NP-completeness

- Many many problems are NP-complete
- If you solve one of them efficiently, you solve all of them efficiently
- We don't know how to solve any of them efficiently

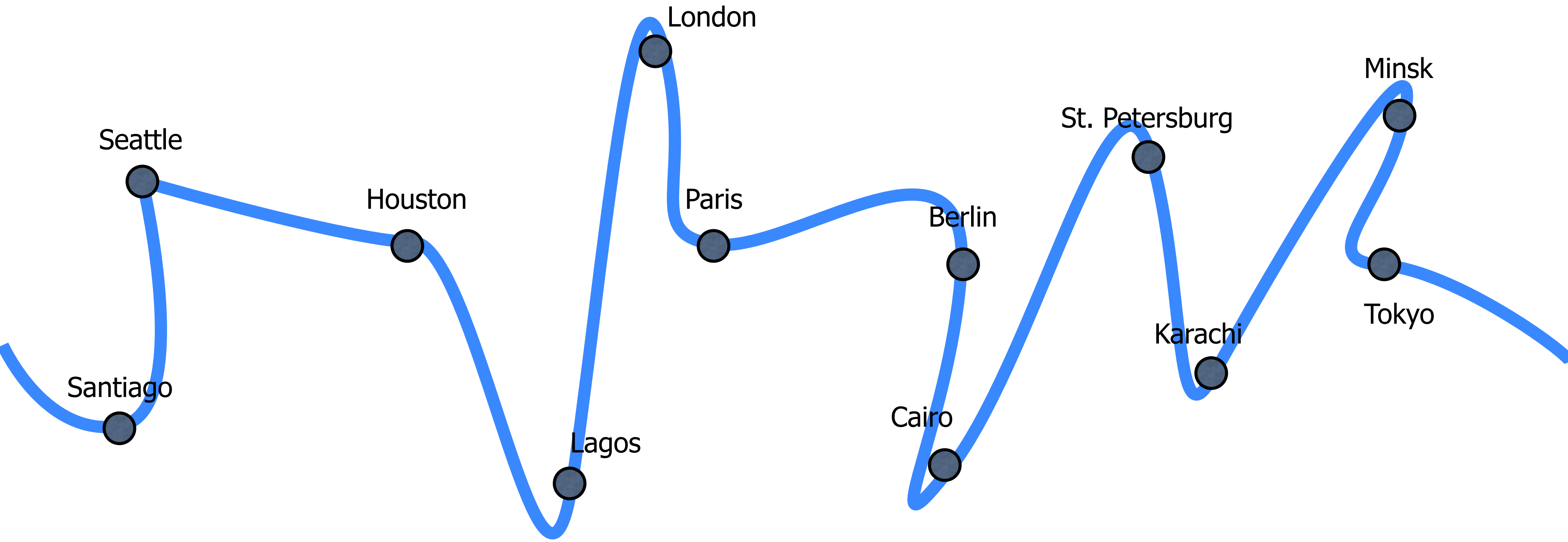
Approximation Algorithms

- So it's unlikely we'll solve one of these soon :(
- Instead of finding the best solution, we'll find a solution that is close :)

Traveling Salesman

Given: n cities with distances

Goal: Compute shortest tour to visit them



Traveling Salesman

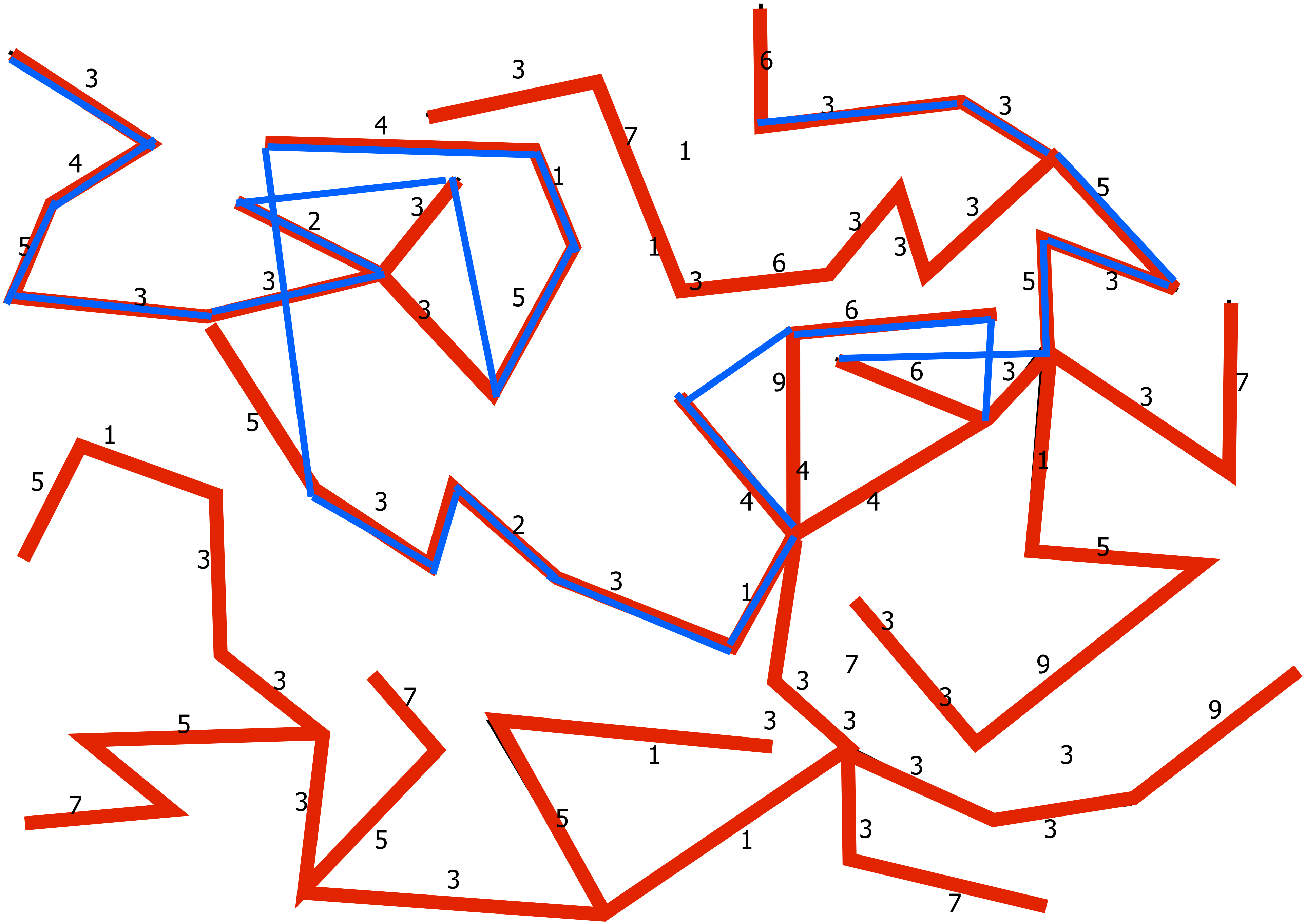
Given: n cities with distances
Goal: Compute shortest tour to visit them

Metric TSP: distances satisfy triangle inequality:

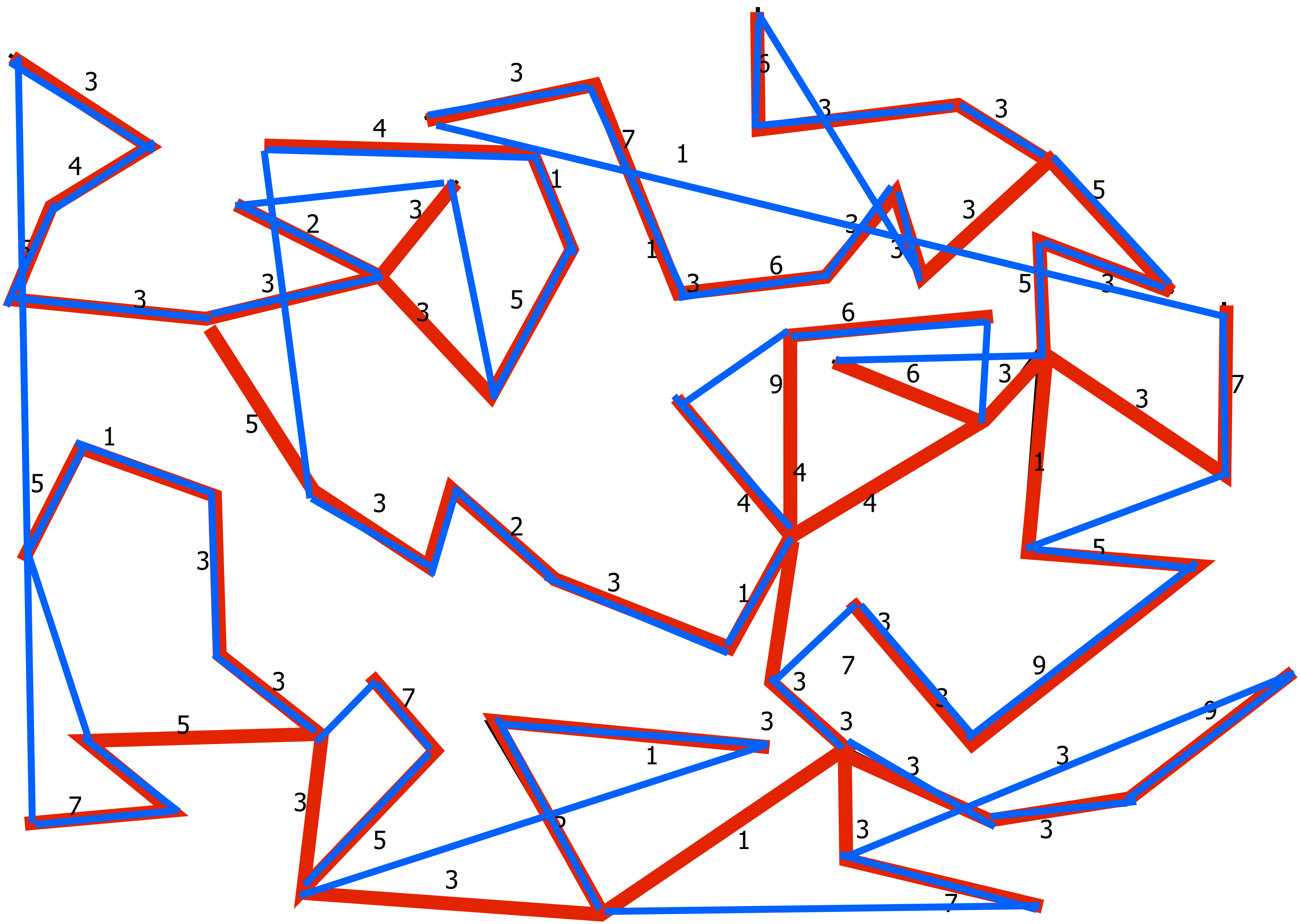
distance(a,c) ≤ distance(a,b) + distance(b,c)

Idea: use MST!
Prove: tour within factor 2 of best possible

MST tour: Show that it is within factor 2!



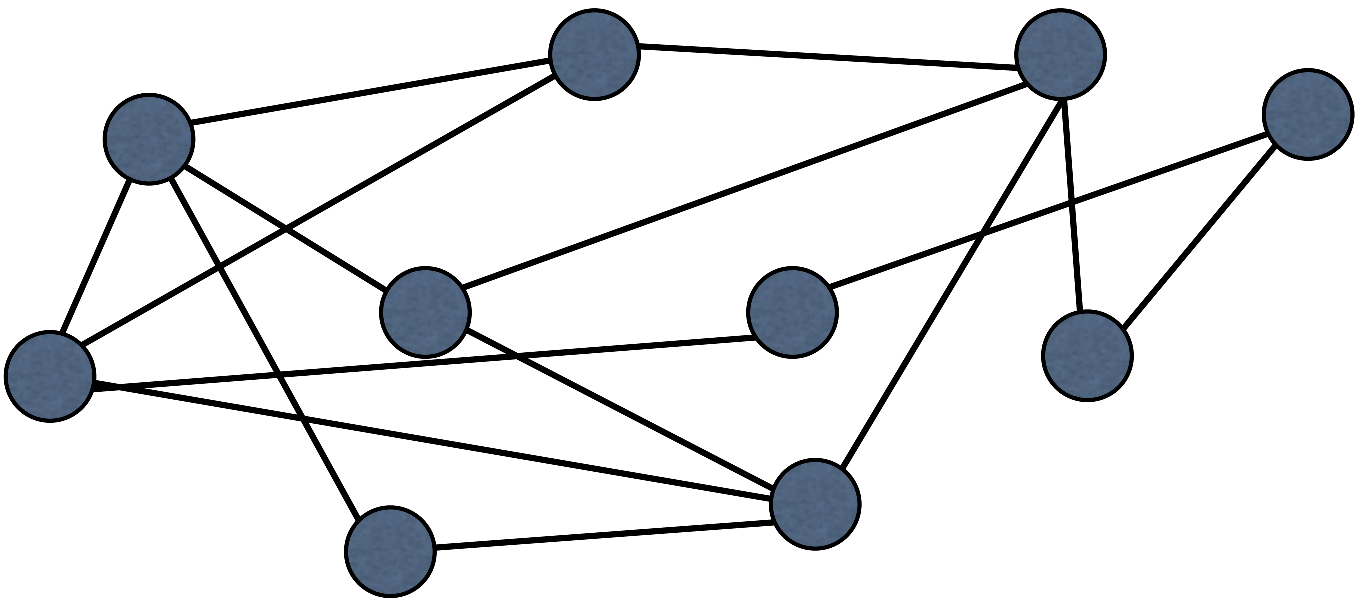
MST tour: Take the Euler tour of tree.



