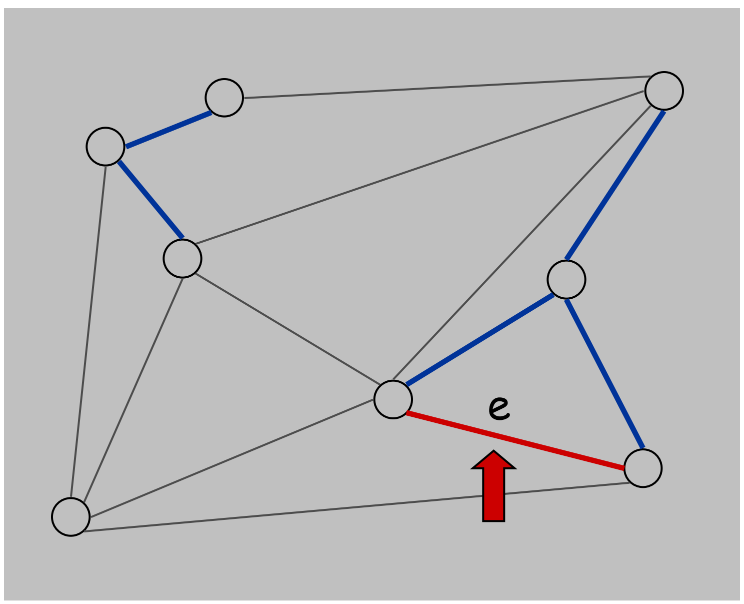
- Suppose f belongs to T*.
- Deleting f from T* cuts T* into two connected components.
- There exists another edge, say e, that is in the cycle and connects the components.
- $T' = T^* \cup \{ e \} - \{ f \}$ is also a spanning tree.
- Since $c_e < c_f$, $\text{cost}(T') < \text{cost}(T^*)$.
- This is a contradiction. ▪



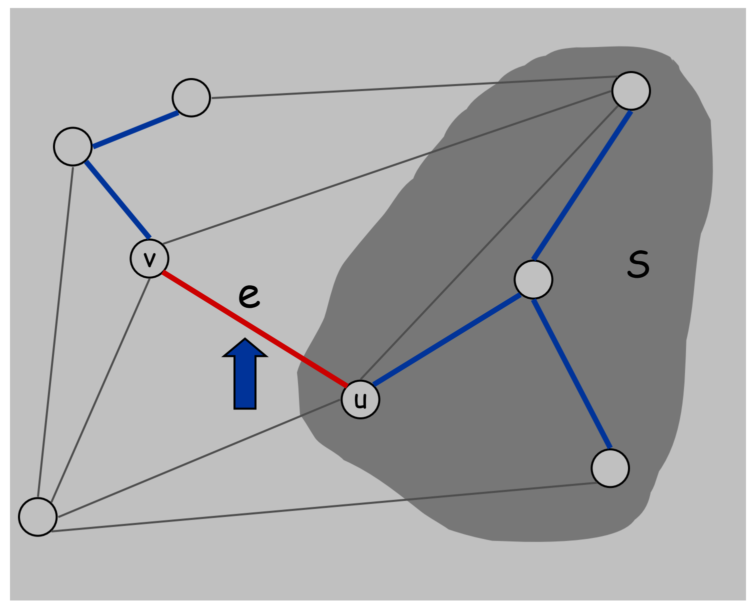
Kruskal's Algorithm: Proof of Correctness

Kruskal's algorithm. [Kruskal, 1956]

- Consider edges in ascending order of weight.
- Case 1: If adding e to T creates a cycle, discard e according to cycle property.
- Case 2: Otherwise, insert e = (u, v) into T according to cut property where S = set of nodes in u's connected component.



Case 1



Case 2

Implementation: Kruskal's Algorithm

Implementation. Use the union-find data structure.

- Build set T of edges in the MST.
- Maintain set for each connected component.
- O(m log n) for sorting and O(m log n) for union-find.

