Linear Programming
A really very extremely big hammer
**Given:** a polytope

**Find:** the *lowest* point in the polytope
**Given:** a polytope

**Find:** the *lowest* point in the polytope
Given: a polytope

Find: the lowest point in the polytope

\[
\begin{align*}
\text{maximize} & \quad z_1 + 2z_3 \\
\text{subject to} & \quad 2z_1 - z_2 + 3z_3 \leq 1 \\
& \quad -z_1 + z_2 - z_3 \leq 5
\end{align*}
\]
Given: a polytope

Find: the lowest point in the polytope

We have fast algorithms for this!

\[
\begin{align*}
\text{maximize} & \quad z_1 + 2z_3 \\
\text{subject to} & \quad 2z_1 - z_2 + 3z_3 \leq 1 \\
& \quad -z_1 + z_2 - z_3 \leq 5
\end{align*}
\]
Linear Algebra primer

\( a, x \in \mathbb{R}^n \), think of them as column vectors.

\[ a^\top x = a_1 x_1 + \ldots + a_n x_n \]

The set of \( x \) satisfying \( a^\top x = 0 \) is a hyperplane.
\[ a^\top x = 0 \]
\[ a^\top x \leq 0 \]
$a^\top x \leq b$
Given: a polytope
Find: the *lowest* point in the polytope
**Given:** a polytope

**Find:** the *lowest* point in the polytope
Linear Algebra primer

\( a, x \in \mathbb{R}^n \), think of them as column vectors.

\[ a^\top x = a_1 x_1 + \ldots + a_n x_n \]

\[ A_1 x \]
\[ A_2 x \]
\[ A_3 x \]
\[ A_m x \]
Given: a polytope

Find: the lowest point in the polytope

maximize \( c^T x \)
subject to
\[ Ax \leq b \]

\( Ax \leq b \) means
\[ (Ax)_i \leq b_i \]
for all \( i \)
Standard form

\[ \text{maximize } c^\top x \]
subject to
\[ Ax \leq b \]
\[ x \geq 0 \]
Standard form

\[
\begin{align*}
\text{maximize} & \quad c^T x \\
\text{subject to} & \quad Ax \leq b \\
& \quad x \geq 0
\end{align*}
\]

\[
\begin{align*}
\text{maximize} & \quad z_1 + 2z_3 \\
\text{subject to} & \quad 2z_1 - z_2 + 3z_3 \leq 1 \\
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Standard form

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\text{maximize } & \quad c^\top x \\
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\begin{align*}
\text{maximize } & \quad z_1 + 2z_3 \\
\text{subject to } & \quad 2z_1 - z_2 + 3z_3 \leq 1 \\
& \quad -z_1 + z_2 - z_3 \leq 5
\end{align*}

\begin{align*}
\text{maximize } & \quad (x_{1,a} - x_{1,b}) + 2(x_{3,a} - x_{3,b}) \\
\text{subject to } & \quad 2(x_{1,a} - x_{1,b}) - (x_{2,a} - x_{2,b}) + 3(x_{3,a} - x_{3,b}) \leq 1 \\
& \quad -(x_{1,a} - x_{1,b}) + (x_{2,a} - x_{2,b}) - (x_{3,a} - x_{3,b}) \leq 5 \\
& \quad x \geq 0
\end{align*}
Max Flow

Given: a flow network

maximize flow out of s

subject to

Respecting capacities and conservation
Max Flow

Given: a flow network

maximize flow out of $s$

subject to Respecting capacities and conservation

maximize $\sum_{e\text{ out of } s} x_e$

subject to
Max Flow

Given: a flow network

maximize flow out of s

subject to

Respecting capacities and conservation

maximize \( \sum_{e \text{ out of } s} x_e \)

subject to

for all \( e \),

\( 0 \leq x_e \leq c(e) \)
Max Flow

**Given:** a flow network

**maximize**  flow out of $s$

subject to

Respecting capacities and conservation

$$\text{maximize} \quad \sum_{e \text{ out of } s} x_e$$

subject to

for all $e$,

$$0 \leq x_e \leq c(e)$$

for all intermediate $v$,

$$\sum_{e \text{ out of } v} x_e = \sum_{e \text{ into } v} x_e$$
maximize \[ \sum_{e \text{ out of } s} x_e \]
subject to
for all \( e \),
\( 0 \leq x_e \leq c(e) \)
for all intermediate \( v \),
\[ \sum_{e \text{ out of } v} x_e = \sum_{e \text{ into } v} x_e \]
maximize \sum_{e \text{ out of } s} x_e

subject to

for all e,
\quad 0 \leq x_e \leq c(e)

for all intermediate v,
\quad \sum_{e \text{ out of } v} x_e = \sum_{e \text{ into } v} x_e

1. \quad c_e = \begin{cases} 
1 & \text{if } e \text{ out of } s \\
0 & \text{otherwise.} 
\end{cases}
maximize \( \sum_{e \text{ out of } s} x_e \) subject to

for all \( e \),

\( 0 \leq x_e \leq c(e) \)

for all intermediate \( v \),

\( \sum_{e \text{ out of } v} x_e = \sum_{e \text{ into } v} x_e \)