Claim: Every tour costs at least as much as MST.
Pf: Every tour contains a spanning tree

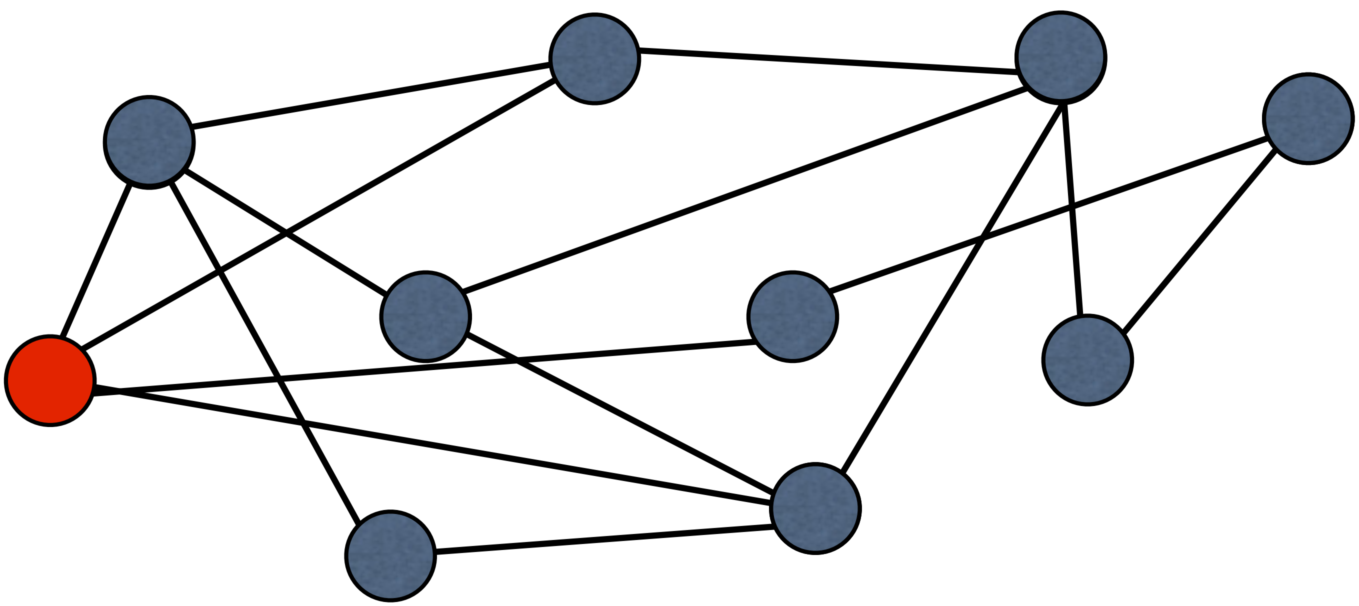
Claim: Euler tour costs at most 2 MST.
Pf: Can carry out Euler tour using each edge at most 2 times.

Vertex Cover



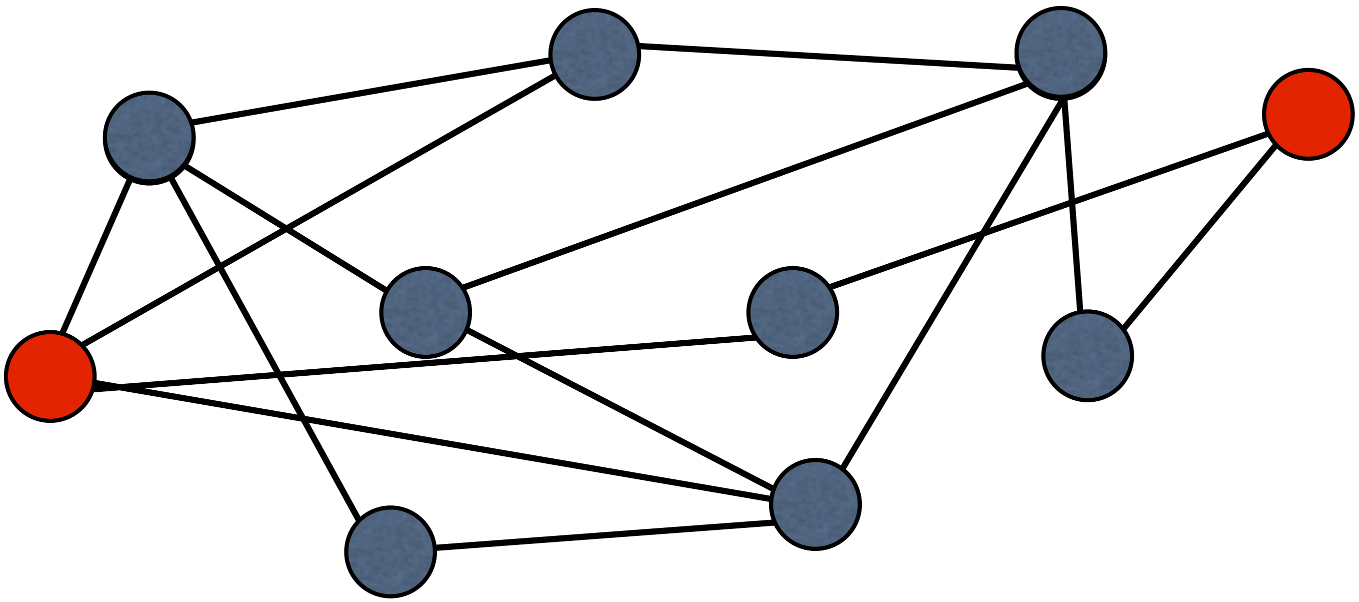
Find smallest set of vertices touching every edge

Vertex Cover



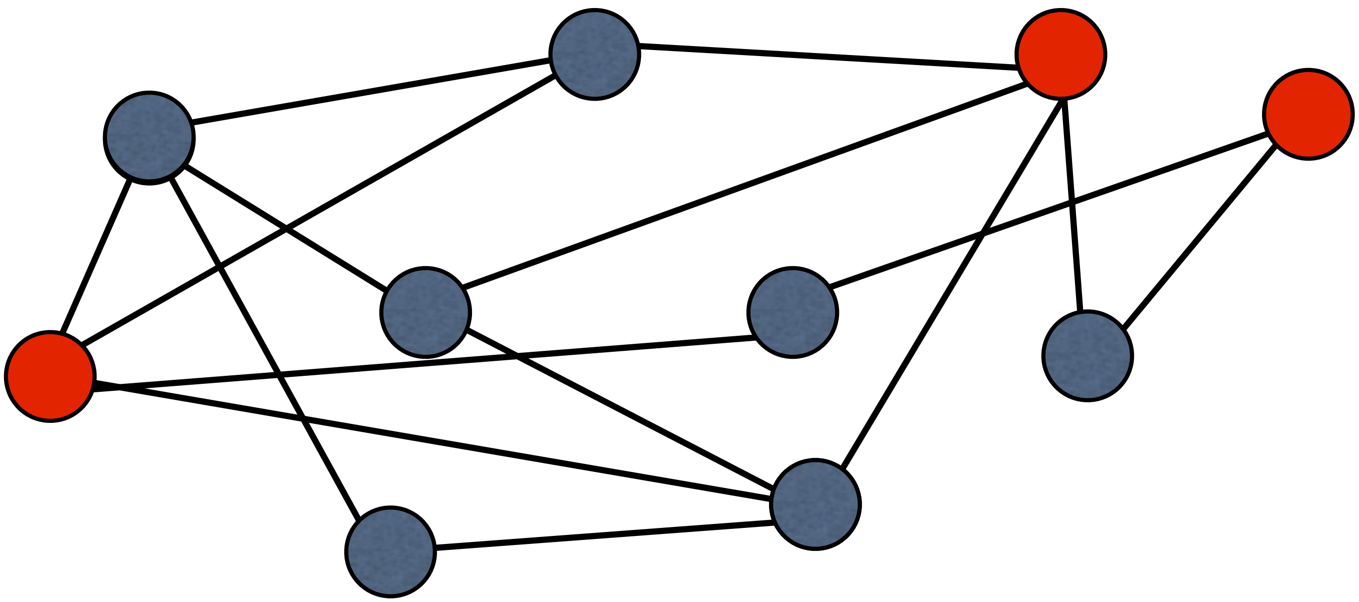
Find smallest set of
vertices touching
every edge

Vertex Cover



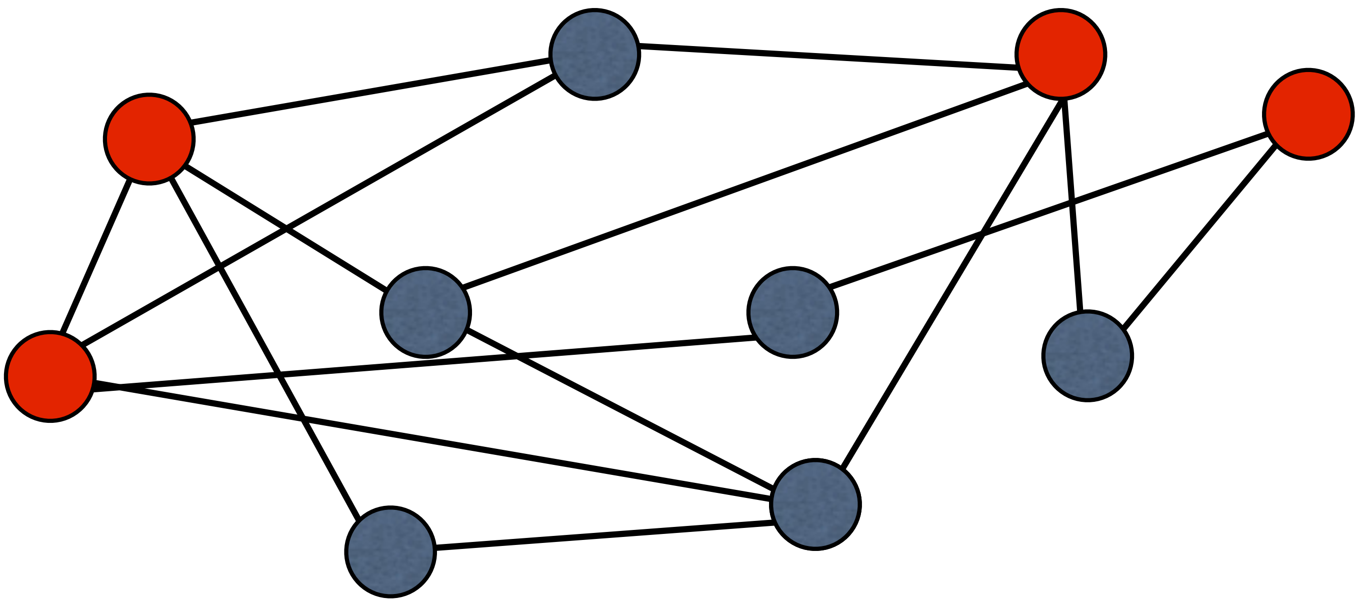
Find smallest set of vertices touching every edge

Vertex Cover



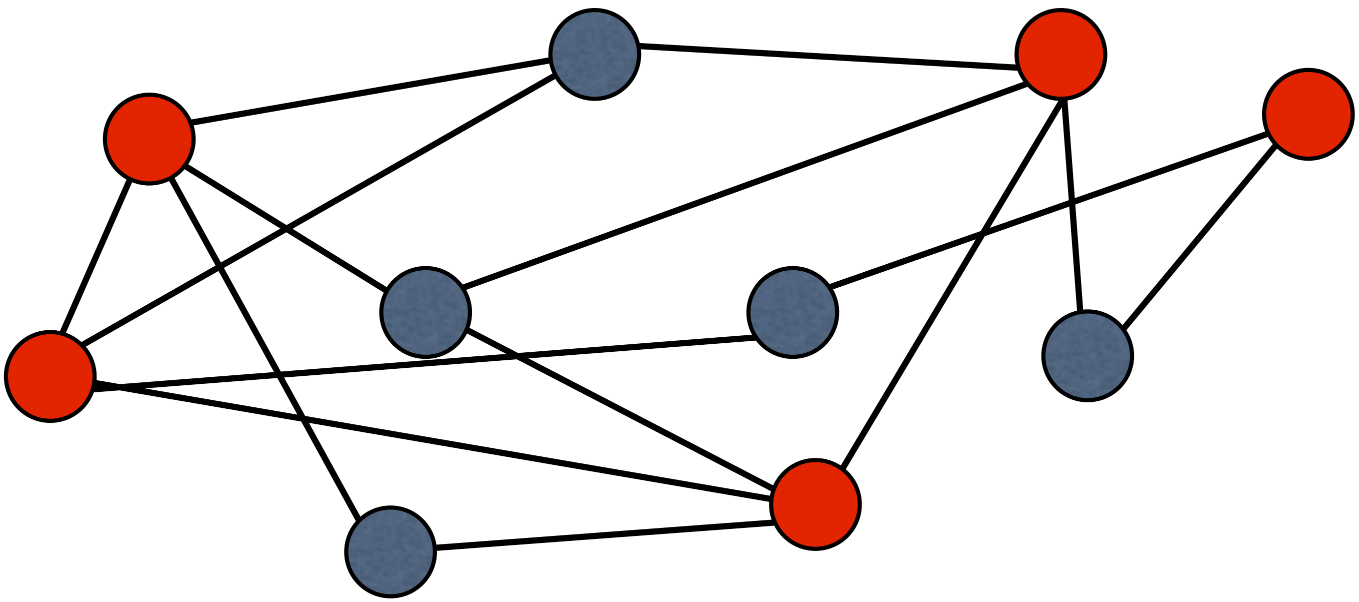
Find smallest set of
vertices touching
every edge

