# Analysis

- How to reason about the performance of algorithms

# Defining Efficiency

"Runs fast on typical real problem instances"

Pro:

sensible, bottom-line-oriented

Con:

moving target (diff computers, compilers)

highly subjective (how fast is "fast"? What's "typical"?)

# Efficiency

We want a general theory of “efficiency” that is

- Simple
- Objective
- Relatively independent of changing technology
- But still predictive – “theoretically bad” algorithms should be bad in practice and vice versa

# Measuring efficiency

Time: # of instructions executed in a simple programming language

- only simple operations (+,*,-,=,if,call,…)
- each operation takes one time step
- each memory access takes one time step
- no fancy stuff (add these two matrices, copy this long string,…) built in; write it/charge for it as above

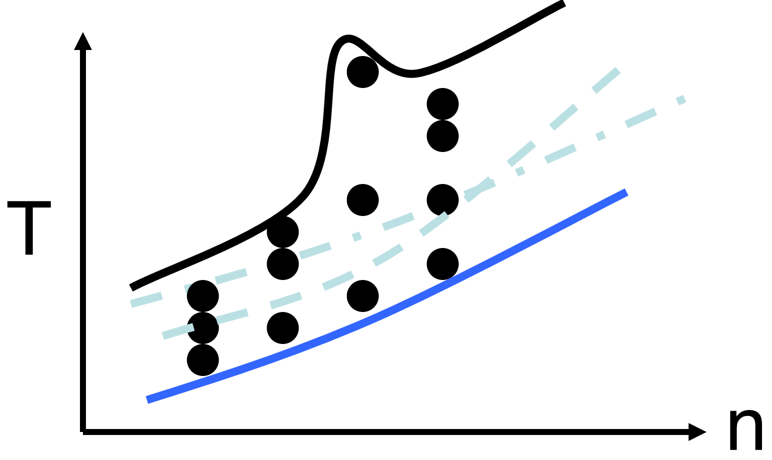
# We left out things but...

- Things we've dropped
  - memory hierarchy
    - disk, caches, registers have many orders of magnitude differences in access time
  - not all instructions take the same time in practice (+, ÷)
  - communication
  - different computers have different primitive instructions
- However,
  - one can usually tune implementations so that the hierarchy, etc., is not a huge factor

# Problem

- Algorithms can have different running times on different inputs!
- Smaller inputs take less time, larger inputs take more time.

# Solution



Measure performance on problem size n

- Average-case complexity: avg # steps algorithm takes on inputs of size n
- Worst-case complexity: max # steps algorithm takes on any input of size n

# Pros and cons:

Average-case

- over what probability distribution? (different settings may have different "average" problems)
- analysis often hard

Worst-case

+ a fast algorithm has a comforting guarantee
+ analysis easier
+ useful in real-time applications (space shuttle, nuclear reactors)

- may be too pessimistic

# General Goals

Characterize *growth rate* of (worst-case) run time as a function of problem size, up to a constant factor

Why not try to be more precise?