---

maximize \( c^\top x \) subject to

\( Ax \leq b \)

\( x \geq 0 \)

1. \( c_e = \begin{cases} 1 & \text{if } e \text{ out of } s \\ 0 & \text{otherwise.} \end{cases} \)

2. \( u^\top x \geq r \equiv (-u)^\top x \leq -r \)
maximize \[ \sum_{e \text{ out of } s} x_e \]
subject to
for all \( e \),
\[ 0 \leq x_e \leq c(e) \]
for all intermediate \( v \),
\[ \sum_{e \text{ out of } v} x_e = \sum_{e \text{ into } v} x_e \]
maximize \[ c^\top x \]
subject to
\[ Ax \leq b \]
\[ x \geq 0 \]
1. \( c_e = \begin{cases} 
1 & \text{if } e \text{ out of } s \\
0 & \text{otherwise.} 
\end{cases} \)
2. \( u^\top x \geq r \equiv (-u)^\top x \leq -r \)
3. \( u^\top x = r \equiv u^\top x \leq r, u^\top x \geq r \)
maximize \( \sum_{e \text{ out of } s} x_e \) subject to

for all \( e \),

\[ 0 \leq x_e \leq c(e) \]

for all intermediate \( v \),

\[ \sum_{e \text{ out of } v} x_e = \sum_{e \text{ into } v} x_e \]

maximize \( c^T x \) subject to

\[ A x \leq b \]
\[ x \geq 0 \]

1. \( c_e = \begin{cases} 1 & \text{if } e \text{ out of } s \\ 0 & \text{otherwise.} \end{cases} \)

2. \( u^T x \geq r \equiv (-u)^T x \leq -r \)

3. \( u^T x = r \equiv u^T x \leq r, u^T x \geq r \)

4. maximize \( c^T x \equiv \text{minimize } (-c)^T x \)
Shortest paths

**Given**: a directed graph

**Find**: shortest path from $s$ to $t$
Shortest paths

**Given:** a directed graph

**Find:** shortest path from $s$ to $t$

**Claim:** Length of the shortest path is solution to program.

\[
\begin{align*}
\text{minimize} \quad & \sum_e x_e \\
\text{subject to} \quad & \text{for all } e, \quad x_e \geq 0, \\
& \sum_{e \text{ out of } s} x_e - \sum_{e \text{ in to } s} x_e = 1, \\
& \sum_{e \text{ in to } t} x_e - \sum_{e \text{ out of } t} x_e = 1, \\
& \text{for all } v \neq s, t, \\
& \sum_{e \text{ out of } v} x_e = \sum_{e \text{ into } v} x_e
\end{align*}
\]
Shortest paths

**Given:** a directed graph

**Find:** shortest path from $s$ to $t$

**Claim:** Length of the shortest path is solution to program.

**Proof sketch:** Optimal solution must be a combination of flows on shortest paths. Indeed, if there is a path using edges with $x_e > 0$ that is not a shortest path, delete the flow on this path and reroute it on a shortest path to get a better solution.

minimize $\sum_e x_e$

subject to

for all $e$, $x_e \geq 0$,

$\sum_{e \text{ out of } s} x_e - \sum_{e \text{ in to } s} x_e = 1,$

$\sum_{e \text{ in to } t} x_e - \sum_{e \text{ out of } t} x_e = 1,$

for all $v \neq s, t$,

$\sum_{e \text{ out of } v} x_e = \sum_{e \text{ into } v} x_e$
Vertex Cover

**Given:** an undirected graph

**Find:** smallest set of vertices touching all edges
Vertex Cover

**Given:** an undirected graph

**Find:** smallest set of vertices touching all edges

\[
\begin{align*}
\text{minimize} & \quad \sum_v x_v \\
\text{subject to} & \quad \\
& \quad \text{for all } v, \\
& \quad 0 \leq x_v \leq 1, \\
& \quad \text{for all } e = \{u, v\} \\
& \quad x_u + x_v \geq 1
\end{align*}
\]
**Vertex Cover**

**Given:** an undirected graph

**Find:** smallest set of vertices touching all edges

Want

\[ x_v = 0 \text{ or } x_v = 1 \]

minimize \( \sum_v x_v \)

subject to

for all \( v \),

\[ 0 \leq x_v \leq 1, \]

for all \( e = \{u, v\} \)

\[ x_u + x_v \geq 1 \]
**Vertex Cover**

**Given:** an undirected graph

**Find:** smallest set of vertices touching all edges

There is a solution of value $3/2$, even though smallest vertex cover has size 2.