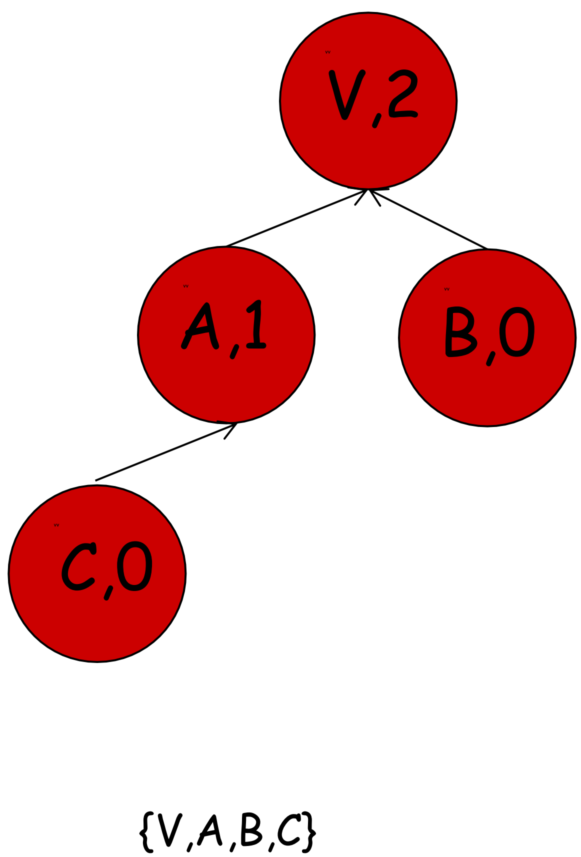
```
Kruskal(G, c) {
   Sort edges weights so that c₁ ≤ c₂ ≤ ... ≤ cₘ.
   T = {}

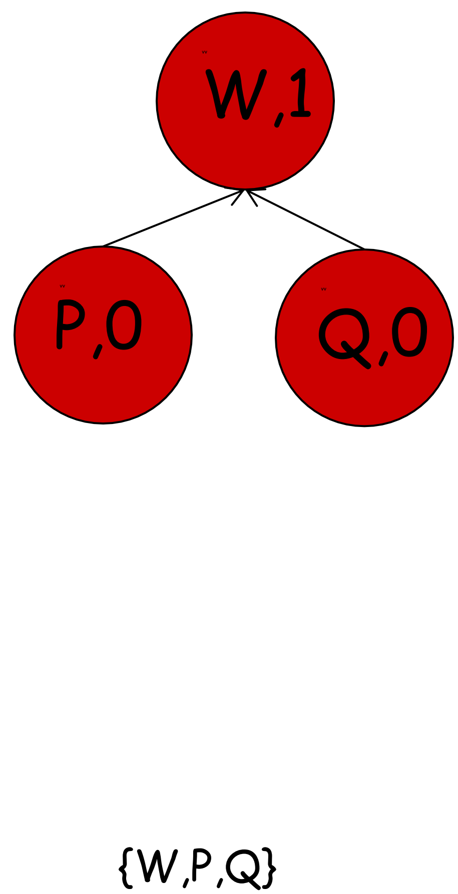
   foreach (u ∈ V) make a set containing singleton u

   for i = 1 to m          are u and v in different connected components?
      (u,v) = eᵢ
      if (u and v are in different sets) {
         T = T ∪ {eᵢ}
         merge the sets containing u and v
      }                     merge two components
   return T
}
```

Union Find Data Structure

- Each set is represented as a tree of pointers, where every vertex is labeled with longest path ending at the vertex