- Technological variations (computer, compiler, OS, …) easily 10x or more

# Complexity

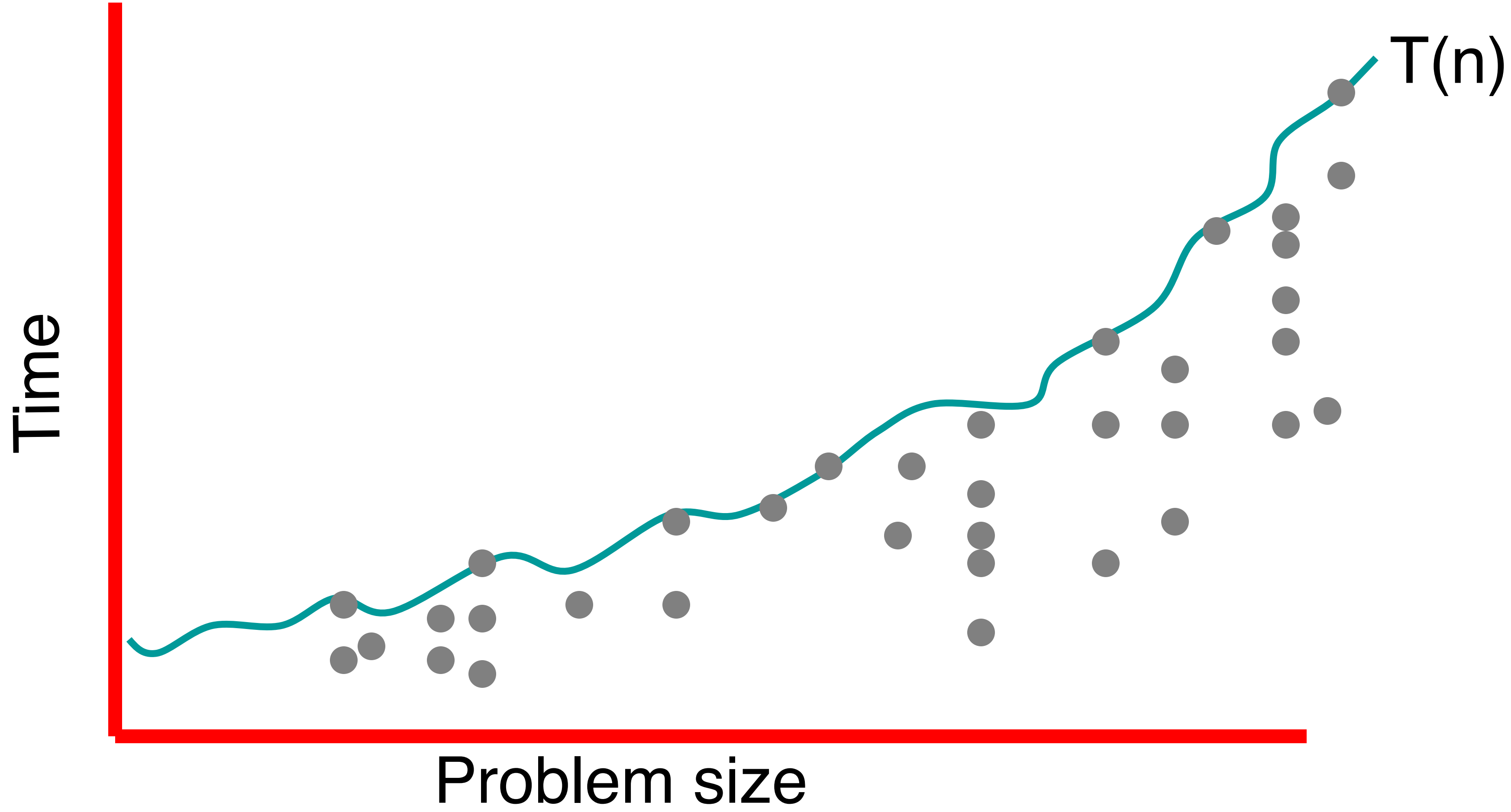
The *complexity* of an algorithm associates a number $T(n)$, the worst-case time the algorithm takes on problems of size $n$, with each problem size $n$.