**Want**

$x_v = 0$ or $x_v = 1$

**minimize** $\sum_v x_v$

subject to

for all $v$, $0 \leq x_v \leq 1$, $\times$

for all $e = \{u, v\}$

$x_u + x_v \geq 1$
Duality

\[
\begin{align*}
\text{maximize} & \quad x_1 + 2x_3 \\
\text{subject to} & \quad 2x_1 - x_2 + 3x_3 \leq 1 \\
& \quad -x_1 + x_2 - x_3 \leq 5 \\
& \quad x \geq 0
\end{align*}
\]
Duality

\[ \text{maximize } x_1 + 2x_3 \]
subject to
\[ 2x_1 - x_2 + 3x_3 \leq 1 \]
\[ -x_1 + x_2 - x_3 \leq 5 \]
\[ x \geq 0 \]

Claim: Optimum \leq 6
Duality

**maximize** $x_1 + 2x_3$
subject to
$2x_1 - x_2 + 3x_3 \leq 1$
$-x_1 + x_2 - x_3 \leq 5$
$x \geq 0$

**Claim:** Optimum $\leq 6$

**Pf:**

$x_1 + 2x_3$

$= (2x_1 - x_2 + 3x_3) + (-x_1 + x_2 - x_3)$

$\leq 6$
Duality

**Claim:** For all non-negative $a, b$, if
$2a - b \geq 1$
$-a + b \geq 0$
$3a - b \geq 2$
then
$\text{opt} \leq a + 5b$

**Pf:**
$x_1 + 2x_3$
$\leq a(2x_1 - x_2 + 3x_3) + b(-x_1 + x_2 - x_3)$
$\leq a + 5b$.

$maximize \ x_1 + 2x_3$
subject to
$a \quad 2x_1 - x_2 + 3x_3 \leq 1$
$b \quad -x_1 + x_2 - x_3 \leq 5$
$x \geq 0$

**Claim:** Optimum $\leq 6$

**Pf:** $x_1 + 2x_3$
$\leq (2x_1 - x_2 + 3x_3) + (-x_1 + x_2 - x_3)$
$\leq 6$
Duality

**maximize** $x_1 + 2x_3$
subject to
\[
\begin{align*}
a & \quad 2x_1 - x_2 + 3x_3 \leq 1 \\
b & \quad -x_1 + x_2 - x_3 \leq 5 \\
x & \quad \geq 0
\end{align*}
\]

subject to
\[
\begin{align*}
\text{primal} & \quad 2a - b \geq 1 \\
& \quad -a + b \geq 0 \\
& \quad 3a - b \geq 2 \\
& \quad a, b \geq 0
\end{align*}
\]

**minimize** $a + 5b$

**Claim:** For all non-negative $a$, $b$, if
\[
\begin{align*}
2a - b & \geq 1 \\
-a + b & \geq 0 \\
3a - b & \geq 2 \\
\text{then} & \\
\text{opt} & \leq a + 5b
\end{align*}
\]

**Pf:**
\[
\begin{align*}
x_1 + 2x_3 & \leq a(2x_1 - x_2 + 3x_3) + b(-x_1 + x_2 - x_3) \\
& \leq a + 5b.
\end{align*}
\]
Duality

\[ \text{maximize } x_1 + 2x_3 \]
\[ \text{subject to } \]
\[ a : 2x_1 - x_2 + 3x_3 \leq 1 \]
\[ b : -x_1 + x_2 - x_3 \leq 5 \]
\[ x \geq 0 \]

Claim: For all non-negative \( a, b \), if
\[ 2a - b \geq 1 \]
\[ -a + b \geq 0 \]
\[ 3a - b \geq 2 \]
then
\[ \text{opt} \leq a + 5b \]

Pf:
\[ x_1 + 2x_3 \]
\[ \leq a(2x_1 - x_2 + 3x_3) + b(-x_1 + x_2 - x_3) \]
\[ \leq a + 5b \]
Duality

**Primal**

maximize $x_1 + 2x_3$
subject to
\[
\begin{align*}
a & \quad 2x_1 - x_2 + 3x_3 \leq 1 \\
b & \quad -x_1 + x_2 - x_3 \leq 5 \\
x & \quad \geq 0
\end{align*}
\]

**Dual**

maximize $-a - 5b$
subject to
\[
\begin{align*}
-2a + b & \leq -1 \\
a - b & \leq 0 \\
a - b & \leq 0 \\
-3a + b & \leq -2 \\
a, b & \geq 0
\end{align*}
\]

What is dual of dual?
Duality

**maximize** \( x_1 + 2x_3 \)
subject to
\[
\begin{align*}
2x_1 - x_2 + 3x_3 & \leq 1 \\
-x_1 + x_2 - x_3 & \leq 5 \\
x & \geq 0
\end{align*}
\]

**maximize** \(-a - 5b\)
subject to
\[
\begin{align*}
-2a + b & \leq -1 \\
a - b & \leq 0 \\
-3a + b & \leq -2 \\
a, b & \geq 0
\end{align*}
\]

What is dual of dual?