Union Find Data Structure

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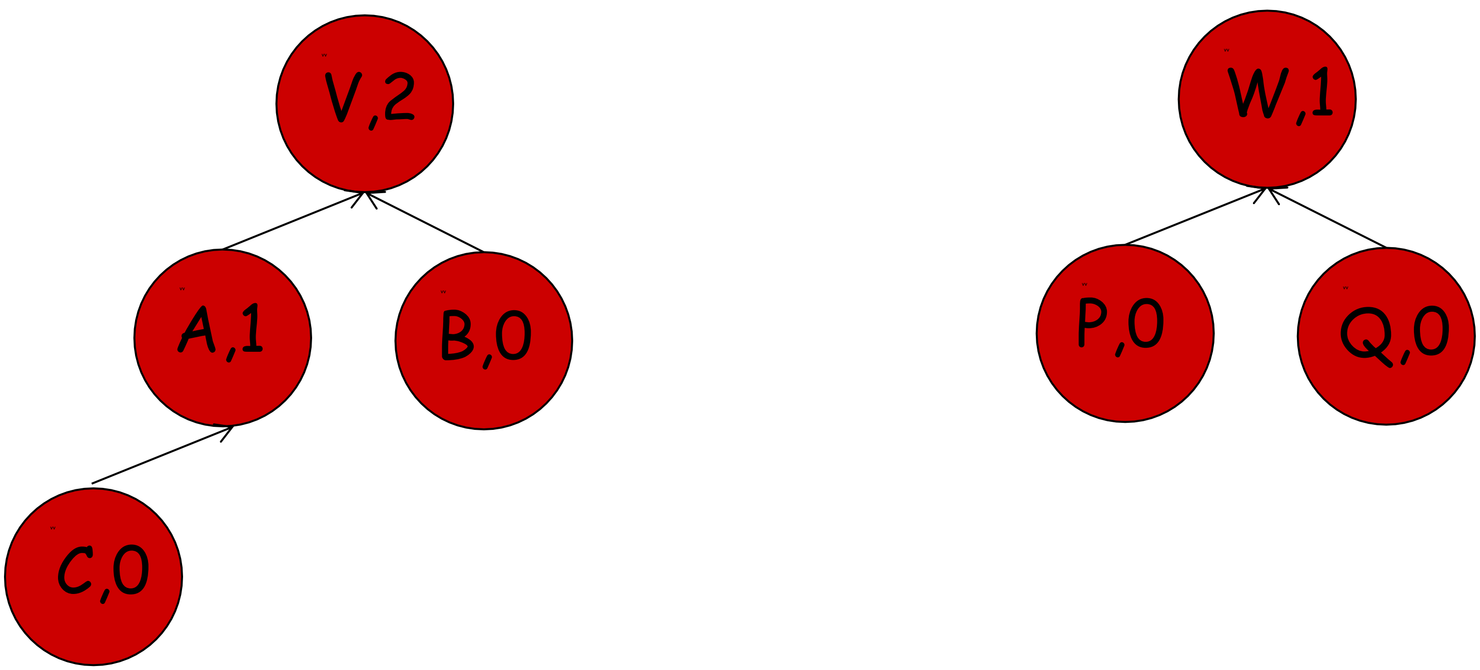
To **check** whether A,Q are in same connected component, follow pointers and check if root is the same.