Vertex Cover



Find smallest set of vertices touching every edge

Vertex Cover



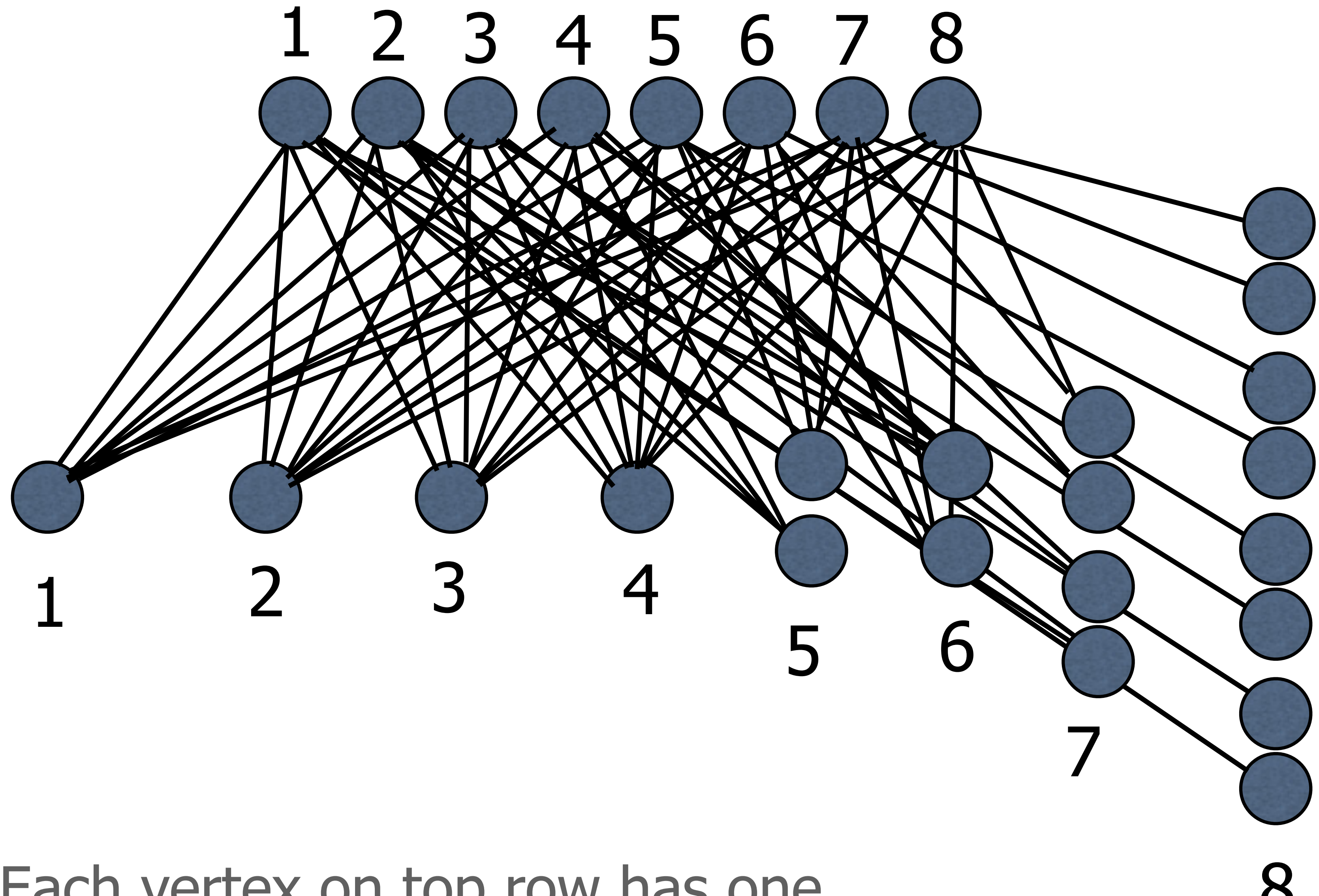
Find smallest set of vertices touching every edge

Vertex Cover size 5

Greedy algorithms?

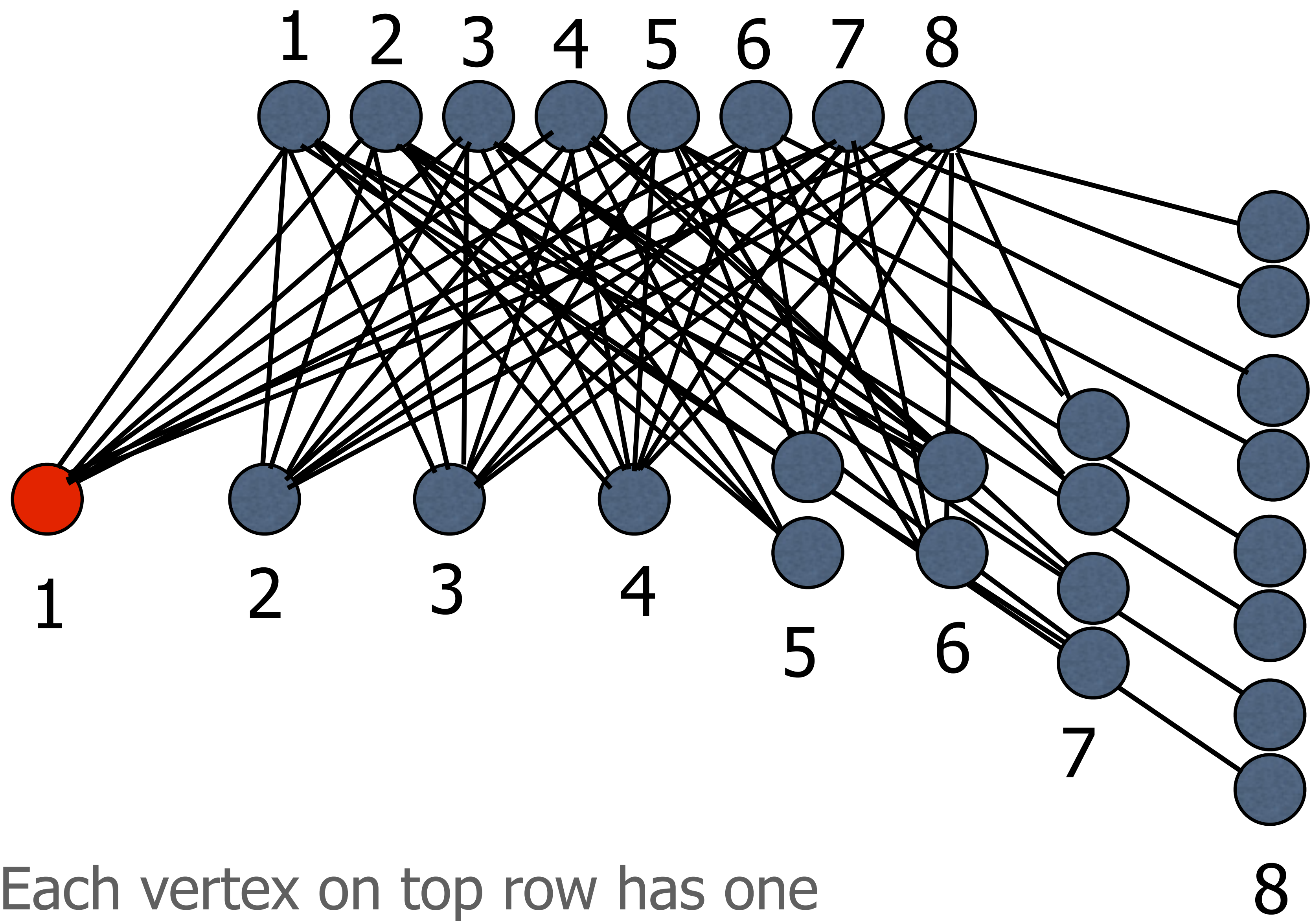
- Include vertex that covers most new edges?

Algorithm: Pick vertex that covers most new edges



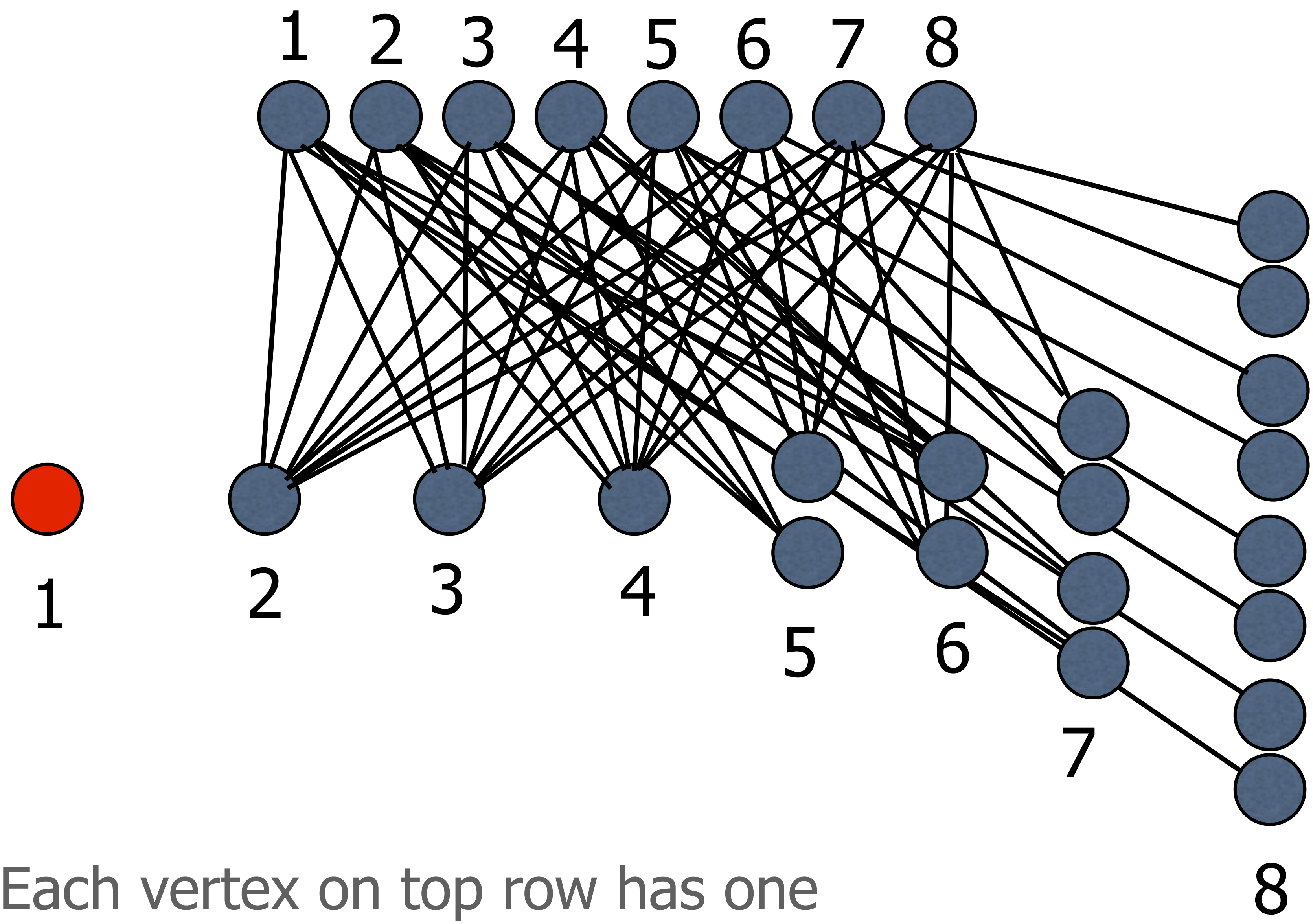
Each vertex on top row has one edge into each of the groups below.

