degrees

Claim: If a graph has m edges, then

$$\sum_{v} \deg(v) = 2m.$$

Proof: Every edge {u,v} contributes exactly 2 to the left hand side, 1 to deg(u), and 1 to deg(v).

degrees

Claim: In a party, the number of people who shake hands with an odd number of people must be even.

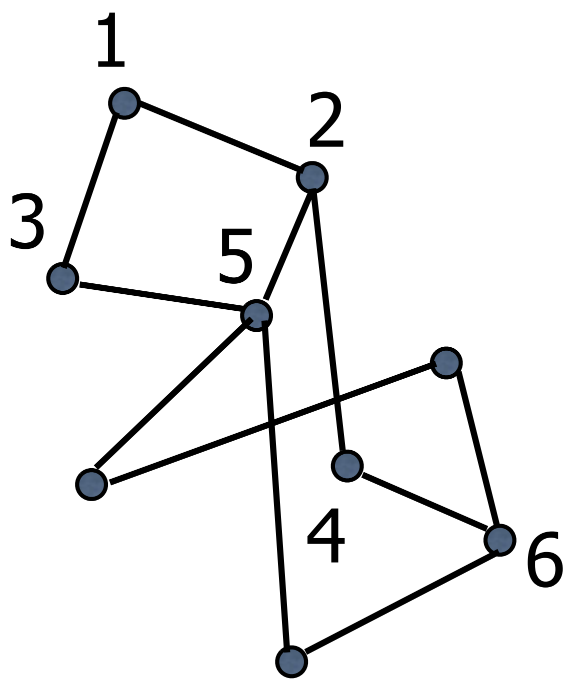
Proof: Construct a graph. Each vertex represents person, put edge between 2 people if they shake hands.

Need to prove: #vertices with odd degree is even.

$$\underbrace{2m}_{\textbf{even}} = \sum \deg(v) = \underbrace{\sum_{v \text{ of even degree}} \deg(v)}_{\textbf{even}} + \sum_{v \text{ of odd degree}} \deg(v)$$

Thus $\sum_{v \text{ of odd degree}} \deg(v)$ is even, so # vertices with odd degree is even.

Claim: If a graph has no cycles, it must have a vertex v of degree ≤ 1.



Path: A sequence of distinct vertices where each vertex is connected to the next by an edge.

Cycle: A path of length > 1 such that the first vertex is connected to the last one.