Union Find Data Structure

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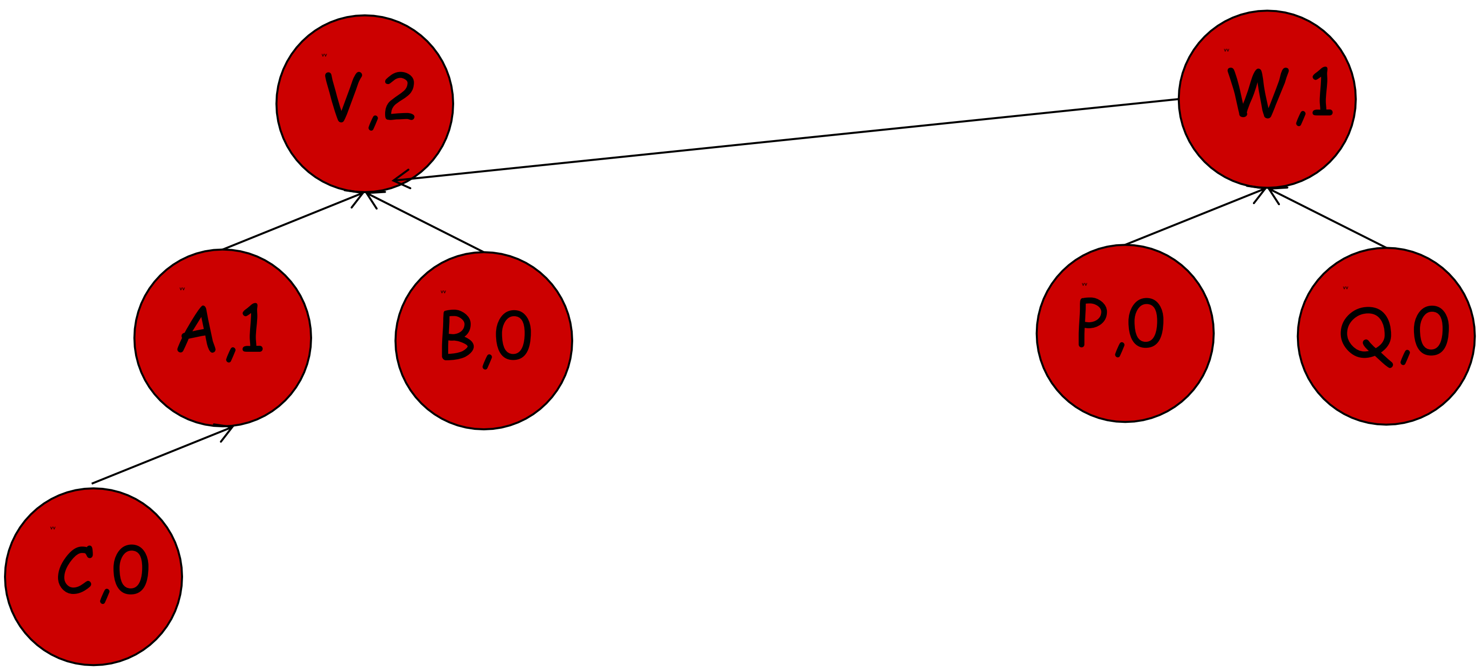
- To **merge** sets, make the root with the smaller label point to the root with the bigger label (adjusting labels if necessary).



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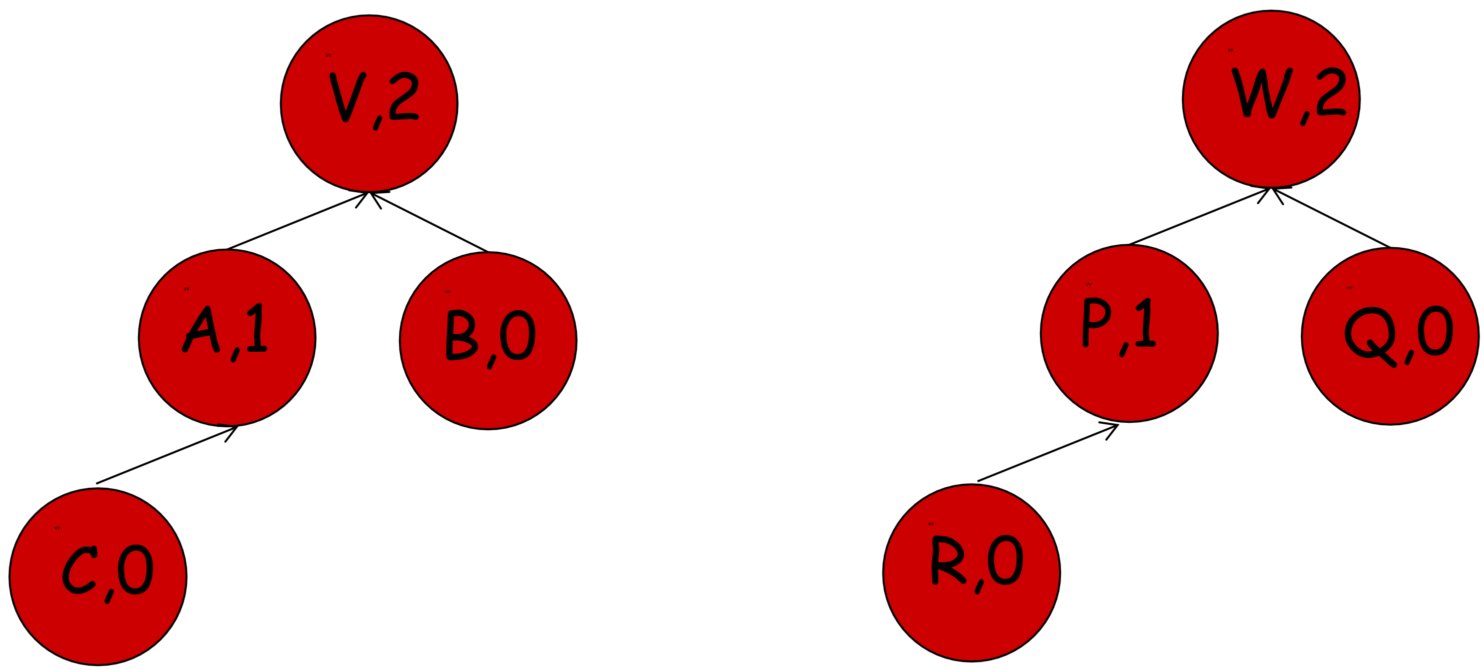

