**Vertex cover**

**VERTEX-COVER.** Given a graph $G = (V, E)$ and an integer $k$, is there a subset of vertices $S \subseteq V$ such that $|S| \leq k$, and for each edge, at least one of its endpoints is in $S$?

**Ex.** Is there a vertex cover of size $\leq 4$?

**Ex.** Is there a vertex cover of size $\leq 3$?

![Graph with vertex and independent set labels]
3-satisfiability reduces to vertex cover

**Theorem.** \(3\text{-SAT} \leq_p \text{VERTEX-COVER} \).  

**Pf.** Given an instance \(\Phi\) of 3-SAT, we construct an instance \((G, k)\) of VERTEX-COVER that has a vertex cover of size \(2k\) iff \(\Phi\) is satisfiable.

**Construction.**  
- \(G\) contains 3 nodes for each clause, one for each literal.  
- Connect 3 literals in a clause in a triangle.

\[
\Phi = (\overline{x_1} \lor x_2 \lor x_3) \land (x_1 \lor \overline{x_2} \lor x_3) \land (\overline{x_1} \lor x_2 \lor x_4)
\]
3-satisfiability reduces to vertex cover

**Theorem.** $3$-**SAT** $\leq_P$ **VERTEX-COVER**.

**Pf.** Given an instance $\Phi$ of $3$-**SAT**, we construct an instance $(G, k)$ of **VERTEX-COVER** that has a vertex cover of size $2k$ iff $\Phi$ is satisfiable.

**Construction.**

- $G$ contains 3 nodes for each clause, one for each literal.
- Connect 3 literals in a clause in a triangle.
- Connect literal to each of its negations.

\[\Phi = (\overline{x_1} \lor x_2 \lor x_3) \land (x_1 \lor \overline{x_2} \lor x_3) \land (\overline{x_1} \lor x_2 \lor \overline{x_4})\]
3-satisfiability reduces to independent set

Lemma. \( G \) contains vertex cover of size \( 2k \) iff \( \Phi \) is satisfiable.

\textbf{Pf.} \( \Rightarrow \) Let \( S \) be a vertex cover of size \( 2k \).

- \( S \) must contain exactly two nodes in each triangle.
- Set the excluded literal to \textit{true} (and remaining variables consistently).
- Truth assignment is consistent and all clauses are satisfied.

\textbf{Pf} \( \Leftarrow \) Given satisfying assignment, select one true literal from each triangle, and exclude that one. This is a vertex cover of size \( 2k \). \( \blacksquare \)

\begin{align*}
\Phi &= \left( \overline{x_1} \lor x_2 \lor x_3 \right) \land \left( x_1 \lor \overline{x_2} \lor x_3 \right) \land \left( \overline{x_1} \lor x_2 \lor x_4 \right) \\
G \end{align*}
**Directed hamilton cycle reduces to hamilton cycle**

**DIR-HAM-CYCLE**: Given a digraph $G = (V, E)$, does there exist a simple directed cycle $\Gamma$ that contains every node in $V$?
Theorem. 3-SAT ≤ \text{p} \text{ DIR-HAM-CYCLE}.

Pf. Given an instance \( \Phi \) of 3-SAT, we construct an instance of \text{DIR-HAM-CYCLE} that has a Hamilton cycle iff \( \Phi \) is satisfiable.

Construction. First, create graph that has \( 2^n \) Hamilton cycles which correspond in a natural way to \( 2^n \) possible truth assignments.
3-satisfiability reduces to directed hamilton cycle

Construction. Given 3-SAT instance $\Phi$ with $n$ variables $x_i$ and $k$ clauses
3-satisfiability reduces to directed hamilton cycle

Construction. Given 3-SAT instance $\Phi$ with $n$ variables $x_i$ and $k$ clauses.
- Intuition: traverse path $i$ from left to right $\iff$ set variable $x_i = true$. 
3-satisfiability reduces to directed hamilton cycle

Construction. Given 3-SAT instance $\Phi$ with $n$ variables $x_i$ and $k$ clauses.

- For each clause, add a node and 6 edges.
3-satisfiability reduces to directed hamilton cycle

Lemma. $\Phi$ is satisfiable iff $G$ has a Hamilton cycle.