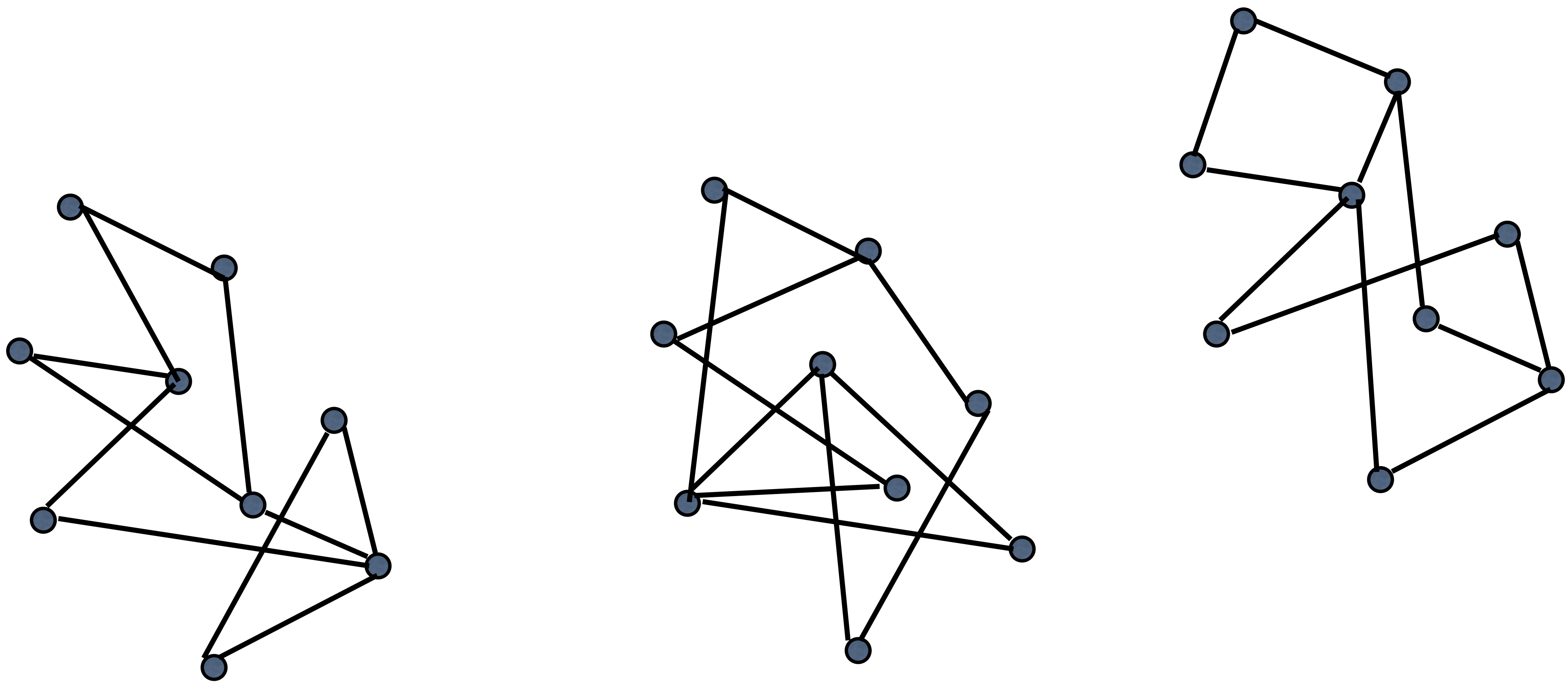
1,2,5,3 is a cycle

6,4,2,1,3,5,2,4 is not a cycle

Claim: If a graph has no cycles, it must have a vertex v of degree ≤ 1.

EQUIVALENT TO

Claim: If every vertex has degree > 1, the graph has a cycle.

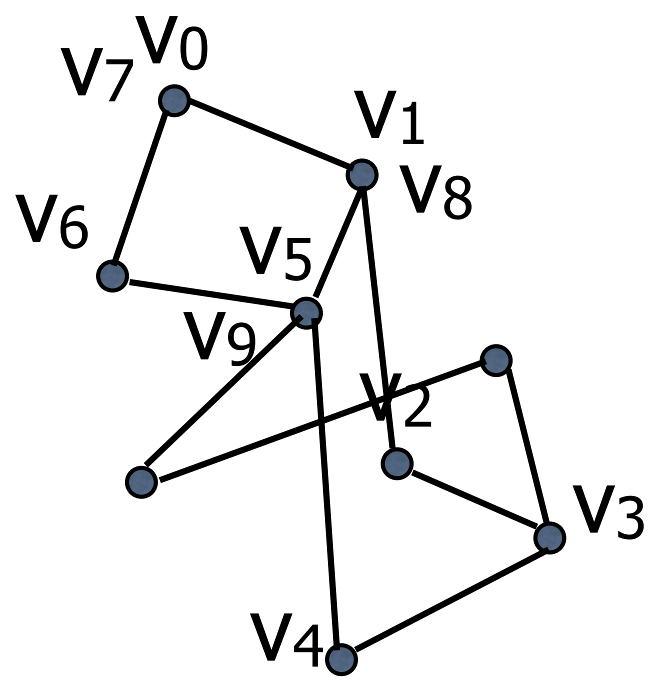


Claim: If a graph has no cycles, it must have a vertex v of degree ≤ 1.

Proof:
Suppose not.

Then there is a graph G that has no cycles, and yet every vertex in the graph has degree > 1.

There must be $v_0,v_1,\ldots,v_n$, a sequence of n+1 vertices such that $v_{i-1} \neq v_{i+1}$, for all i, and adjacent vertices have an edge.



Intuition: this should not be possible! Let's use the degrees of the vertices to find a cycle.