Algorithm: Pick vertex that covers most new edges



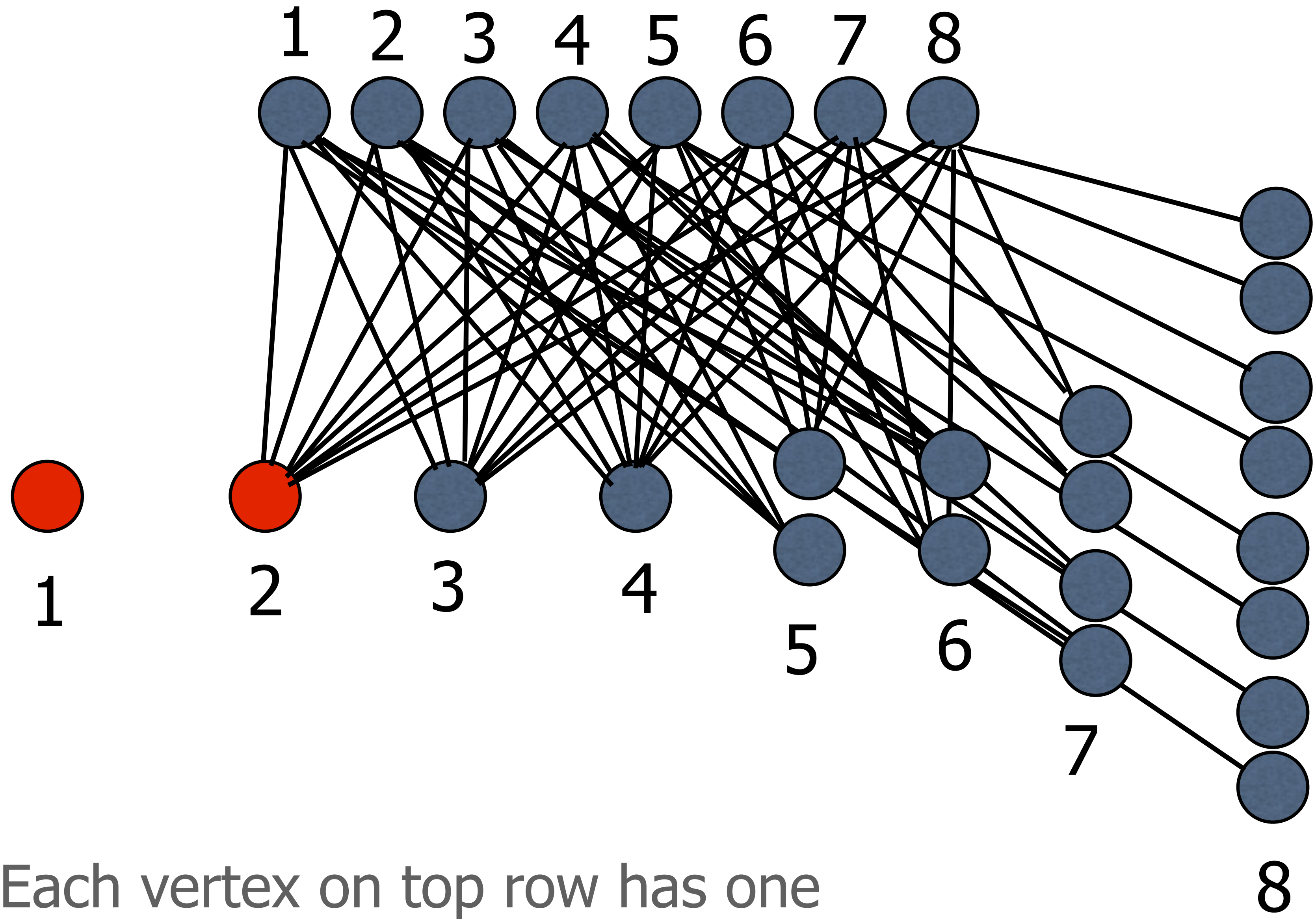
Each vertex on top row has one edge into each of the groups below.

Algorithm: Pick vertex that covers most new edges



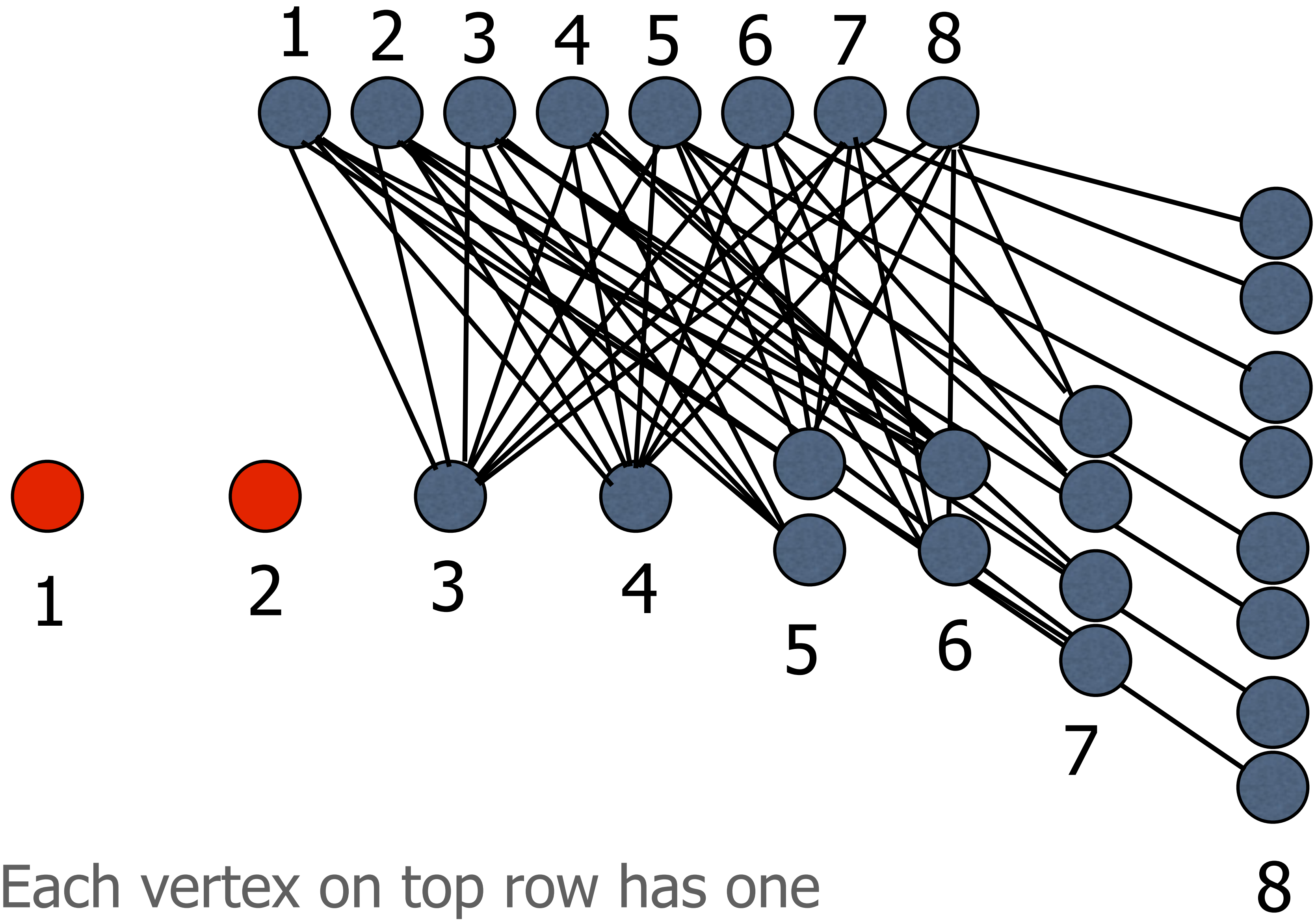
Each vertex on top row has one edge into each of the groups below.

Algorithm: Pick vertex that covers most new edges



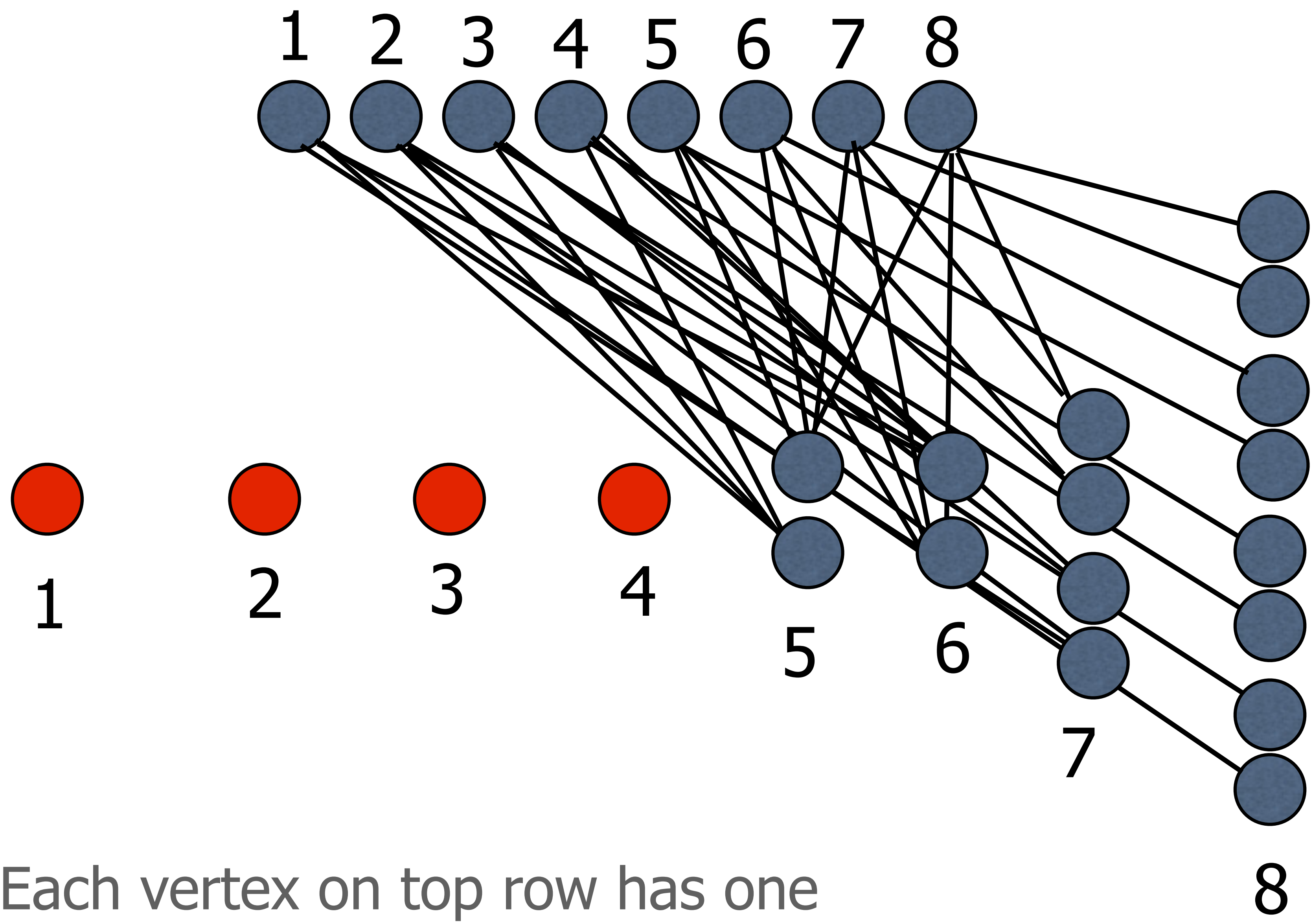
Each vertex on top row has one
edge into each of the groups below.

Algorithm: Pick vertex that covers most new edges



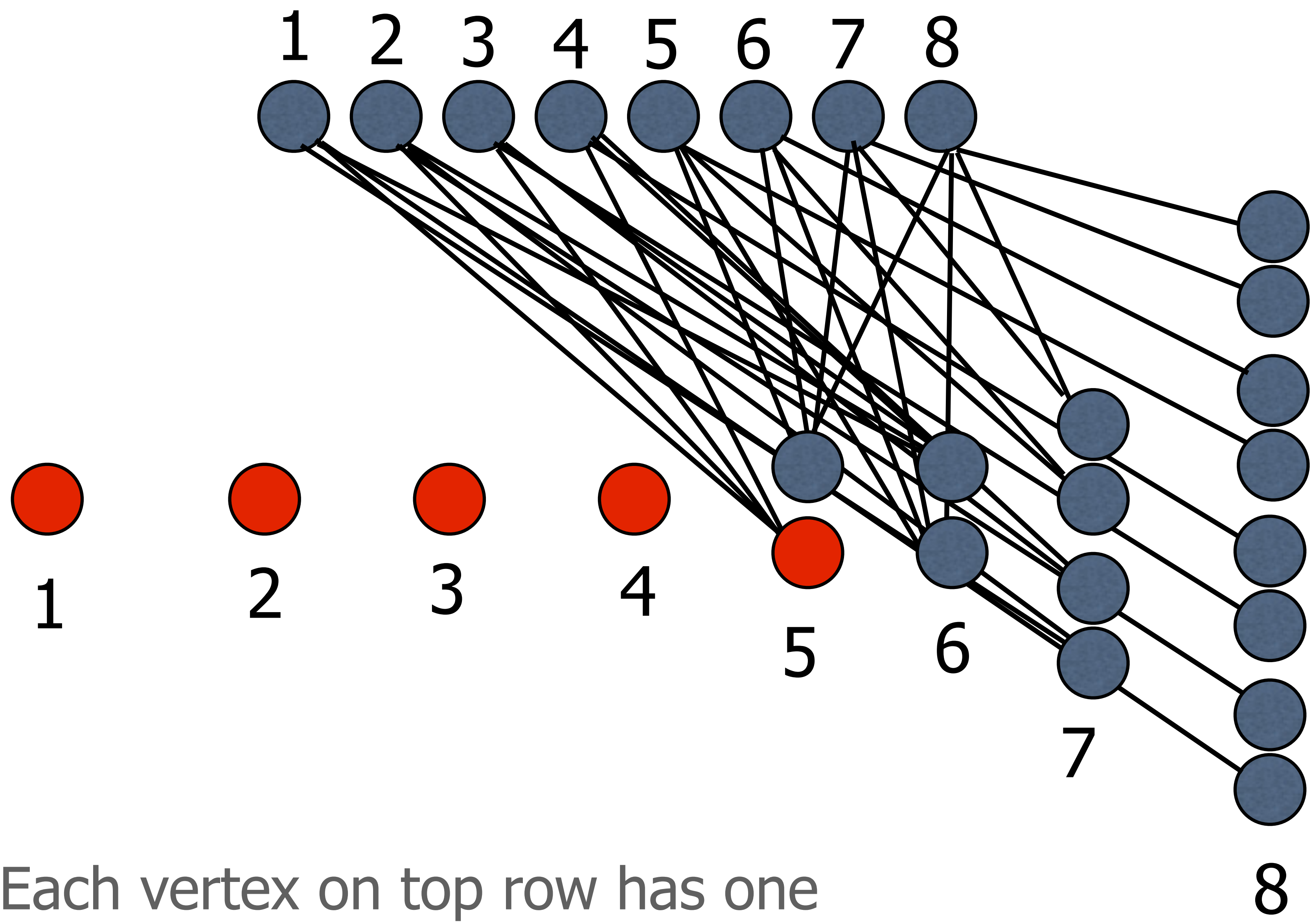
Each vertex on top row has one edge into each of the groups below.