Mathematically,

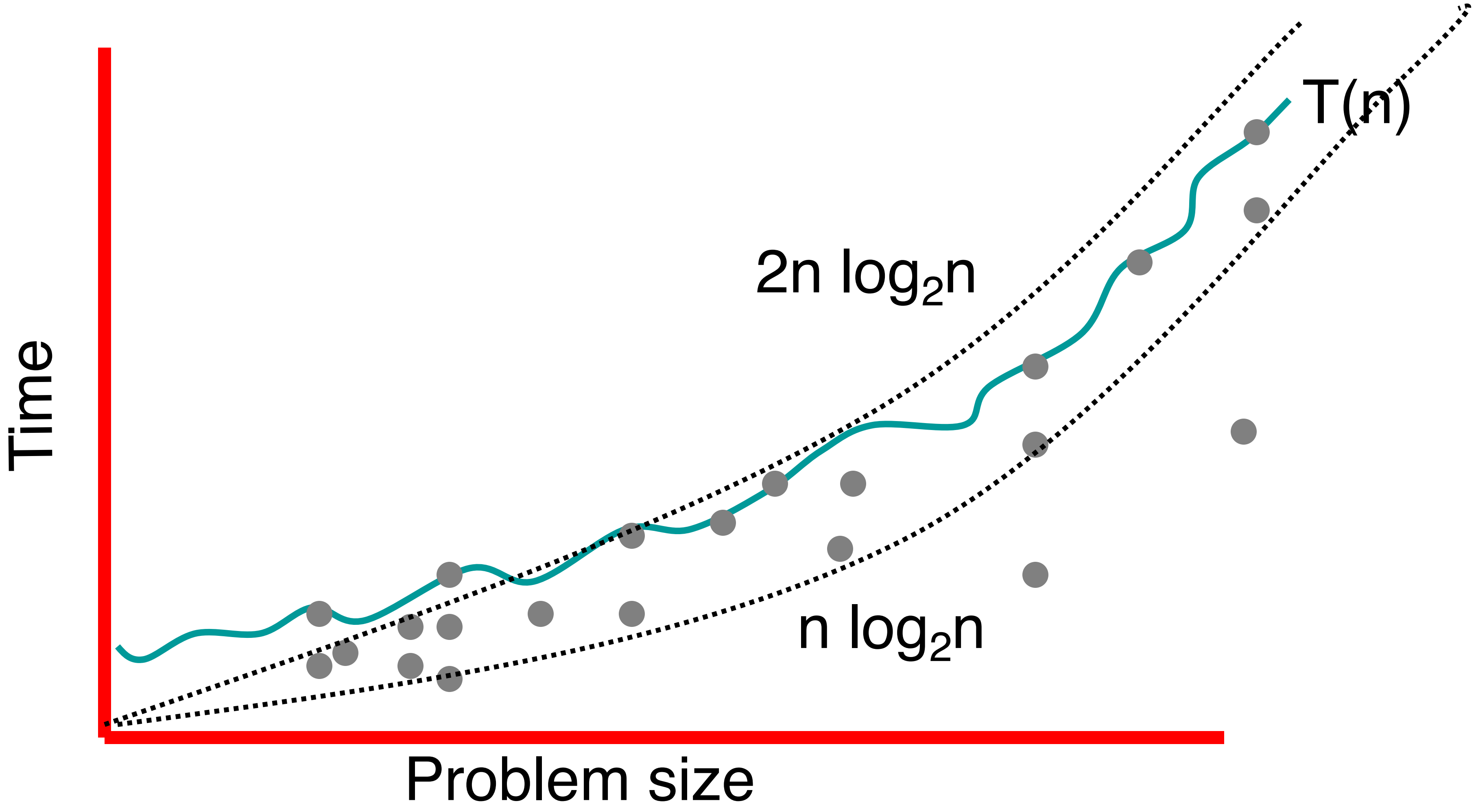
$T: N^+ \text{->} R^+$

I.e., $T$ is a function that maps positive integers (problem sizes) to positive real numbers (number of steps).

# Complexity



# Complexity



# O-notation, etc.

Given two functions f and g:N->R

f(n) is O(g(n)) iff there is a constant c>0 so that
f(n) is eventually always < c g(n)

f(n) is Ω(g(n)) iff there is a constant c>0 so that
f(n) is eventually always > c g(n)

f(n) is Θ(g(n)) iff there are constants $c_1$, $c_2$>0 so that
eventually always $c_1$g(n) < f(n) < $c_2$g(n)

# Examples

$10n^2-16n+100$ is $O(n^2)$ also $O(n^3)$

$10n^2-16n+100 < 10n^2$ for all $n > 10$

$10n^2-16n+100$ is $\Omega(n^2)$ also $\Omega(n)$

$10n^2-16n+100 > 9n^2$ for all $n > 16$

Therefore also $10n^2-16n+100$ is $\Theta(n^2)$

$10n^2-16n+100$ is not $O(n)$ also not $\Omega(n^3)$

# Properties

Transitivity.

If $f = O(g)$ and $g = O(h)$ then $f = O(h)$.

If $f = \Omega(g)$ and $g = \Omega(h)$ then $f = \Omega(h)$.

If $f = \Theta(g)$ and $g = \Theta(h)$ then $f = \Theta(h)$.

Additivity.

If $f = O(h)$ and $g = O(h)$ then $f + g = O(h)$.

If $f = \Omega(h)$ and $g = \Omega(h)$ then $f + g = \Omega(h)$.

If $f = \Theta(h)$ and $g = \Theta(h)$ then $f + g = \Theta(h)$.