**minimize** \(-y_1 - 2y_3\)
subject to
\[
\begin{align*}
-2y_1 + y_2 - 3y_3 & \geq -1 \\
y_1 - y_2 + y_3 & \geq -5 \\
y & \geq 0
\end{align*}
\]
**Duality**

**maximize** \( x_1 + 2x_3 \)

subject to

| \( a \) | \( 2x_1 - x_2 + 3x_3 \leq 1 \) |
|\( b \) | \(-x_1 + x_2 - x_3 \leq 5 \) |
| \( x \geq 0 \) |

**maximize** \(-a - 5b\)

subject to

| \( y_1 \) | \(-2a + b \leq -1 \) |
| \( y_2 \) | \(a - b \leq 0 \) |
| \( y_3 \) | \(-3a + b \leq -2 \) |
| \( a, b \geq 0 \) |

**What is dual of dual?**

**minimize** \(-y_1 - 2y_3\)

subject to

| \(-2y_1 + y_2 - 3y_3 \geq -1 \) |
| \( y_1 - y_2 + y_3 \geq -5 \) |
| \( y \geq 0 \) |

**equivalent to**

**maximize** \( y_1 + 2y_3 \)

subject to

| \( 2y_1 - y_2 + 3y_3 \leq 1 \) |
| \(-y_1 + y_2 - y_3 \leq 5 \) |
| \( y \geq 0 \) |
Duality

\[
\text{maximize } c^\top x \\
\text{subject to } A x \leq b \\
x \geq 0
\]

\[
\text{minimize } b^\top y \\
\text{subject to } A^\top y \geq c \\
y \geq 0
\]

\[
\equiv \\
\text{maximize } (-b)^\top y \\
\text{subject to } (-A)^\top y \leq -c \\
y \geq 0
\]
Duality

\[
\begin{align*}
\text{primal} & \quad \text{dual} & \quad \text{dual} \\
\text{maximize } c^\top x & \quad \text{minimize } b^\top y & \quad \text{maximize } (-b)^\top y \\
\text{subject to} & \quad \text{subject to} & \quad \text{subject to} \\
Ax \leq b & \quad A^\top y \geq c & \quad (-A)^\top y \leq -c \\
x \geq 0 & \quad y \geq 0 & \quad y \geq 0
\end{align*}
\]

**Thm:** The dual of the dual is the primal.
**Duality**

<table>
<thead>
<tr>
<th>Primal</th>
<th>Dual</th>
<th>Dual of Dual</th>
</tr>
</thead>
<tbody>
<tr>
<td>\textbf{maximize} : c^\mathsf{T} x</td>
<td>\textbf{minimize} : b^\mathsf{T} y</td>
<td>\textbf{maximize} : (-b)^\mathsf{T} y</td>
</tr>
<tr>
<td>subject to : Ax \leq b</td>
<td>subject to : A^\mathsf{T} y \geq c</td>
<td>subject to : (-A)^\mathsf{T} y \leq -c</td>
</tr>
<tr>
<td>: x \geq 0</td>
<td>: y \geq 0</td>
<td>: y \geq 0</td>
</tr>
</tbody>
</table>

**Thm:** The dual of the dual is the primal.

\textbf{dual of dual}

\textbf{minimize} \: (-c)^\mathsf{T} x

subject to \: ((-A)^\mathsf{T})^\mathsf{T} x \geq -b

\: x \geq 0
Duality

**primal**

maximize $c^\top x$
subject to
$Ax \leq b$
$x \geq 0$

**dual**

minimize $b^\top y$
subject to
$A^\top y \geq c$
$y \geq 0$

**dual of dual**

maximize $(-b)^\top y$
subject to
$((-A)^\top)^\top y \leq -c$
$y \geq 0$

**Thm:** The dual of the dual is the primal.

dual of dual

minimize $(-c)^\top x$
subject to
$((-A)^\top)^\top x \geq - b$
$x \geq 0$

maximize $c^\top x$
subject to
$Ax \leq b$
$x \geq 0$
Duality

**Primal**

\[
\begin{align*}
\text{maximize} & \quad c^\top x \\
\text{subject to} & \quad Ax \leq b \\
x & \geq 0
\end{align*}
\]

**Dual**

\[
\begin{align*}
\text{minimize} & \quad b^\top y \\
\text{subject to} & \quad A^\top y = c \\
y & \geq 0
\end{align*}
\]

**Thm:** The dual of the dual is the primal.

**Thm: (Weak Duality)** Every solution to primal is at most every solution to dual.
Duality

**Primal**

\[
\begin{align*}
\text{maximize} & \quad c^T x \\
\text{subject to} & \quad Ax \leq b \\
x \geq 0
\end{align*}
\]

**Dual**

\[
\begin{align*}
\text{minimize} & \quad b^T y \\
\text{subject to} & \quad A^T y = c \\
y \geq 0
\end{align*}
\]

**Thm:** The dual of the dual is the primal.

**Thm: (Weak Duality)** Every solution to primal is at most every solution to dual.

**Thm: (Strong Duality)** If primal has solution of finite value, then value is equal to optimal solution of dual.
Duality