Algorithm: Pick vertex that covers most new edges



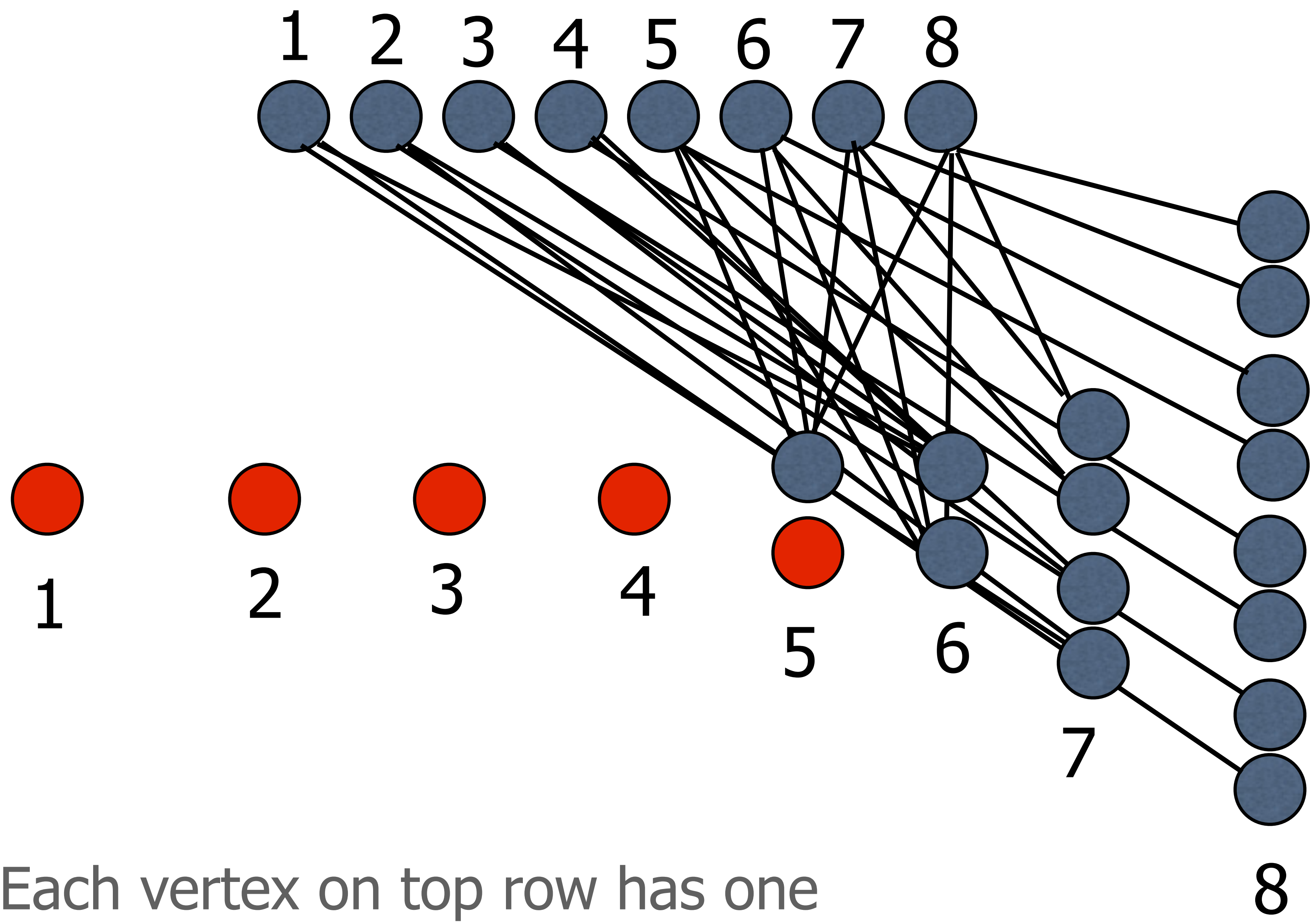
Each vertex on top row has one
edge into each of the groups below.

Algorithm: Pick vertex that covers most new edges



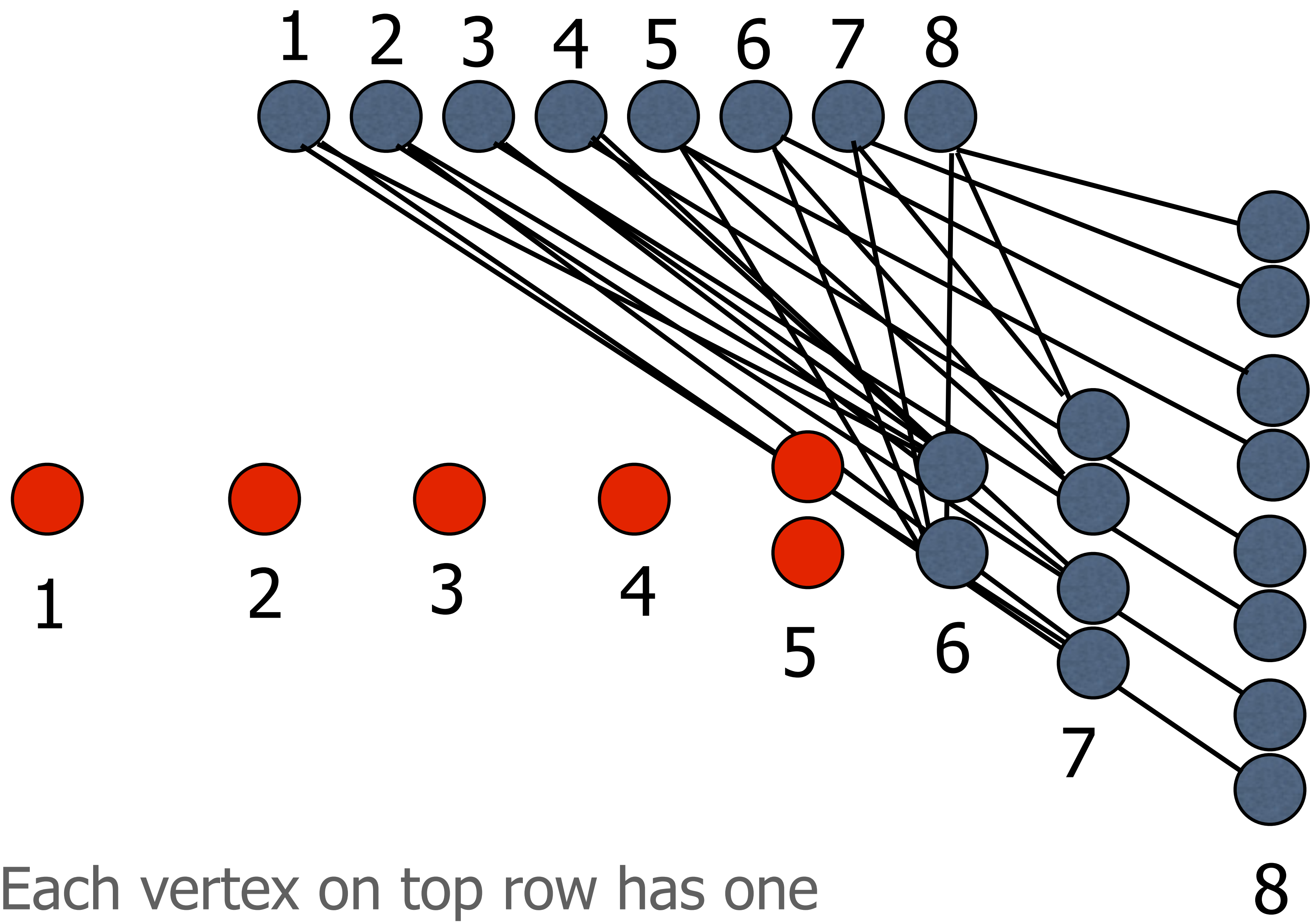
Each vertex on top row has one edge into each of the groups below.

Algorithm: Pick vertex that covers most new edges



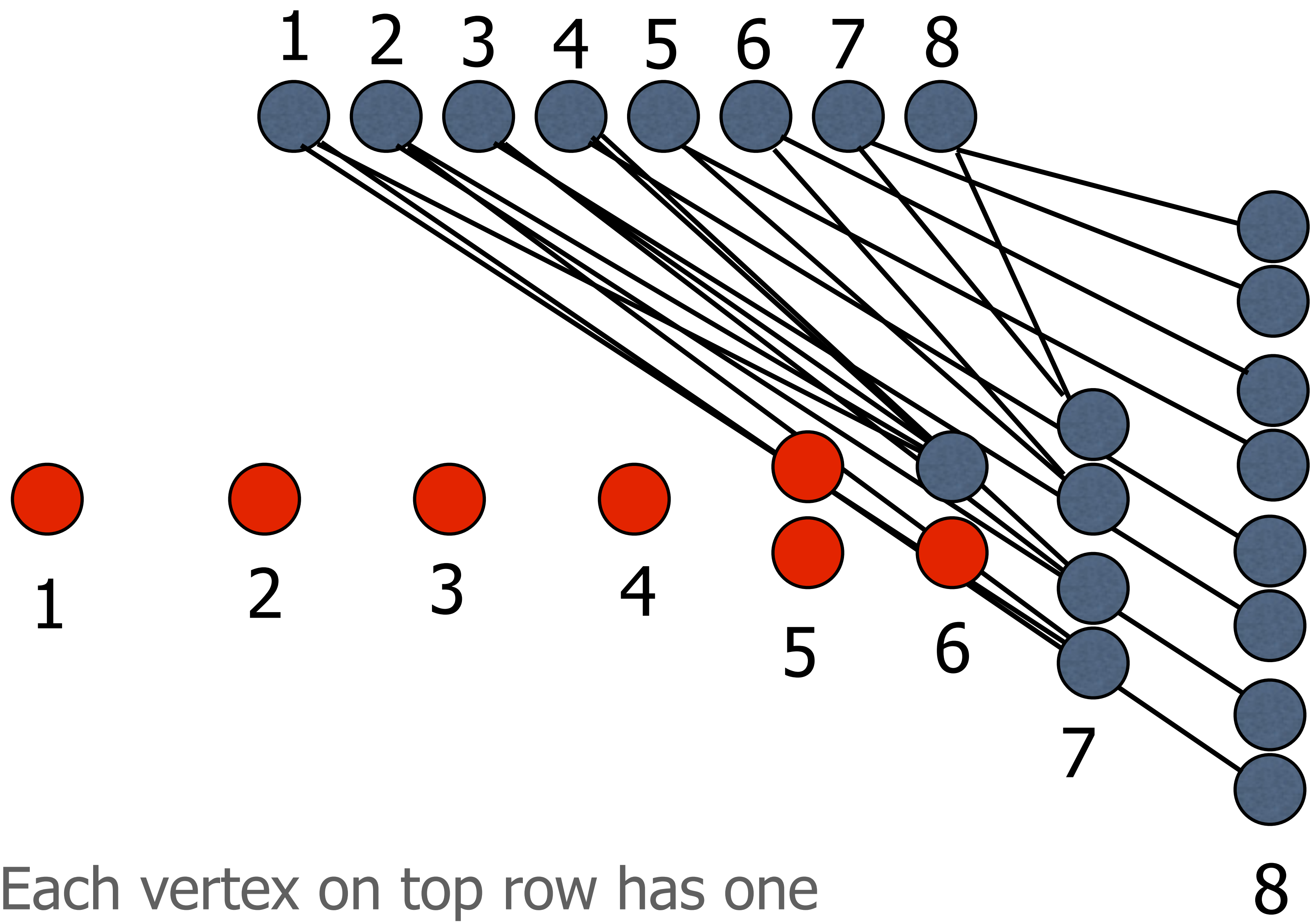
Each vertex on top row has one edge into each of the groups below.