# Asymptotic Bounds for Some Common Functions

Polynomials:

$a_0 + a_1 n + \ldots + a_d n^d$ is $\Theta(n^d)$ if $a_d > 0$

Logarithms:

$\log_a n = \Theta(\log_b n)$ for any constants $a, b > 1$

Logarithms:

For all $x > 0$, $\log n = O(n^x)$

*log grows slower than every polynomial*

# “One-Way Equalities”

$2 + 2$ is $4$

$2 + 2 = 4$

$4 = 2 + 2$

$2n^2 + 5n$ is $O(n^3)$

$2n^2 + 5n = O(n^3)$

~~$O(n^3) = 2n^2 + 5n$~~

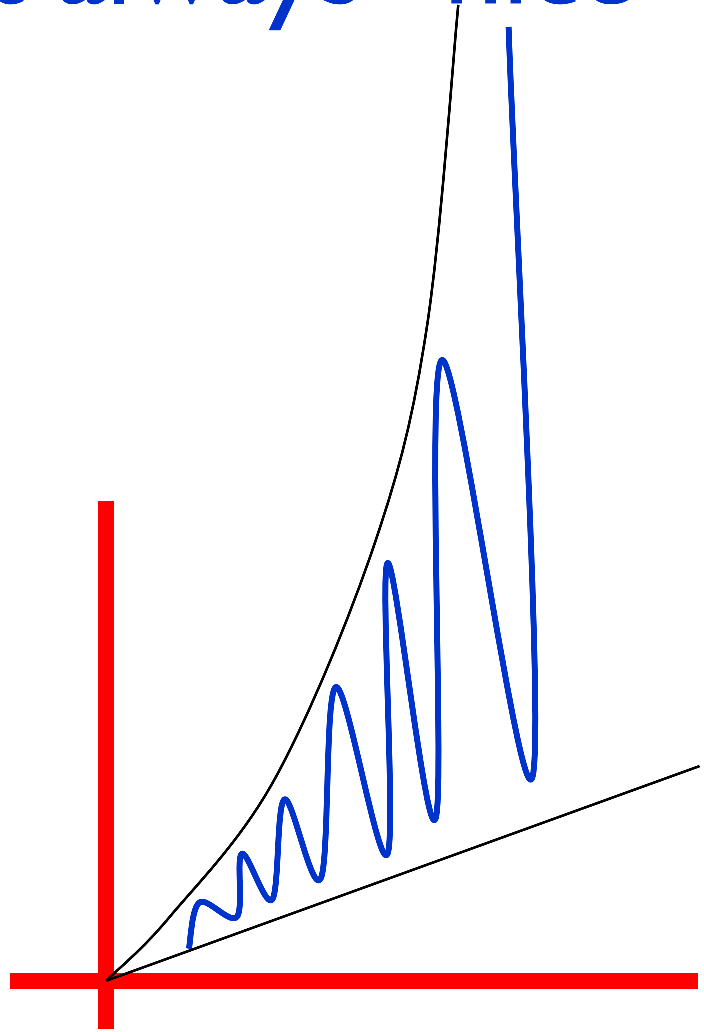
Bottom line:

OK to put big-O in R.H.S. of equality, but not left.

[Better, but uncommon, notation: $T(n) < O(f(n))$.]