**Primal**
- **Maximize**: $c^T x$
- Subject to: $Ax \leq b$
- $x \geq 0$

**Dual**
- **Minimize**: $b^T y$
- Subject to: $A^T y \leq c$
- $y \geq 0$

**Thm: (Strong Duality)** If primal has solution of finite value, then value is equal to optimal solution of dual.

**Fact:** A vertex is a point for which $n$ of the inequalities become tight.

**By physics:**
- There must be $y_i, y_j \geq 0$
- $y_i A_i + y_j A_j = c$
- If $\hat{A} x = \hat{b}$ correspond to sides touching $x$, $A^T y = \hat{A}^T \hat{y} = c$
- Then $b^T y = \hat{b}^T \hat{y} = (\hat{A} x)^T y = x^T \hat{A}^T \hat{y} = x^T c = c^T x$
Duality of Max flow

maximize \[ \sum_{e \text{ out of } s} x_e \]

subject to

for all \( e \),

\[ 0 \leq x_e \leq c(e) \]

for all intermediate \( v \),

\[ \sum_{e \text{ out of } v} x_e = \sum_{e \text{ into } v} x_e \]

minimize \[ c^\top a \]

subject to

for all \( e = (s, v) \),

\[ a_e + b_v \geq 1 \]

for all \( e = (u, t) \),

\[ a_e - b_u \geq 0 \]

for all other \( e = (u, v) \),

\[ a_e - b_u + b_v \geq 0 \]

for all \( e \)

\[ a_e \geq 0 \]
Duality of Max flow

maximize \[ \sum_{e \text{ out of } s} x_e \]
subject to
for all \( e = (s, v) \),
\[ 0 \leq x_e \leq c(e) \]
for all \( e = (u, t) \),
\[ a_e + b_v \geq 1 \]
for all other \( e = (u, v) \),
\[ a_e - b_u \geq 0 \]
\[ a_e - b_u + b_v \geq 0 \]
for all \( e \)
\[ a_e \geq 0 \]

\[ \sum_{e \text{ out of } v} x_e = \sum_{e \text{ into } v} x_e \]

minimize \[ c^\top a \]
subject to
\[ b_s = 1, b_t = 0 \]
for all \( e = (u, v) \),
\[ a_e \geq b_u - b_v \]
for all \( e \)
\[ a_e \geq 0 \]
minimize $c^T a$

subject to

for all $e = (s, v)$,
$a_e + b_v \leq 1$

for all $e = (u, t)$,
$a_e - b_u \leq 0$

for all other $e = (u, v)$,
$a_e - b_u + b_v \leq 0$

for all $e$
$a_e \geq 0$

$\equiv$

minimize $c^T a$

subject to

for all $e = (u, v)$,
$b_s = 1, b_t = 0$

for all $e = (u, v)$,
$a_e \geq b_u - b_v$

for all $e$
$a_e \geq 0$

$\equiv$

minimize $c^T a$

subject to

for all $e = (u, v)$,
$b_s = 1, b_t = 0$

for all $e = (u, v)$,
$a_e = \max\{0, b_u - b_v\}$
**Claim:** Opt is achieved with \( 1 \geq b_u \geq 0 \).

**Pf:** Take any solution and move the extreme values up/down. The solution only improves.

\[
\begin{align*}
\text{minimize} & \quad c^\top a \\
\text{subject to} & \quad b_s = 1, b_t = 0 \\
& \quad 0 \leq b_u \leq 1 \\
& \quad \text{for all } e = (u, v), \\
& \quad a_e = \max\{0, b_u - b_v\}
\end{align*}
\]
\textbf{minimize} \; c^\top a

subject to

\begin{align*}
    b_s &= 1, b_t = 0 \\
    0 &\leq b_u \leq 1 \\
    \text{for all } e = (u, v), \\
    a_e &= \max\{0, b_u - b_v\}
\end{align*}
\[ \text{minimize } c^\top a \]

subject to

\[ b_s = 1, b_t = 0 \]
\[ 0 \leq b_u \leq 1 \]

for all \( e = (u, v) \),
\[ a_e = \max\{0, b_u - b_v\} \]

Claim: Opt is achieved with \( b_u = 0/1 \).

Pf: Pick \( 0 \leq t \leq 1 \) uniformly at random. If \( b_u \geq t \), set \( b_u = 1 \), otherwise set it to 0. The expected value of resulting solution is the same as original!
\begin{align*}
\text{minimize} & \quad c^\top a \\
\text{subject to} & \quad b_s = 1, b_t = 0 \\
& \quad b_u \in \{0,1\} \\
& \quad \text{for all } e = (u, v), \\
& \quad a_e = \max\{0, b_u - b_v\}
\end{align*}
Duality of Shortest Path

minimize \( \sum_{e} x_e \)
subject to

for all \( e \),

\[ x_e \geq 0, \]

\[ \sum_{e \text{ out of } s} x_e - \sum_{e \text{ in to } s} x_e = 1, \]

\[ \sum_{e \text{ out of } t} x_e - \sum_{e \text{ in to } t} x_e = -1, \]