Algorithm: Pick vertex that covers most new edges



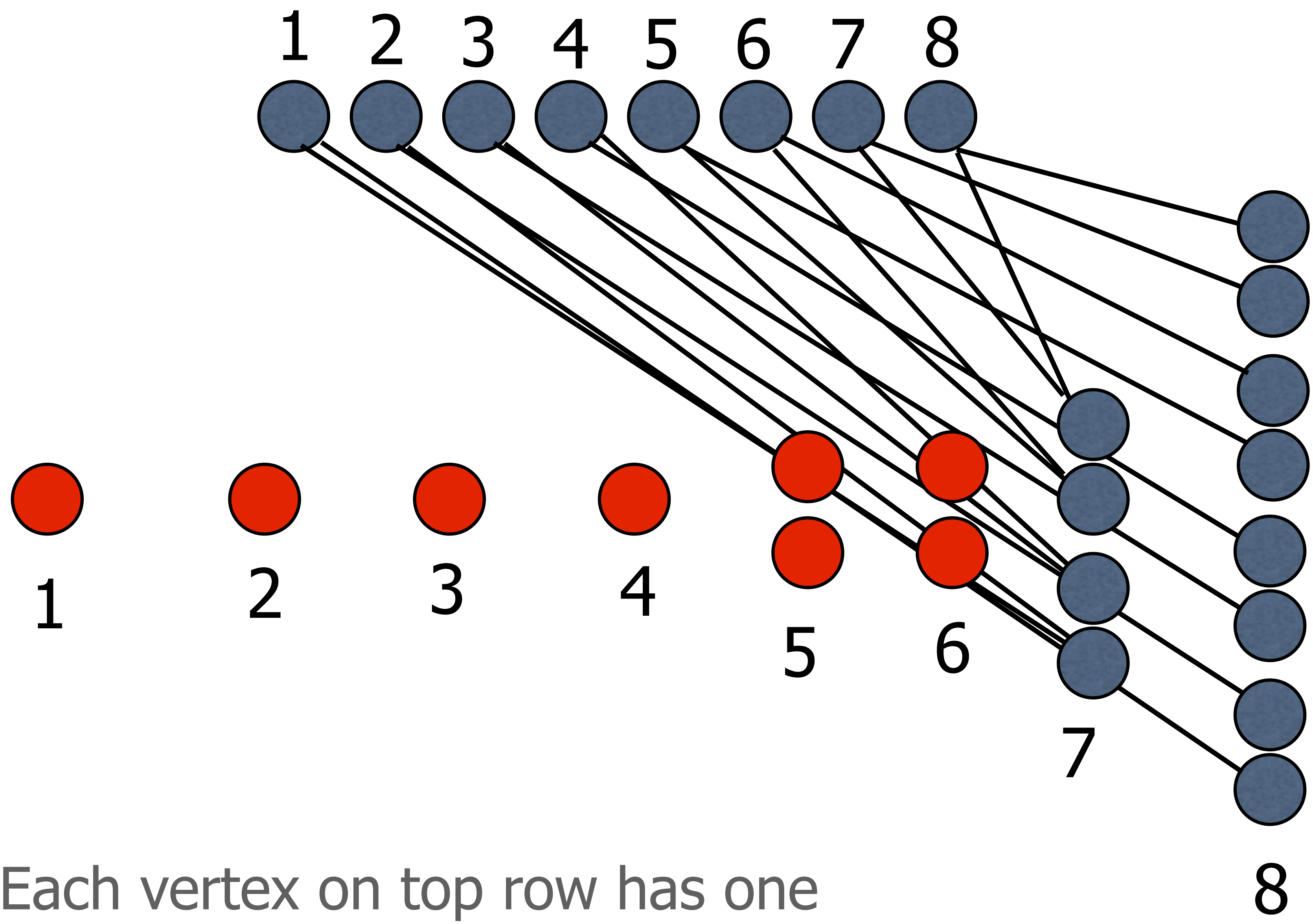
Each vertex on top row has one
edge into each of the groups below.

Algorithm: Pick vertex that covers most new edges



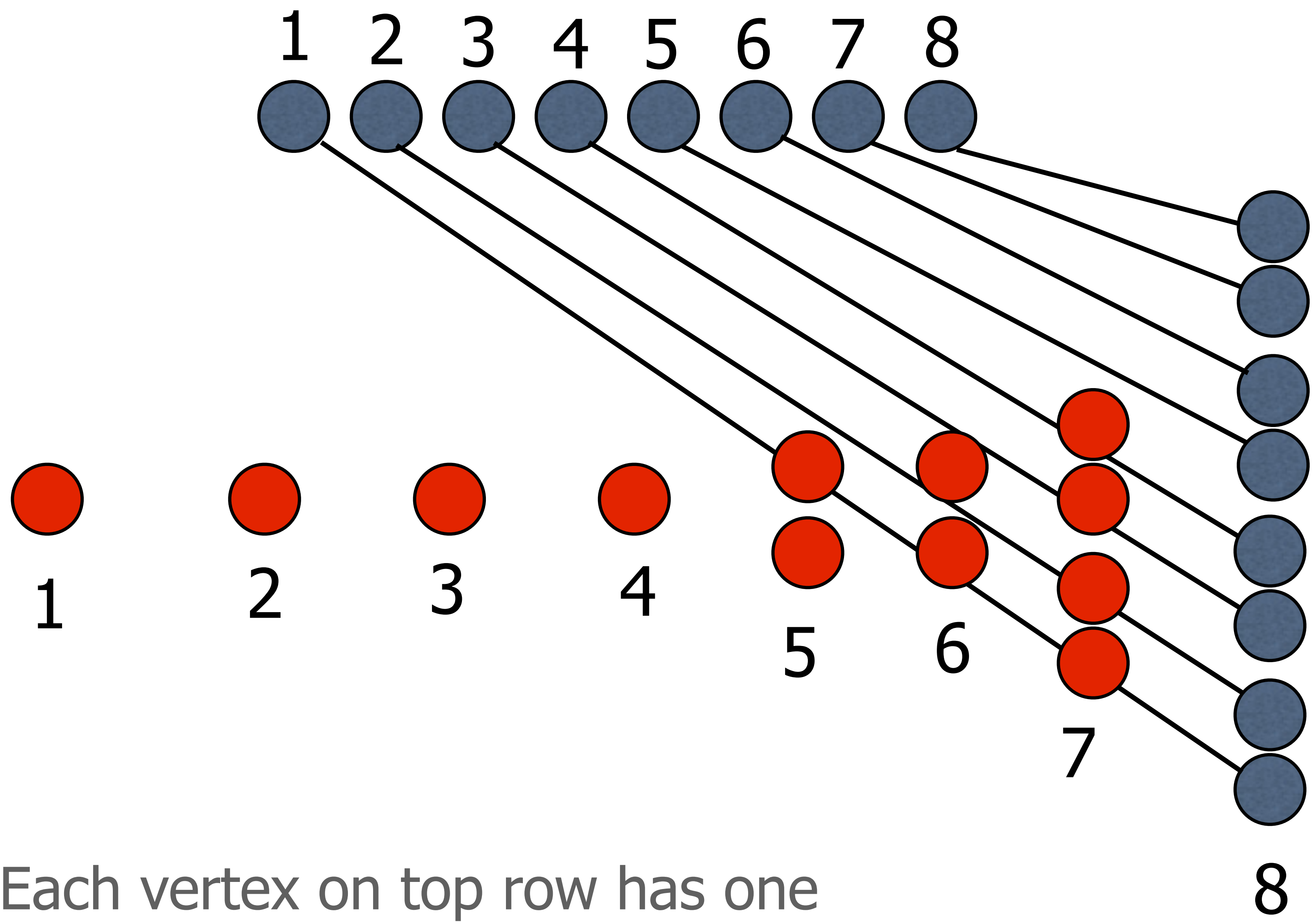
Each vertex on top row has one edge into each of the groups below.

Algorithm: Pick vertex that covers most new edges



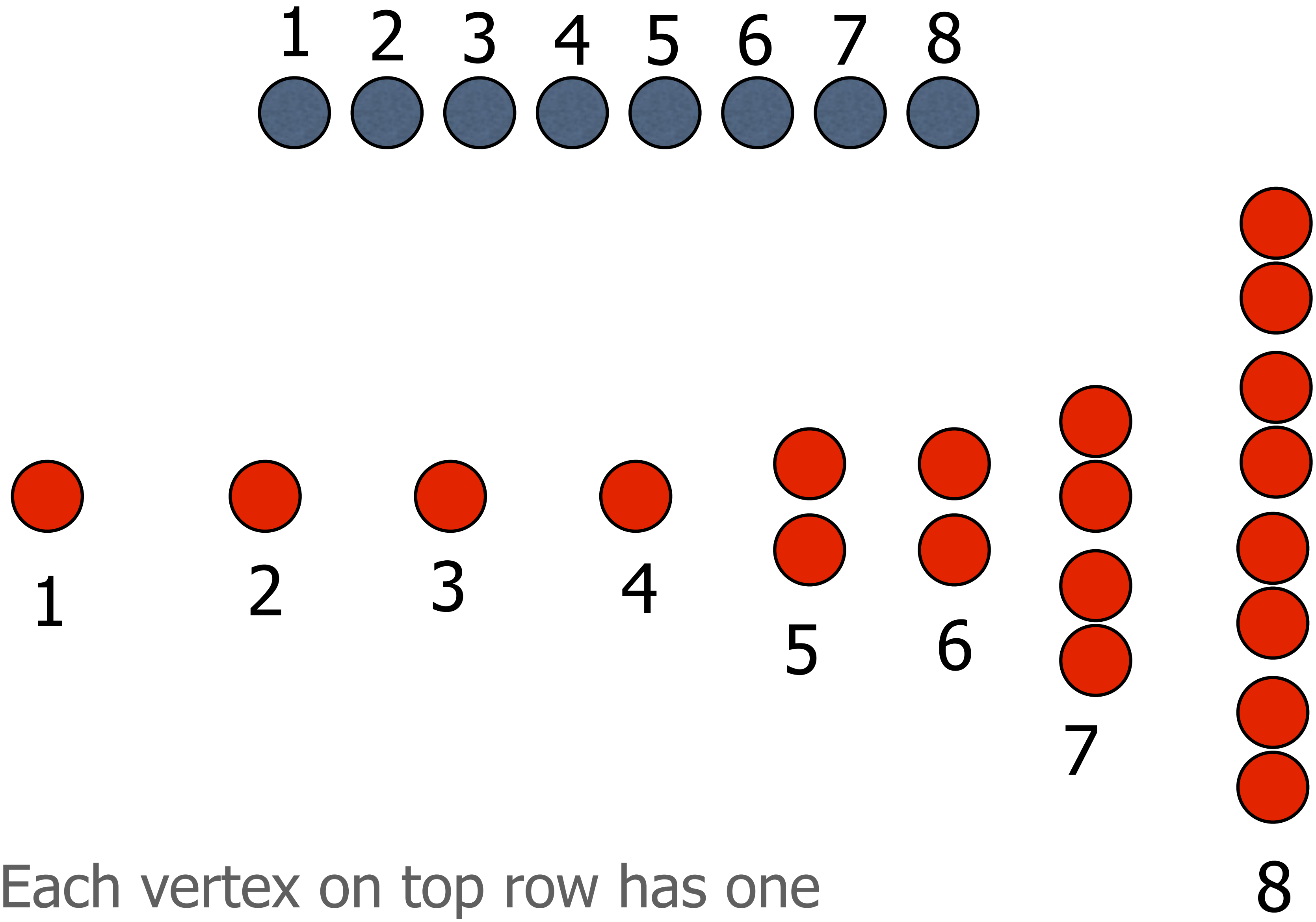
Each vertex on top row has one edge into each of the groups below.