# Big-Theta, etc. not always "nice"

$$f(n) = \begin{cases} n^2, & n \text{ even} \\ n, & n \text{ odd} \end{cases}$$



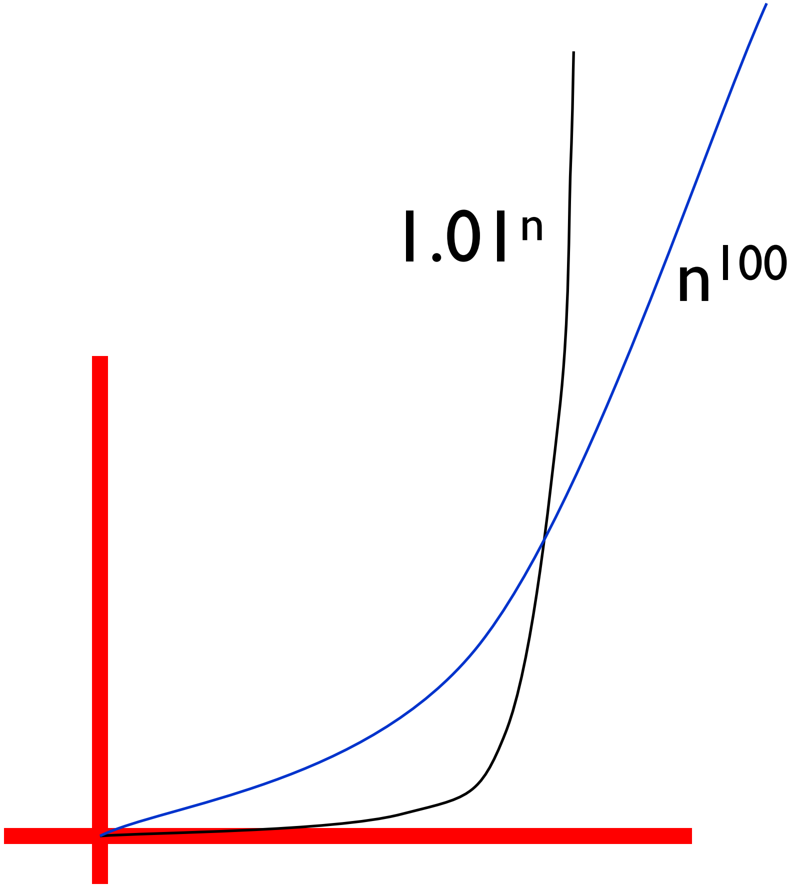
*f(n) is not* Θ(*n*$^a$) for any *a*.

Fortunately, such nasty cases are rare

# Asymptotic Bounds for Some Common Functions

Exponentials. For all $r > 1$ and all $d > 0$, $n^d = O(r^n)$.

*every exponential grows faster than every polynomial*