Pf. $\Rightarrow$

- Suppose 3-SAT instance has satisfying assignment $x^*$.
- Then, define Hamilton cycle in $G$ as follows:
  - if $x^*_i = true$, traverse row $i$ from left to right
  - if $x^*_i = false$, traverse row $i$ from right to left
  - for each clause $C_j$, there will be at least one row $i$ in which we are going in "correct" direction to splice clause node $C_j$ into cycle
    (and we splice in $C_j$ exactly once)
Lemma. \( \Phi \) is satisfiable iff \( G \) has a Hamilton cycle.

Pf. \( \iff \)

- Suppose \( G \) has a Hamilton cycle \( \Gamma \).
- If \( \Gamma \) enters clause node \( C_j \), it must depart on mate edge.
  - nodes immediately before and after \( C_j \) are connected by an edge \( e \in E \)
  - removing \( C_j \) from cycle, and replacing it with edge \( e \) yields Hamilton cycle on \( G - \{ C_j \} \)
- Continuing in this way, we are left with a Hamilton cycle \( \Gamma' \) in \( G - \{ C_1, C_2, \ldots, C_k \} \).
- Set \( x^*_i = true \) iff \( \Gamma' \) traverses row \( i \) left to right.
- Since \( \Gamma \) visits each clause node \( C_j \), at least one of the paths is traversed in "correct" direction, and each clause is satisfied. \( \blacksquare \)
3-colorability

**3-COLOR.** Given an undirected graph $G$, can the nodes be colored red, green, and blue so that no adjacent nodes have the same color?
Application: register allocation

Register allocation. Assign program variables to machine register so that no more than \( k \) registers are used and no two program variables that are needed at the same time are assigned to the same register.

Interference graph. Nodes are program variables names; edge between \( u \) and \( v \) if there exists an operation where both \( u \) and \( v \) are "live" at the same time.

Observation. [Chaitin 1982] Can solve register allocation problem iff interference graph is \( k \)-colorable.

Fact. \( 3\text{-COLOR} \leq_P K\text{-REGISTER-ALLOCATION} \) for any constant \( k \geq 3 \).
3-satisfiability reduces to 3-colorability

**Theorem.** $3$-SAT $\leq_p 3$-COLOR.

**Pf.** Given $3$-SAT instance $\Phi$, we construct an instance of $3$-COLOR that is 3-colorable iff $\Phi$ is satisfiable.
3-satisfiability reduces to 3-colorability

Construction.

(i) Create a graph $G$ with a node for each literal.
(ii) Connect each literal to its negation.
(iii) Create 3 new nodes $T$, $F$, and $B$; connect them in a triangle.
(iv) Connect each literal to $B$.
(v) For each clause $C_j$, add a gadget of 6 nodes and 13 edges.

true

F

T

base

B

\[x_1, x_1, x_2, x_2, x_3, x_3, x_n, x_n\]
3-satisfiability reduces to 3-colorability

**Lemma.** Graph $G$ is 3-colorable iff $\Phi$ is satisfiable.

**Pf.** $\Rightarrow$ Suppose graph $G$ is 3-colorable.
- Consider assignment that sets all $T$ literals to true.
- (iv) ensures each literal is $T$ or $F$.
- (ii) ensures a literal and its negation are opposites.
3-satisfiability reduces to 3-colorability

Lemma. Graph $G$ is 3-colorable iff $\Phi$ is satisfiable.

Pf. $\Rightarrow$ Suppose graph $G$ is 3-colorable.

• Consider assignment that sets all $T$ literals to true.
• (iv) ensures each literal is $T$ or $F$.
• (ii) ensures a literal and its negation are opposites.
• (v) ensures at least one literal in each clause is $T$.

$$C_j = x_1 \lor \overline{x_2} \lor x_3$$
Lemma. Graph $G$ is 3-colorable iff $\Phi$ is satisfiable.

Pf. $\Rightarrow$ Suppose graph $G$ is 3-colorable.

- Consider assignment that sets all $T$ literals to true.
- (iv) ensures each literal is $T$ or $F$.
- (ii) ensures a literal and its negation are opposites.
- (v) ensures at least one literal in each clause is $T$.

$C_j = x_1 \lor \overline{x_2} \lor x_3$
Lemma. Graph $G$ is 3-colorable iff $\Phi$ is satisfiable.

Pf. $\iff$ Suppose 3-SAT instance $\Phi$ is satisfiable.

- Color all true literals $T$.
- Color node below green node $F$, and node below that $B$.
- Color remaining middle row nodes $B$.
- Color remaining bottom nodes $T$ or $F$ as forced. □

$$C_j = x_1 \lor \overline{x}_2 \lor x_3$$

3-satisfiability reduces to 3-colorability