Algorithm: Pick vertex that covers most new edges



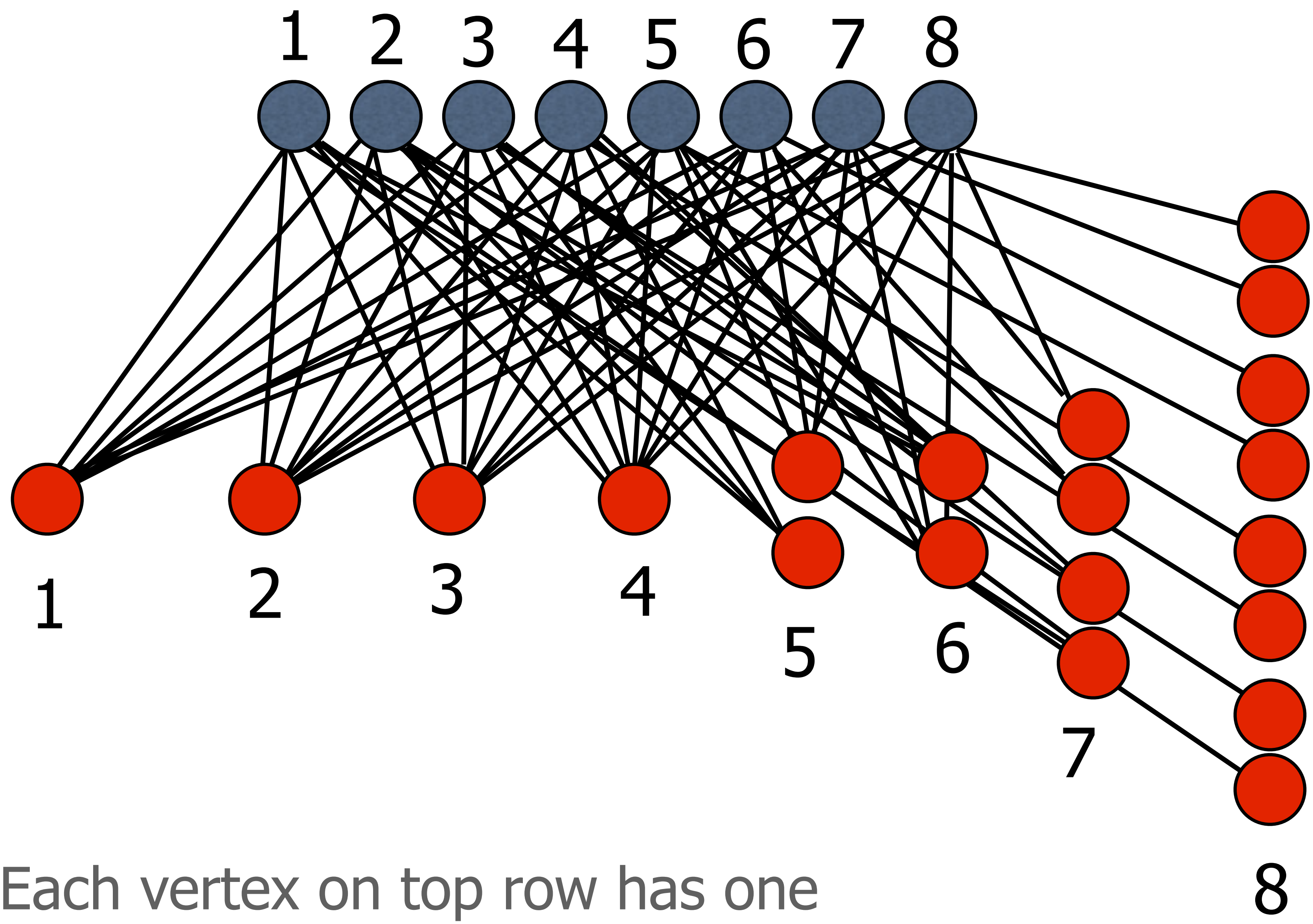
Each vertex on top row has one edge into each of the groups below.

Algorithm: Pick vertex that covers most new edges



Each vertex on top row has one
edge into each of the groups below.

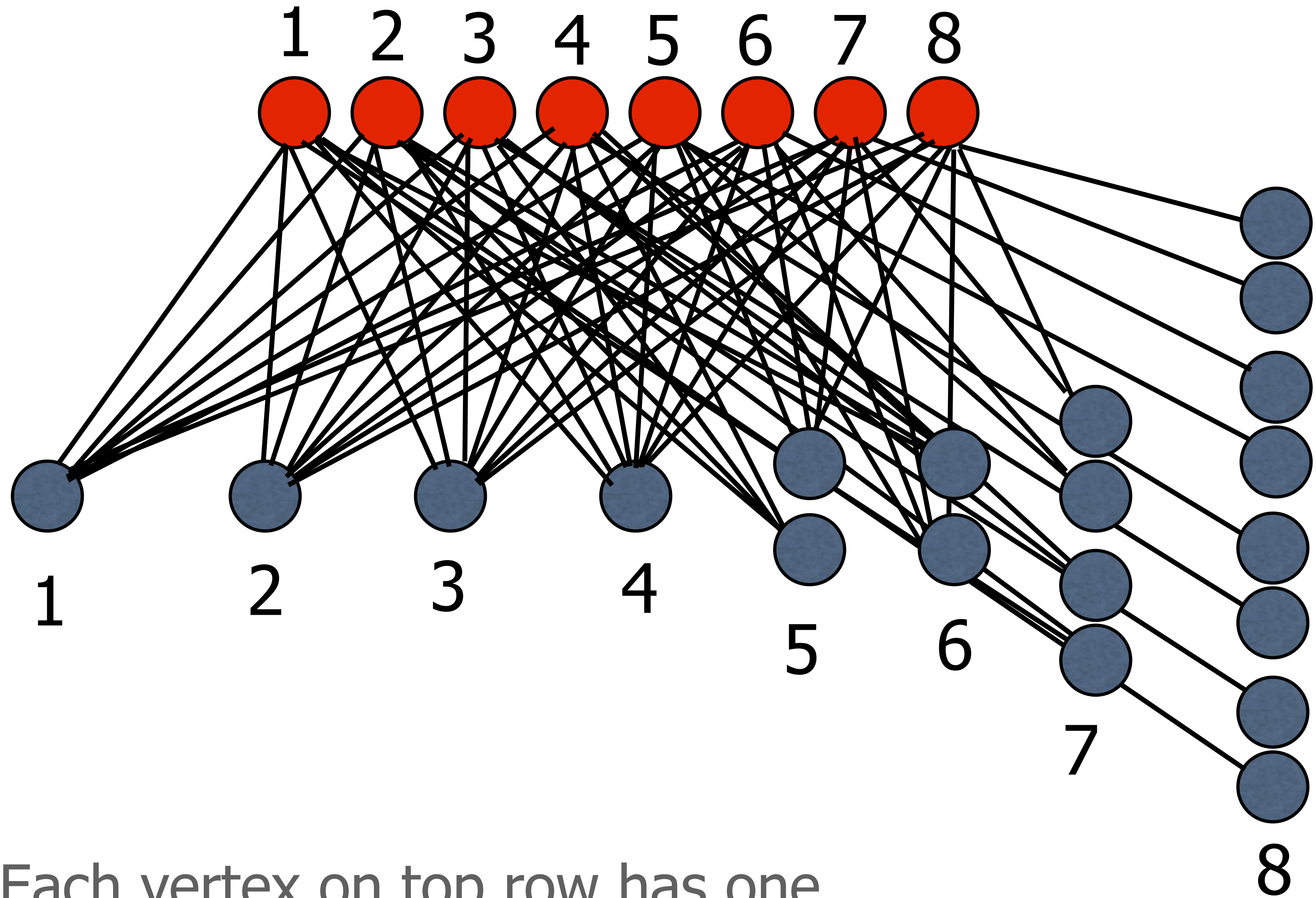
Algorithm: Pick vertex that covers most new edges



Each vertex on top row has one edge into each of the groups below.

Vertex Cover size 20

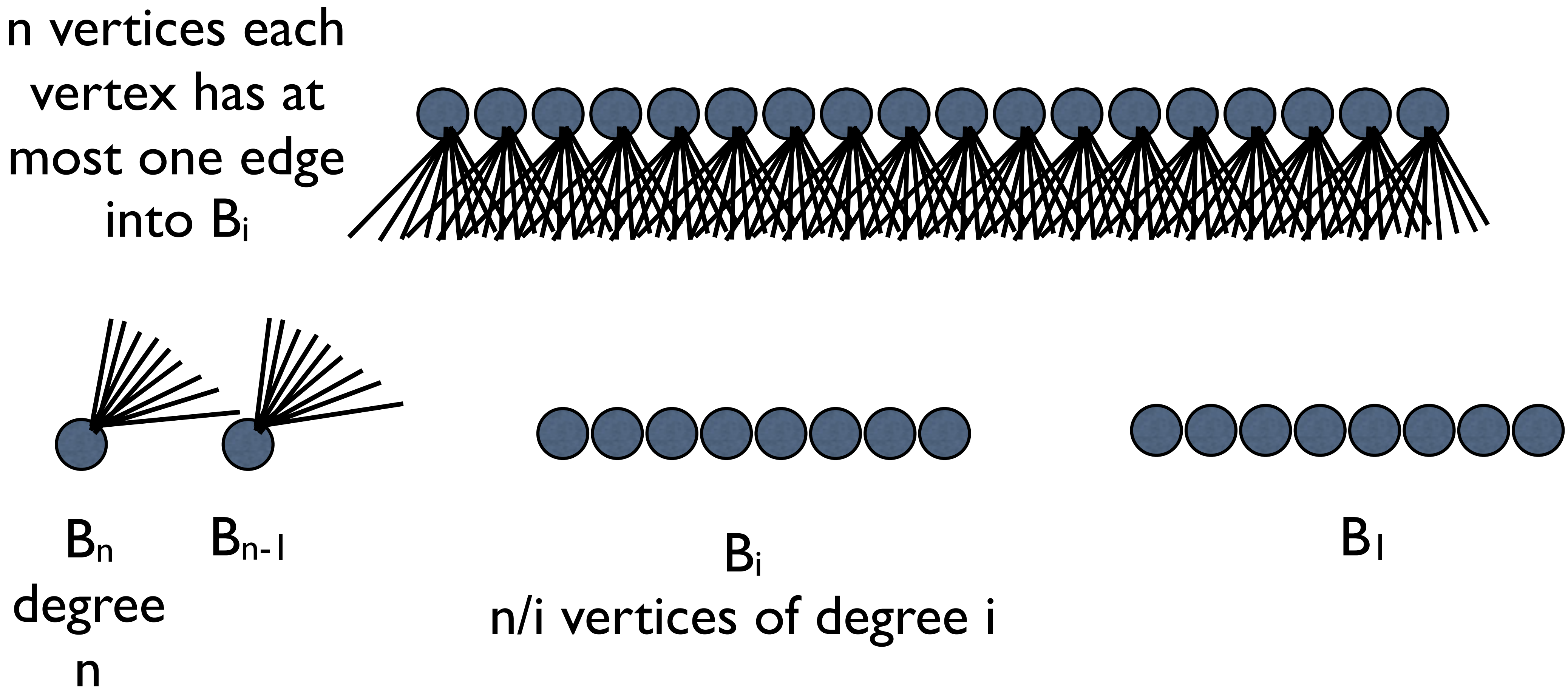
Algorithm: Pick vertex that covers most new edges



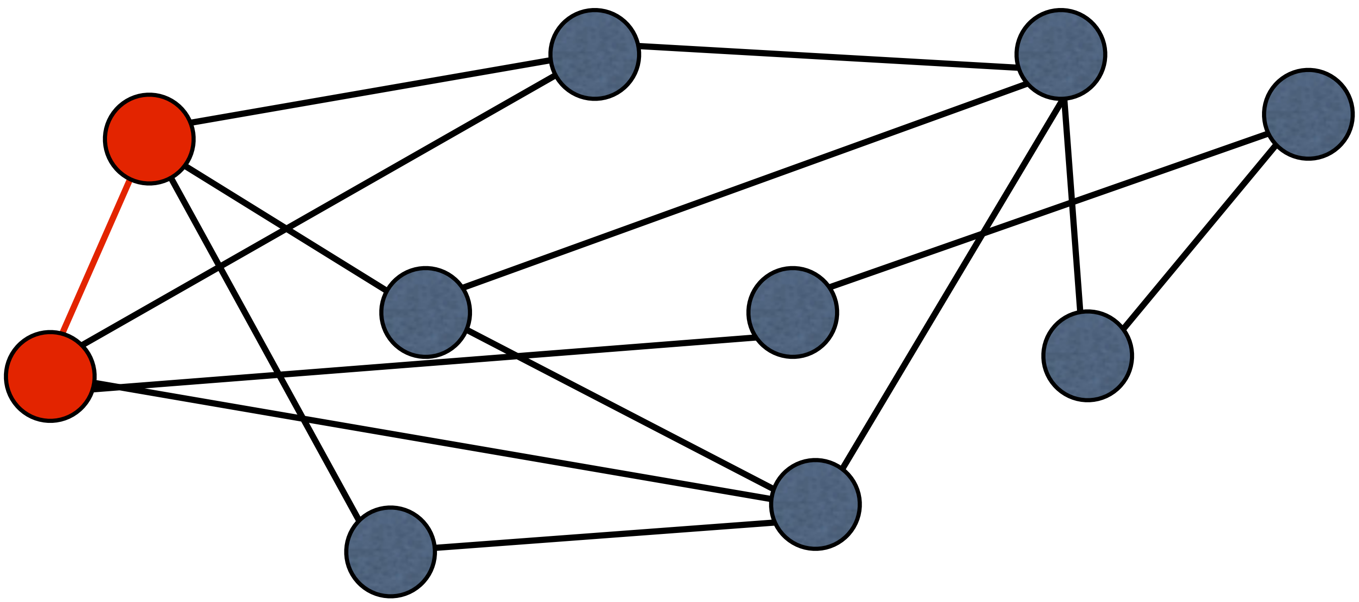
Each vertex on top row has one edge into each of the groups below.