# Polynomial time

P: Running time is $O(n^d)$ for some constant d independent of the input size n.

*Nice scaling property*: there is a constant c s.t. *doubling* n, time increases only by a factor of c.

(E.g., $c \sim 2^d$)

Contrast with exponential: For any constant c, there is a d such that $n \rightarrow n+d$ increases time by a factor of more than c.

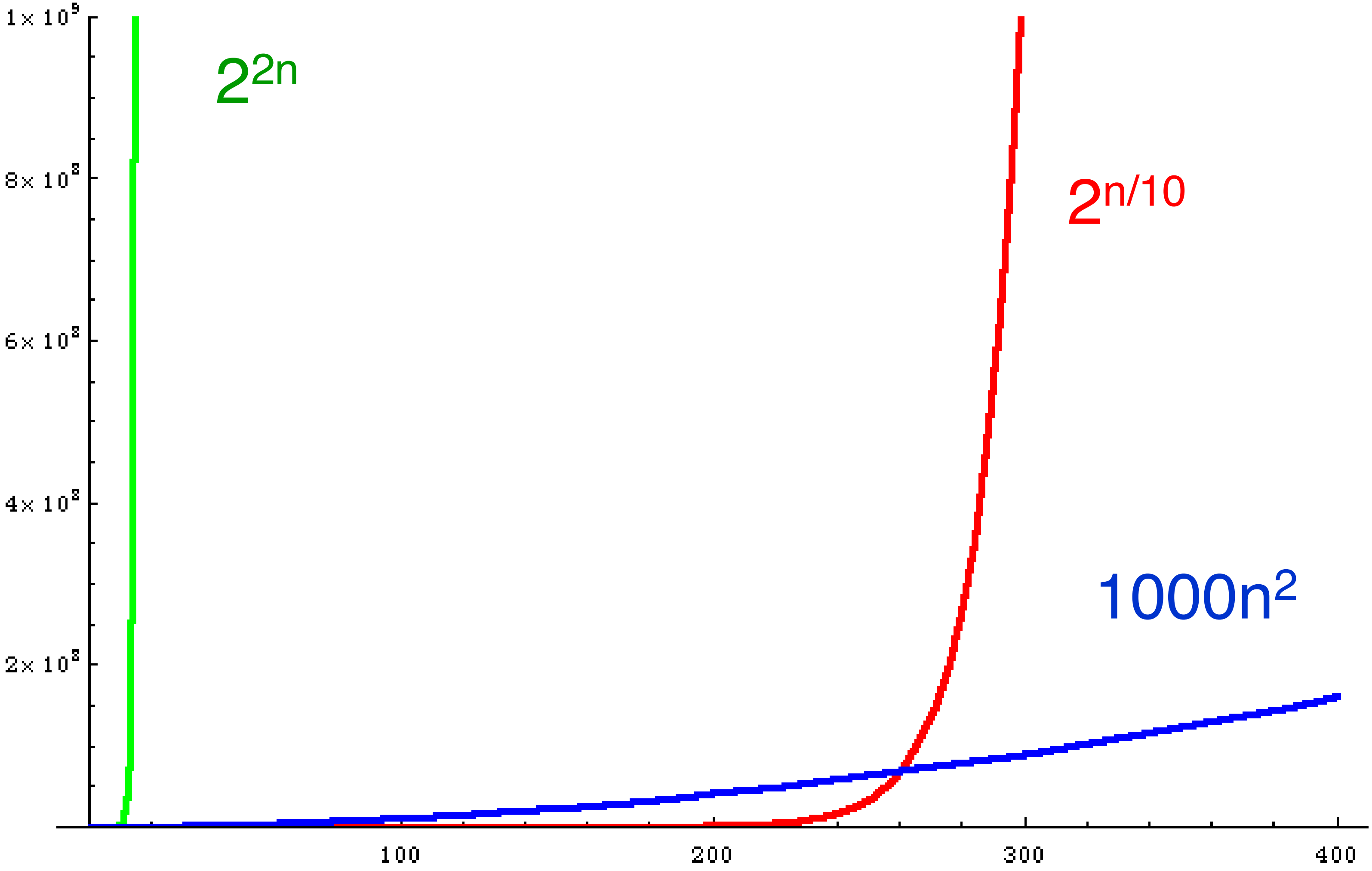
(E.g., $2^n$ vs $2^{n+1}$)