for all \( v \neq s, t \),

\[ \sum_{e \text{ out of } v} x_e - \sum_{e \text{ into } v} x_e = 0 \]
Duality of Shortest Path

minimize \[ \sum_{e} x_e \]
subject to
for all \( e \),
\( x_e \geq 0 \),
\[ \sum_{e \text{ out of } s} x_e - \sum_{e \text{ in to } s} x_e = 1, \]
\[ \sum_{e \text{ out of } t} x_e - \sum_{e \text{ in to } t} x_e = -1, \]
for all \( v \neq s, t \),
\[ \sum_{e \text{ out of } v} x_e - \sum_{e \text{ into } v} x_e = 0. \]

\[ \text{dual} \]

maximize \( a_s - a_t \)
subject to
for all edges \( e = (u, v) \),
\( a_u - a_v \leq 1 \)
Duality of Shortest Path

minimize \sum_e x_e

subject to

for all \( e \),
\( x_e \geq 0 \),
\[
\sum_{e \text{ out of } s} x_e - \sum_{e \text{ in to } s} x_e = 1,
\]
\[
\sum_{e \text{ out of } t} x_e - \sum_{e \text{ in to } t} x_e = -1,
\]
for all \( v \neq s, t \),
\[
\sum_{e \text{ out of } v} x_e - \sum_{e \text{ into } v} x_e = 0
\]

\textbf{dual}

maximize \( a_s - a_t \)

subject to

for all edges \( e = (u, v) \),
\( a_u - a_v \leq 1 \)
Duality and zero-sum games

Two player zero-sum game:
an $m \times n$ matrix $G$

$G_{i,j}$: payoff to row player, assuming row player uses
strategy $i$, and column player uses strategy $j$.
$-G_{i,j}$: payoff to column player.

Example: Chess
$i$: specifies how white would move in every possible
board configuration.
$j$: specifies how black would move.

$$G_{i,j} = \begin{cases} 
1 & \text{if white wins} \\
-1 & \text{if black wins} \\
0 & \text{stalemate}
\end{cases}$$

Randomized strategy:
probability distribution on row strategies
A column vector $x$ with
$x_i \geq 0$, $\sum_i x_i = 1$

probability distribution on column strategies
$y_j \geq 0$, $\sum_j y_j = 1$

expected payoff to row player
$x^\top G y$
**Who decides on their strategy first?**

If row player commits to \( x \)

Row player will get payoff

\[
\min_{y} x^\top G y = \min (x^\top G)_j
\]

So, if row player has to play first:

\[
\max_{x} \min_{y} x^\top G y
\]

If column player commits to \( y \)

Row player will get payoff

\[
\max_{x} \min_{y} x^\top G y = \max (G y)_i
\]

So, if column player has to play first

\[
\min_{y} \max_{x} x^\top G y
\]

**Randomized strategy:**

- Probability distribution on row strategies
  - A column vector \( x \) with
  \[
x_i \geq 0, \sum_i x_i = 1
\]
- Probability distribution on column strategies
  - \( y_i \geq 0, \sum_j y_j = 1 \)
- Expected payoff to row player
  \[
x^\top G y
\]
von-Neumann’s min-max Theorem

If row player commits to $x$

Row player will get payoff

$$\min x^\top Gy = \min_{y} (x^\top G)_j$$

So, if row player has to play first:

$$\max \min x^\top Gy$$

If column player commits to $y$

Row player will get payoff

$$\max x^\top Gy = \max_{x} (Gy)_i$$

So, if column player has to play first

$$\min \max x^\top Gy$$

Doesn’t matter who plays first:

**Thm:**

$$\max \min x^\top Gy = \min \max x^\top Gy.$$
Using strong duality

**Thm:** \( \max_{x} \min_{y} x^\top G y = \min_{y} \max_{x} x^\top G y. \)

\[
\begin{align*}
\max_{x} \min_{y} (&x^\top G y)_{j} = \min_{y} \max_{x} (G y)_{i} \\
\text{primal} & \quad \text{dual}
\end{align*}
\]

**maximize** \( z \)

subject to

\[w \quad x_{1} + \ldots + x_{m} = 1\]

for all \( j \),

\[y_{j} \quad z \leq (x^\top G)_{j}\]

\[x \geq 0\]

**minimize** \( w \)

subject to

\[y_{1} + \ldots + y_{m} = 1\]

for all \( i \),

\[w \geq (G y)_{i}\]

\[y \geq 0\]
Algorithms for Linear programs

Simplex Algorithm

- Simple
- Often fast in practice
- Not polynomial time (on pathological counterexamples)

Ellipsoid Algorithm

- More complicated
- Polynomial time, but not always fast
Simplex

Start with a vertex
In each step,
move to a lower vertex

Problem: Number of vertices
on this path can be
exponential!
Simplex: how to find initial vertex?