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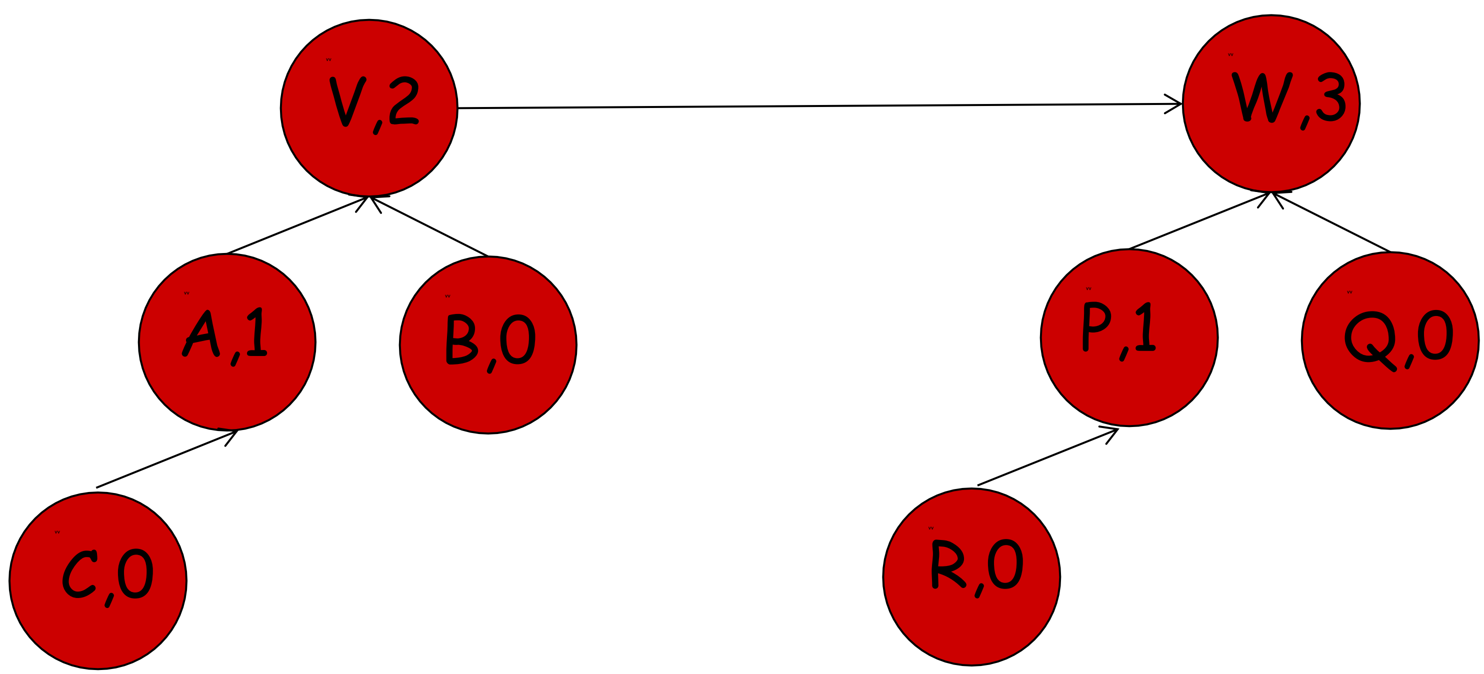
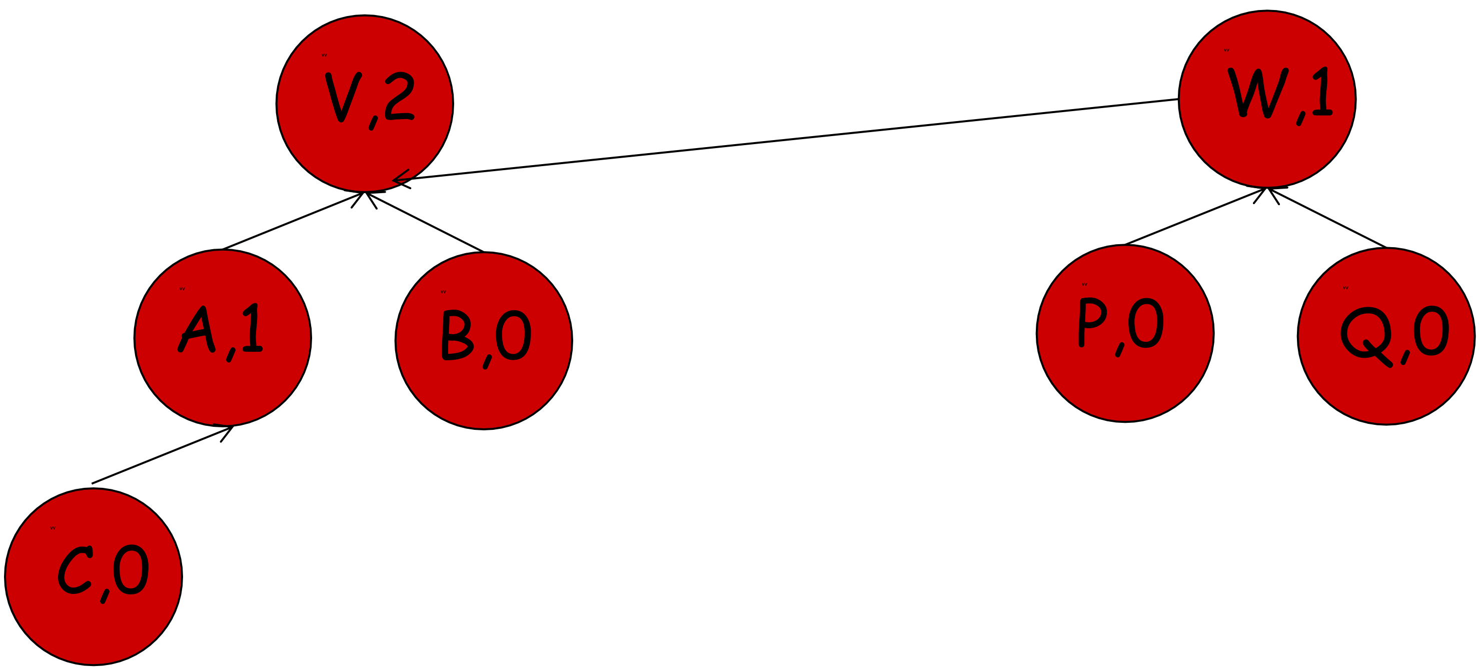