# Polynomial time

P: Running time is $O(n^d)$ for some constant d independent of the input size n.

Behaves well under composition: if algorithm has a polynomial running time with polynomial number of calls to a subroutine that has polynomial running time, then overall running time is still polynomial.

# polynomial vs exponential growth



# Why It Matters

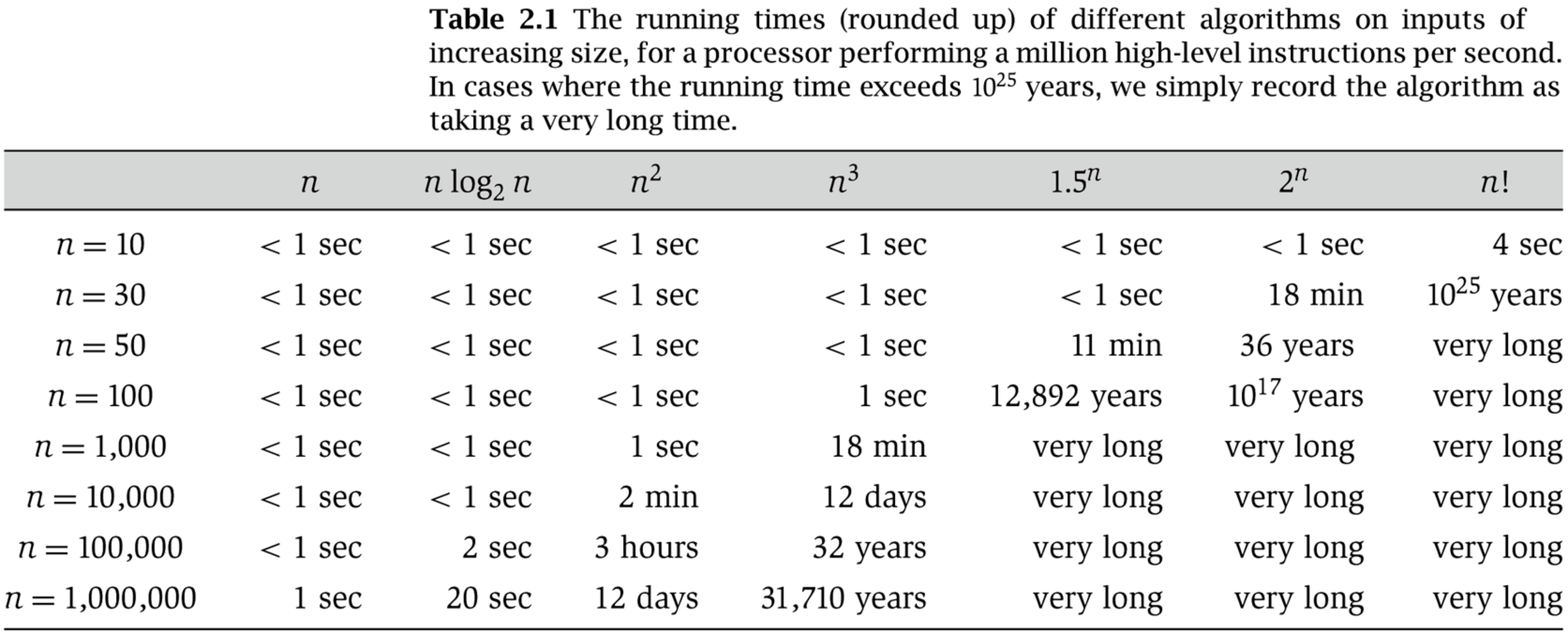


**Table 2.1** The running times (rounded up) of different algorithms on inputs of increasing size, for a processor performing a million high-level instructions per second. In cases where the running time exceeds $10^{25}$ years, we simply record the algorithm as taking a very long time.

| | $n$ | $n \log_2 n$ | $n^2$ | $n^3$ | $1.5^n$ | $2^n$ | $n!$ |
|---|---|---|---|---|---|---|---|
| $n = 10$ | < 1 sec | < 1 sec | < 1 sec | < 1 sec | < 1 sec | < 1 sec | 4 sec |
| $n = 30$ | < 1 sec | < 1 sec | < 1 sec | < 1 sec | < 1 sec | 18 min | $10^{25}$ years |
| $n = 50$ | < 1 sec | < 1 sec | < 1 sec | < 1 sec | 11 min | 36 years | very long |
| $n = 100$ | < 1 sec | < 1 sec | < 1 sec | 1 sec | 12,892 years | $10^{17}$ years | very long |
| $n = 1{,}000$ | < 1 sec | < 1 sec | 1 sec | 18 min | very long | very long | very long |
| $n = 10{,}000$ | < 1 sec | < 1 sec | 2 min | 12 days | very long | very long | very long |
| $n = 100{,}000$ | < 1 sec | 2 sec | 3 hours | 32 years | very long | very long | very long |
| $n = 1{,}000{,}000$ | 1 sec | 20 sec | 12 days | 31,710 years | very long | very long | very long |

# another view of poly vs exp

Next year's computer will be 2x faster. If I can solve problem of size $n_0$ today, how large a problem can I solve in the same time next year?

| Complexity | Increase | E.g. $T=10^{12}$ | | |
|---|---|---|---|---|
| $O(n)$ | $n_0 \rightarrow 2n_0$ | $10^{12}$ | → | $2 \times 10^{12}$ |
| $O(n^2)$ | $n_0 \rightarrow \sqrt{2}\, n_0$ | $10^6$ | → | $1.4 \times 10^6$ |
| $O(n^3)$ | $n_0 \rightarrow \sqrt[3]{2}\, n_0$ | $10^4$ | → | $1.25 \times 10^4$ |
| $2^{n/10}$ | $n_0 \rightarrow n_0+10$ | 400 | → | 410 |
| $2^n$ | $n_0 \rightarrow n_0+1$ | 40 | → | 41 |

# Domination

$f(n)$ is $o(g(n))$ iff $\lim_{n\to\infty} f(n)/g(n)=0$

that is $g(n)$ *dominates* $f(n)$

If $a < b$ then $n^a$ is $O(n^b)$

If $a > b$ then $n^a$ is $o(n^b)$

Note:
if $f(n)$ is $\Omega(g(n))$ then it cannot be $o(g(n))$

# Summary

Typical initial goal for algorithm analysis is to find a

reasonably tight ← i.e., $\Theta$ if possible

*asymptotic* ← i.e., $O$ or $\Theta$

bound on ← usually upper bound

*worst case* running time

as a function of problem *size*

This is rarely the last word, but often helps separate good algorithms from blatantly poor ones - so you can concentrate on the good ones!