Note: The argument should work for every such graph!

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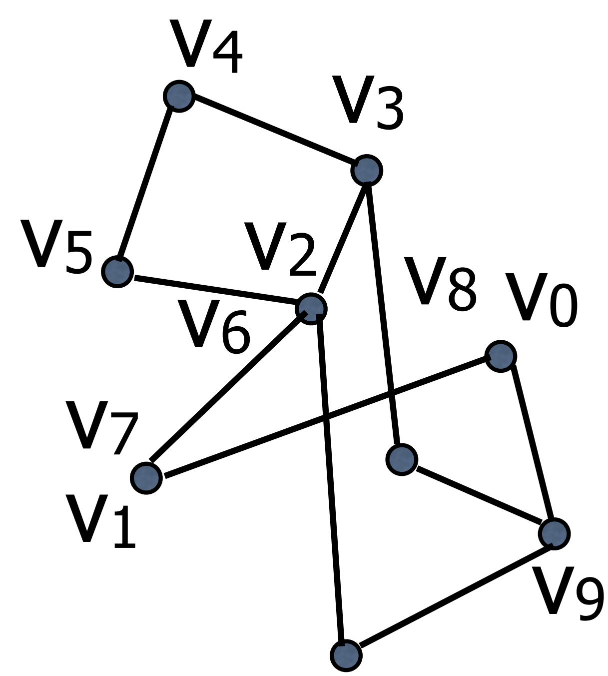
By the pigeonhole principle, there must $i<j$ s.t. $v_i = v_j$.

Let $i<j$ be the closest such pair. Then $v_i, v_{i+1}, \ldots, v_{j-1}$ form a cycle.



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Lemma: Every tree on n vertices has exactly n-1 edges.

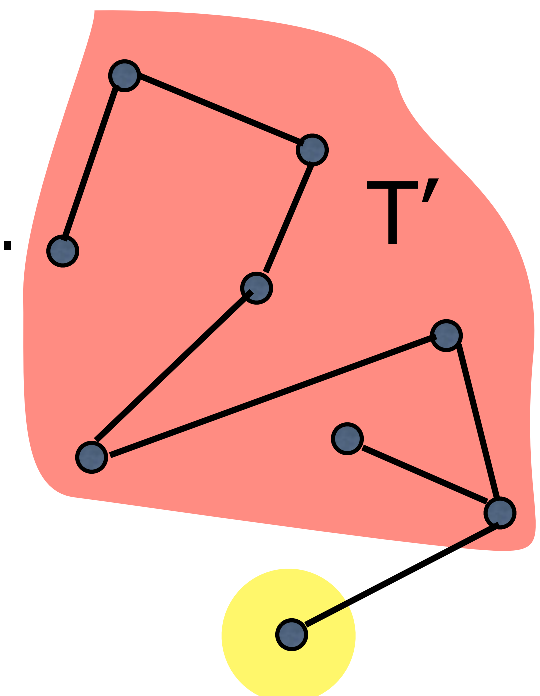
Tree: A connected graph that has no cycles.

Proof: By induction on n.

Base case: n=1. Every graph with 1 vertex has 1-1 = 0 edges. The claim holds.

Inductive step: n>1. Let T be a tree with n vertices.

Since T has no cycles, there is a vertex v, deg(v) ≤ 1 by previous claim. deg(v)=1 since T is connected means there are no degree 0 vertices.



T′ = T-v is connected and has no cycles, so it is a tree. T′ has n-1 vertices. So by induction, T′ has n-2 edges. Thus T has n-1 edges.

Mantel's Theorem: If a graph on $2n$ vertices has n^2+1 edges, then it must have a triangle (3-cycle).

Proof: By induction on n.