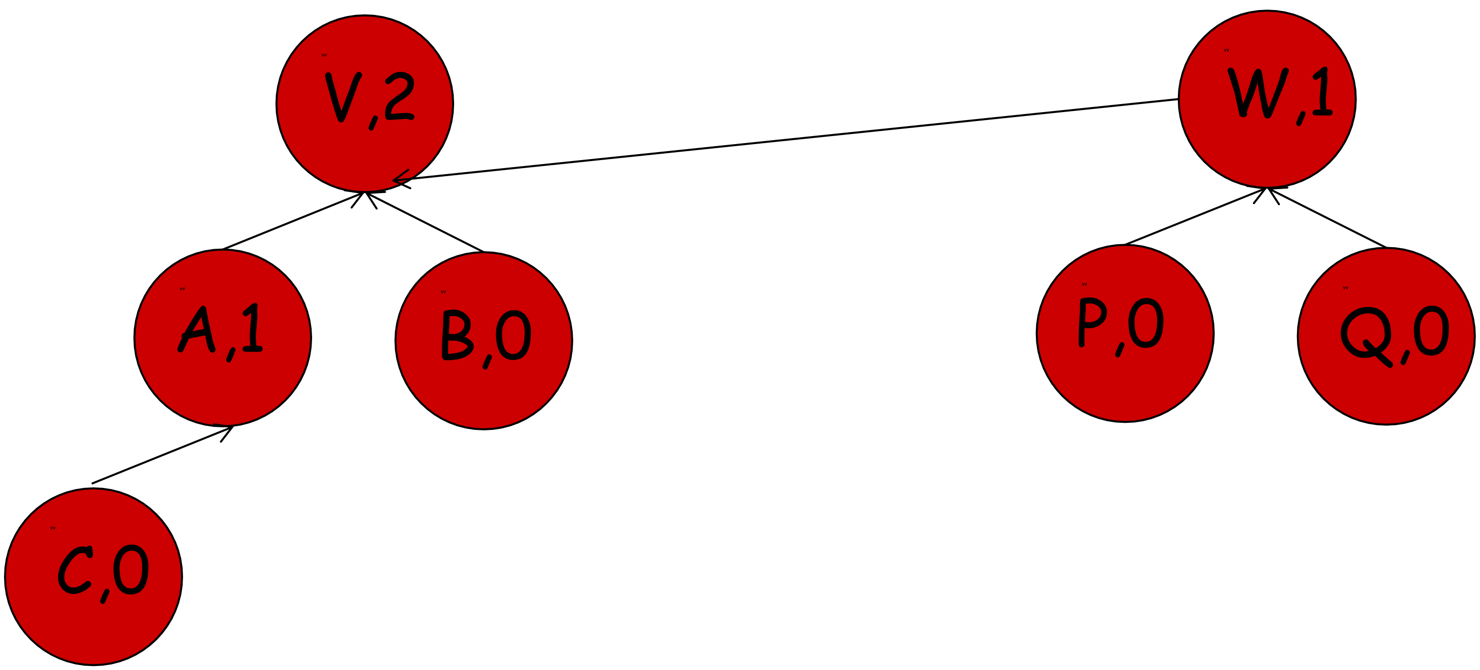
Union Find Data Structure

- To merge sets, make the root with the smaller label point to the root with the bigger label (adjusting labels if necessary).
- **Claim:** If the label of a root is k, there are at least 2^k elements in the set. (Therefore the depth of any tree in algorithm is at most log n)



Union Find Data Structure

- Claim: If the label of a root is k, there are at least 2^k elements in the set. (Therefore the depth of any tree in algorithm is at most log n)
- Pf: By induction on k. When k = 0, this is true. If we merge roots with labels k1 > k2, the number of vertices only increases while the label stays the same. If k1 = k2, the merged tree has label k1+1, and by induction, at least $2^{k1} + 2^{k2} = 2^{k1+1}$ elements.



Implementation: Kruskal's Algorithm

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      if (u and v are in different sets) {
         T = T ∪ {eᵢ}
         merge the sets containing u and v
      }
   return T
}
```

Removing the assumption that edge weights are distinct

Suppose edge weights are not distinct, and Kruskal's algorithm sorts edges so
$w(e1) \leq w(e2) \leq \ldots \leq w(em)$

Suppose Kruskal finds MST T of weight w(T), but the optimal solution T* has weight $w(T^*) < w(T)$.

Perturb each of the weights by a very small amount so that

$w'(e1) < w'(e2) < \ldots < w'(em)$

If the perturbation is small enough, $w'(T^*) < w'(T)$. However, this contradicts the correctness of Kruskal's algorithm, since the algorithm will still find T!

Greedy Algorithms

Kruskal's algorithm. Start with T = {}. Consider edges in ascending order of cost. Insert edge e in T unless doing so would create a cycle.

Reverse-Delete algorithm. Start with T = E. Consider edges in descending order of cost. Delete edge e from T unless doing so would disconnect T.

Prim's algorithm. Start with some root node s and greedily grow a tree T from s outward. At each step, add the cheapest edge e to T that has exactly one endpoint in T.

Remark. All three algorithms produce an MST.