\[
\begin{align*}
\text{maximize} \quad & c^\top x \\
\text{subject to} \quad & Ax \leq b \\
\quad & x \geq 0
\end{align*}
\]

\[
\begin{align*}
\text{minimize} \quad & z_1 + z_2 + \ldots \\
\text{subject to} \quad & Ax \leq b + z \\
\quad & x, z \geq 0
\end{align*}
\]

For this program, \( z_i = \max\{0, -b_i\}, x = 0 \) is a vertex. Run simplex to find a solution with \( z = 0 \). The \( x \) value of solution will be a vertex of original program!
Simplex: how to go to better vertex?

\[
\begin{align*}
\text{maximize } & c^\top x \\
\text{subject to } & Ax \leq b \\
& x \geq 0
\end{align*}
\]

1. There must be $\hat{A}x = \hat{b}$.
2. Find $y$ satisfying $n - 1$ of the equations, $c^\top y > 0$.
3. Change $x = x + \epsilon y$, until some new equation becomes tight.
Ellipsoid method

*Ellipsoid*: a squished ball

\[ x^2 + y^2 \leq 1 \]
Ellipsoid method

Ellipsoid: a squished ball

\[ x^2 + y^2 \leq 1 \]

\[ (2x)^2 + (y/2)^2 \leq 1 \]
**Ellipsoid method**

*Ellipsoid*: a squished ball

\[ x^2 + y^2 \leq 1 \]

\[ (2x)^2 + (y/2)^2 \leq 1 \]
Ellipsoid method

*Ellipsoid*: a squished ball

\[ x^2 + y^2 \leq 1 \]

\[ (2x)^2 + \left(\frac{y}{2}\right)^2 \leq 1 \]

Ratio of area of ellipsoid to sphere:

\[ \frac{1}{2} \cdot \frac{2}{1} = 1 \]
Ellipsoid method

Ellipsoid: a squished ball

\[
x^2 + y^2 \leq 1
\]

\[
(2x - 1)^2 + \left(\frac{y - 1}{2}\right)^2 \leq 1
\]

Ratio of area of ellipsoid to sphere:

\[
\frac{1}{2} \cdot \frac{2}{1} = 1
\]
Ellipsoid method

Ellipsoid: a squished ball

Let $U^{-1}$ be the linear transformation corresponding to a rotation.

$$\frac{1}{2} \cdot \frac{2}{1} = 1$$

$$\frac{1}{2} \cdot \frac{2}{1} = 1$$

$$\frac{1}{2} \cdot \frac{2}{1} = 1$$

Ratio of area of ellipsoid to sphere:

$$\frac{1}{2} \cdot \frac{2}{1} = 1$$

$$(2(U_1(x, y) - 1))^2 + ((U_2(x, y) - 1)/2)^2 \leq 1$$
The desired solution is bounded

**Fact:** If the solution is finite, then its magnitude is at most $2^{O(\text{poly}(\text{input length}))}$.

**Pf:** If finite, the solution occurs at a vertex. Since every vertex satisfies $Bx = d$, for some $B, d$, we have $x = B^{-1}d$, and the size of coefficients of $B^{-1}$ are polynomially related to the size of coefficients of $A$.

**Fact:** If there is finite solution, then volume of feasible region (i.e. polytope) is at least $2^{-O(\text{poly}(\text{input length}))}$.

**Pf sketch:** The smallest angle that can be generated is $2^{-O(\text{poly}(\text{input length}))}$. 
Ellipsoid method

maximize $c^T x$
subject to
$Ax \leq b$
$x \geq 0$

Claim: If we can find $x$ inside polytope in poly time, we can use binary search to find the best value of $d$ in poly time!

Fact: If the solution is finite, then its magnitude is at most $2^{O(poly(input\ length))}$.

Fact: If there is finite solution, then volume of feasible region (i.e. polytope) is at least $2^{-O(poly(input\ length))}$.

Consequence: We know $-T \leq c^T x \leq T$, where $T \leq 2^{O(poly(input\ length))}$. 

Is there $x$
with
$c^T x \geq d$
$Ax \leq b$
$x \geq 0$
Using binary search

\[ y = T \]

\[ y = -T \]
Check polytope is non-empty

\[ y = T \]

\[ y = -T \]
Add new constraint

\[ y = T \]

\[ y \leq 0 \]

\[ y = -T \]
Find point

\[ y = T \]

\[ y \leq 0 \]

\[ y = -T \]
Add new constraint

\[ y \leq 0 \]

\[ y \leq -\frac{T}{2} \]

\[ y = -T \]
Find point: polytope is empty!

\[ y \leq 0 \]

\[ y \leq -T/2 \]

\[ y = -T \]
Add new constraint

\[ y \leq 0 \]

\[ y \leq -T/4 \]

\[ y \leq -T/2 \]
Add new constraint

\[ y \leq 0 \]
\[ y \leq -T/4 \]
\[ y \leq -T/2 \]
Find point

\[ y \leq 0 \]

\[ y \leq -T/4 \]

\[ y \leq -T/2 \]
\[ y \leq -\frac{T}{4} \]

\[ y \leq -\frac{T}{2} \]
Add new constraint

\[ y \leq -T/4 \]

\[ y \leq -3T/8 \]

\[ y \leq -T/2 \]
Find point: polytope is empty!