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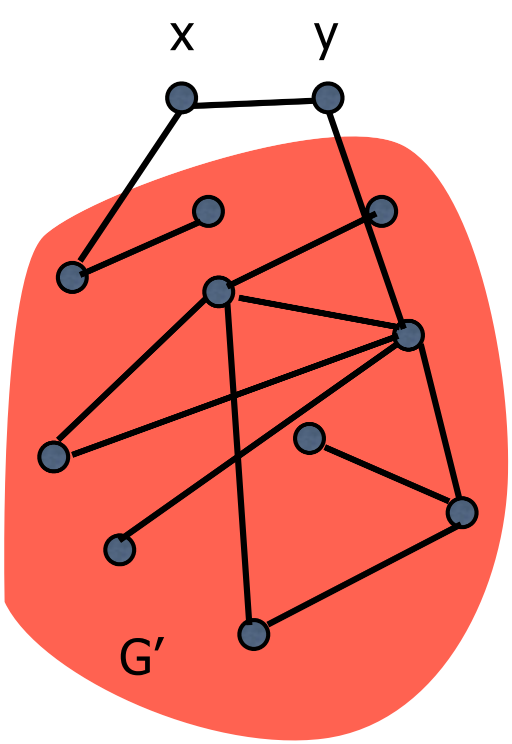
Let {x,y} be an edge of the graph, and let G′ denote the subgraph using the rest of the vertices.



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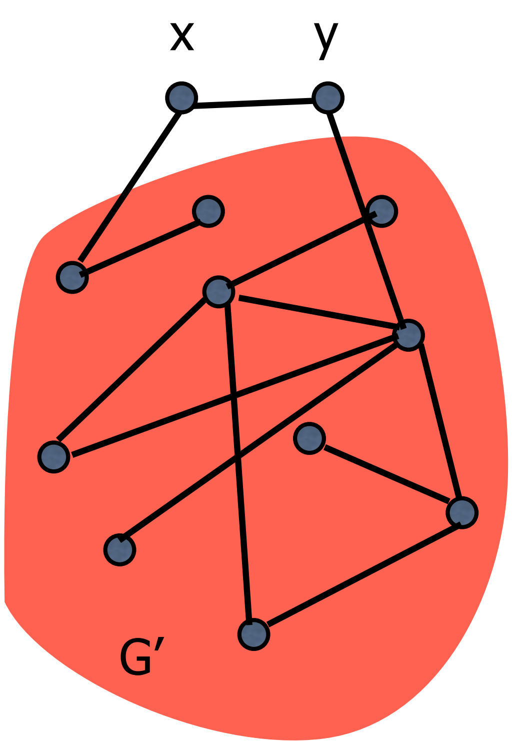
Case 1: At least $(n-1)^2+1$ edges are in G′.
Then by the induction hypothesis, G′ has a triangle.



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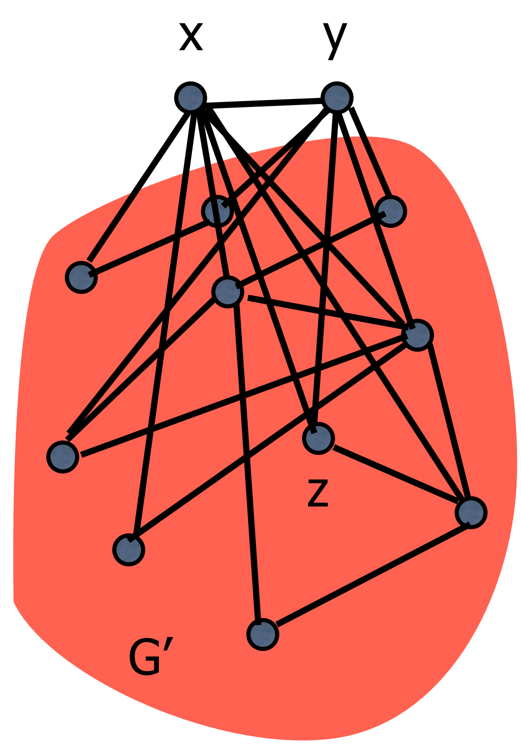

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Case 2: At most $(n-1)^2$ edges are in G′.
There is one edge between x,y, so #edges from {x,y} to G′ $\geq n^2-(n-1)^2 = 2(n-1)+1$. But G′ has 2(n-1) vertices, so there is vertex z such that {x,z},{y,z} are edges. Then x,y,z is a triangle.