\[ y \leq -\frac{T}{4} \]
\[ y \leq -\frac{3T}{8} \]
\[ y \leq -\frac{T}{2} \]
\[ y \leq -\frac{T}{8} \]
\[ y \leq -\frac{3T}{8} \]
Find point

\[ y \leq -T/4 \]
\[ y \leq -3T/8 \]
Conclusion: It is enough to give an algorithm to find a point in a polytope.
Ellipsoid algorithm for finding points in polytopes

**Idea:** Iteratively find ellipsoids where the density of the polytope is larger and larger, until a point is found.
Fact: If the solution is finite, then its magnitude is at most $2^{O(poly(input\ length))}$. 
Check 0
Find violated inequality
Shift inequality to origin
Find ellipsoid containing half-sphere
Find ellipsoid containing half-sphere
Shift to center
Stretch to get sphere
Check 0
Find violated inequality
Shift inequality to origin
Find ellipsoid containing half-sphere
Find ellipsoid containing half-sphere
Shift to center
Stretch to get sphere
Check 0
Ellipsoid method

Algorithm to find element of non-empty $P$:
1. Let $E$ be circle of radius $R$ containing polytope $P$.
2. If $0 \in P$, output $0$.
3. Otherwise half-circle containing $P$, and ellipsoid $E'$ containing half-circle.
4. Scale and shift $E'$ to get $E$, and find element of $P$ using new $E$.

Key Lemma: $\frac{\text{vol}(E')}{\text{vol}(E)} \leq e^{-1} e^{2(n+1)}$

Corollary: After $t$ rounds, $\frac{\text{vol}(P)}{\text{vol}(E')} \geq e^{t} e^{2(n+1)} \cdot \frac{\text{vol}(P)}{\text{vol}(E)}$

Corollary: The algorithm must terminate in poly(input length) steps.
Claim: $E'$ contains right half-ball.

If $x \in E$, $x_1 \geq 0$, then
\[
\left( \frac{n + 1}{n} \right)^2 \left( x_1 - \frac{1}{n + 1} \right)^2 + \frac{n^2 - 1}{n^2} \cdot \sum_{i>2} x_i^2
\]
\[
= \left( \frac{(n + 1)x_1 - 1}{n} \right)^2 + \frac{n^2 - 1}{n^2} \cdot \sum_{i>2} x_i^2
\]
\[
= \frac{(n^2 + 2n + 1)x_1^2 - 2(n + 1)x_1 + 1}{n^2} + \frac{n^2 - 1}{n^2} \cdot \sum_{i>2} x_i^2
\]
\[
= \frac{(2n + 2)x_1^2 - (2n + 2)x_1}{n^2} + \frac{1}{n^2} + \frac{n^2 - 1}{n^2} \cdot \sum_{i} x_i^2
\]
\[
\leq \frac{1}{n^2} + \frac{n^2 - 1}{n^2} \leq 1.
\]
Claim: $\frac{\text{vol}(E')}{\text{vol}(E)} \leq e^{\frac{-1}{2(n+1)}}$

$E$: $\sum x_i^2 \leq 1$

$E'$: 
\[
\left(\frac{n + 1}{n}\right)^2 \left(x_1 - \frac{1}{n + 1}\right)^2 + \frac{n^2 - 1}{n^2} \cdot \sum_{i>2} x_i^2 \leq 1
\]

$\frac{\text{vol}(E')}{\text{vol}(E)}$

\[
= \frac{n}{n + 1} \cdot \left(\sqrt{\frac{n^2}{n^2 - 1}}\right)^{n-1}
\]
\[
= \left(1 - \frac{1}{n + 1}\right) \cdot \left(1 + \frac{1}{n^2 - 1}\right)^{(n-1)/2}
\]

using $1 + z \leq e^z$

\[
\leq e^{-\frac{1}{n+1}} \cdot e^{\frac{(n-1)/2}{n^2 - 1}} = e^{-\frac{1}{n+1}} \cdot e^{\frac{1}{2(n+1)}} = e^{\frac{-1}{2(n+1)}}
\]
Why is linear programming so powerful?

In a sense, every algorithm can be expressed as linear program!
Boolean circuits

\[ x_1 \oplus x_2 \oplus x_3 \]
Fact: If $f : \{0,1\}^n \rightarrow \{0,1\}$ can be computed in time $T$, then it can be computed by a circuit of size $O(T \log T)$.
Fact: If $f : \{0,1\}^n \rightarrow \{0,1\}$ can be computed in time $T$, then it can be computed by a circuit of size $O(T \log T)$.
### Boolean circuits

**Fact:** If $f : \{0,1\}^n \rightarrow \{0,1\}$ can be computed in time $T$, then it can be computed by a circuit of size $O(T \log T)$.

Computing $f$ is equivalent to finding $x$ satisfying these constraints!