Optimal Vertex Cover size 8

Greedy Rule: Pick vertex that covers the most edges
Could pick $B_1, \ldots, B_n$: $n\log(n)$ vertices

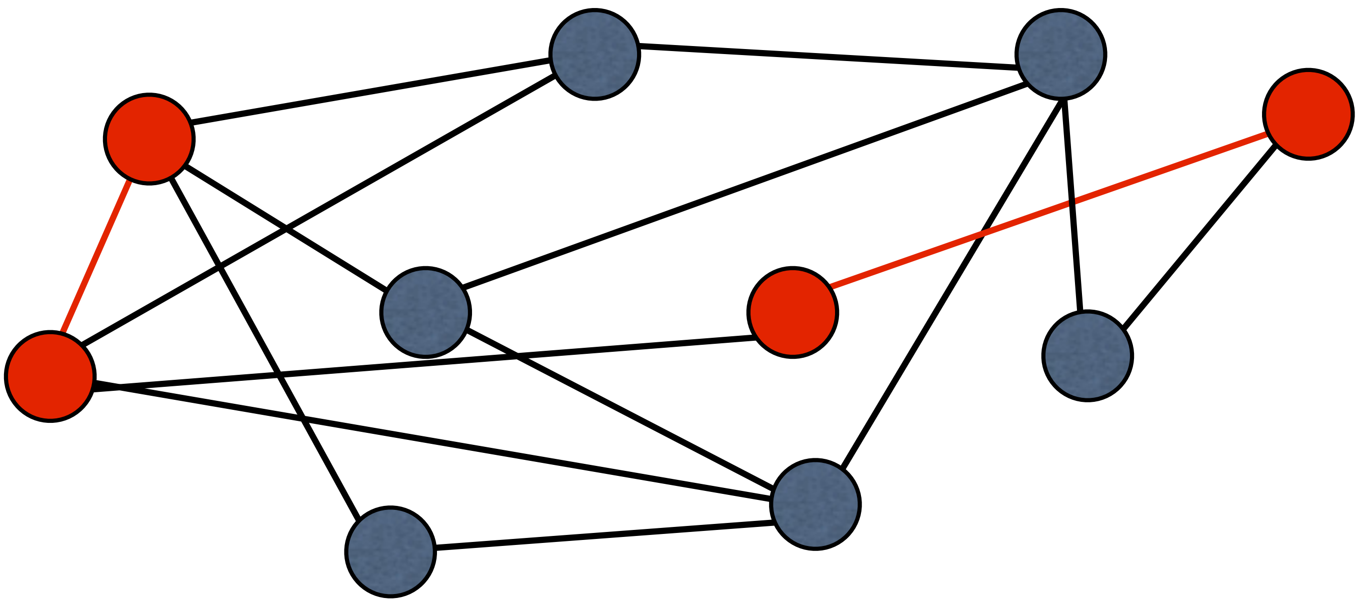


Greedy Rule:
Pick uncovered edge, add its end points



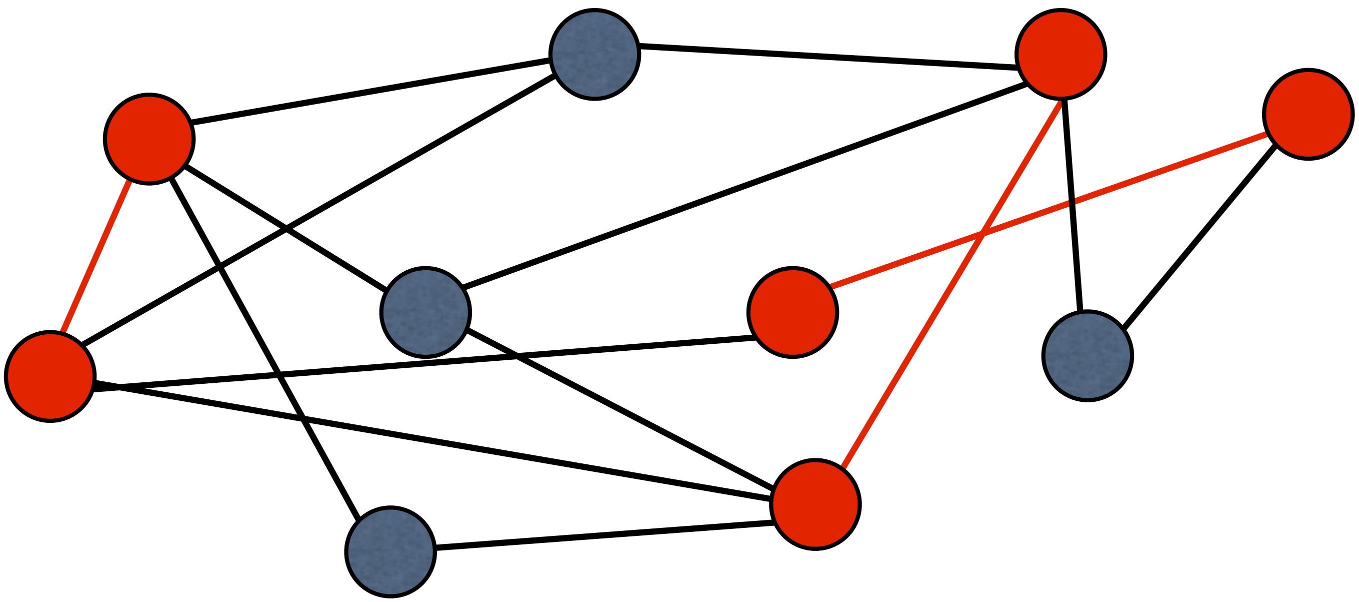
Find smallest set of
vertices touching
every edge

Greedy Rule:
Pick uncovered edge, add its end points



Find smallest set of
vertices touching
every edge

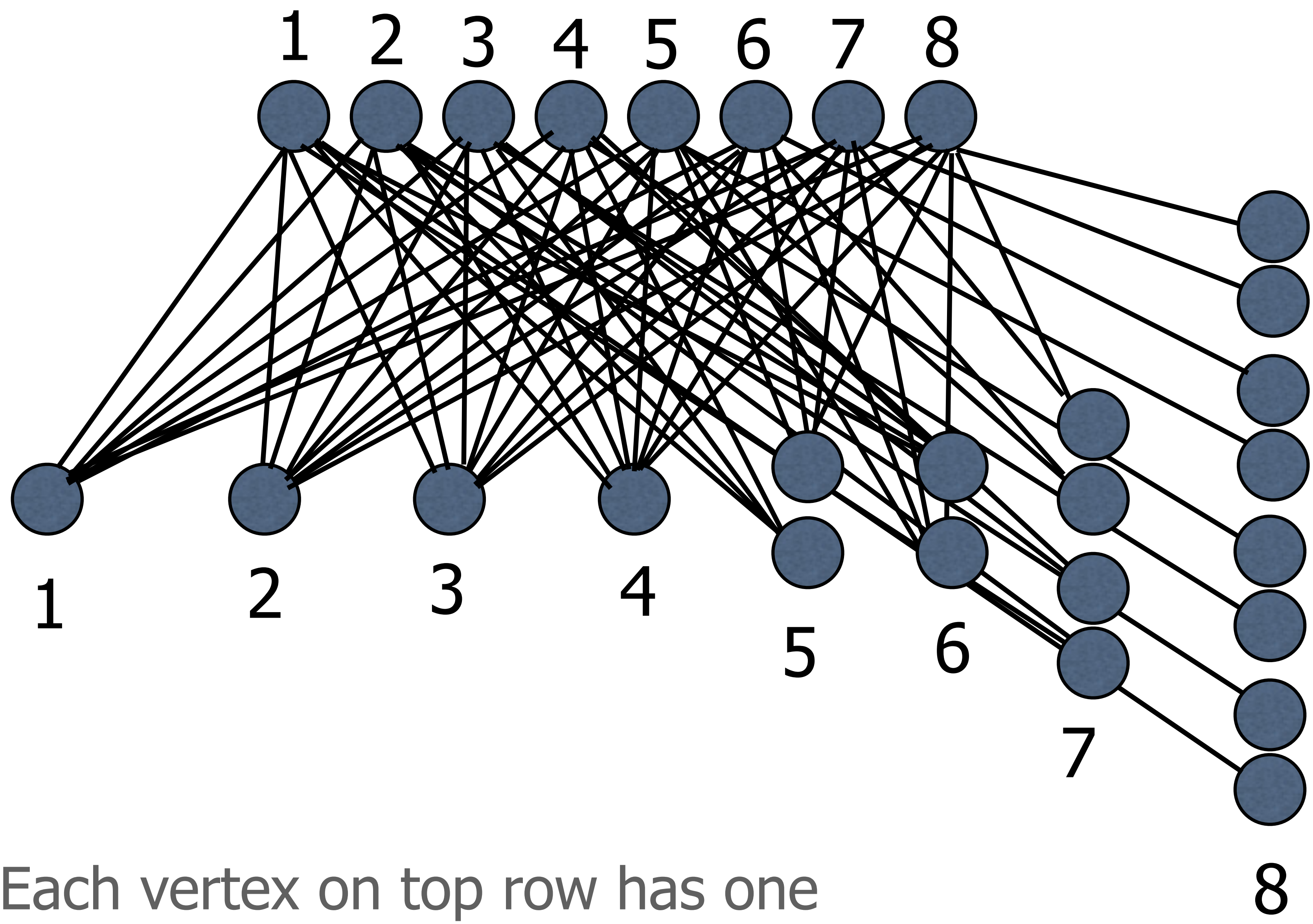
Greedy Rule:
Pick uncovered edge, add its end points



Find smallest set of vertices touching every edge

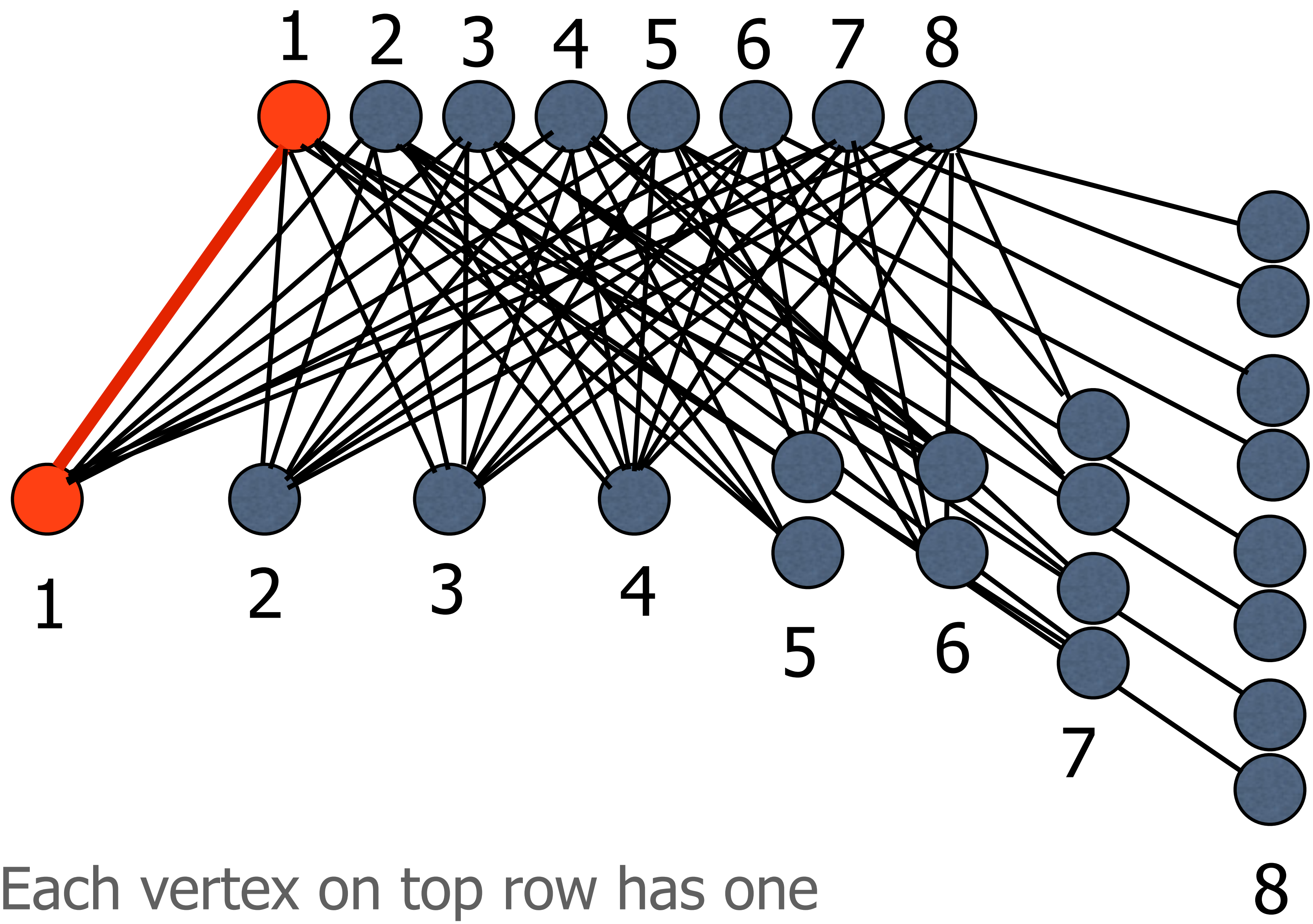
Vertex Cover size 6

Greedy Rule:
Pick uncovered edge, add its end points



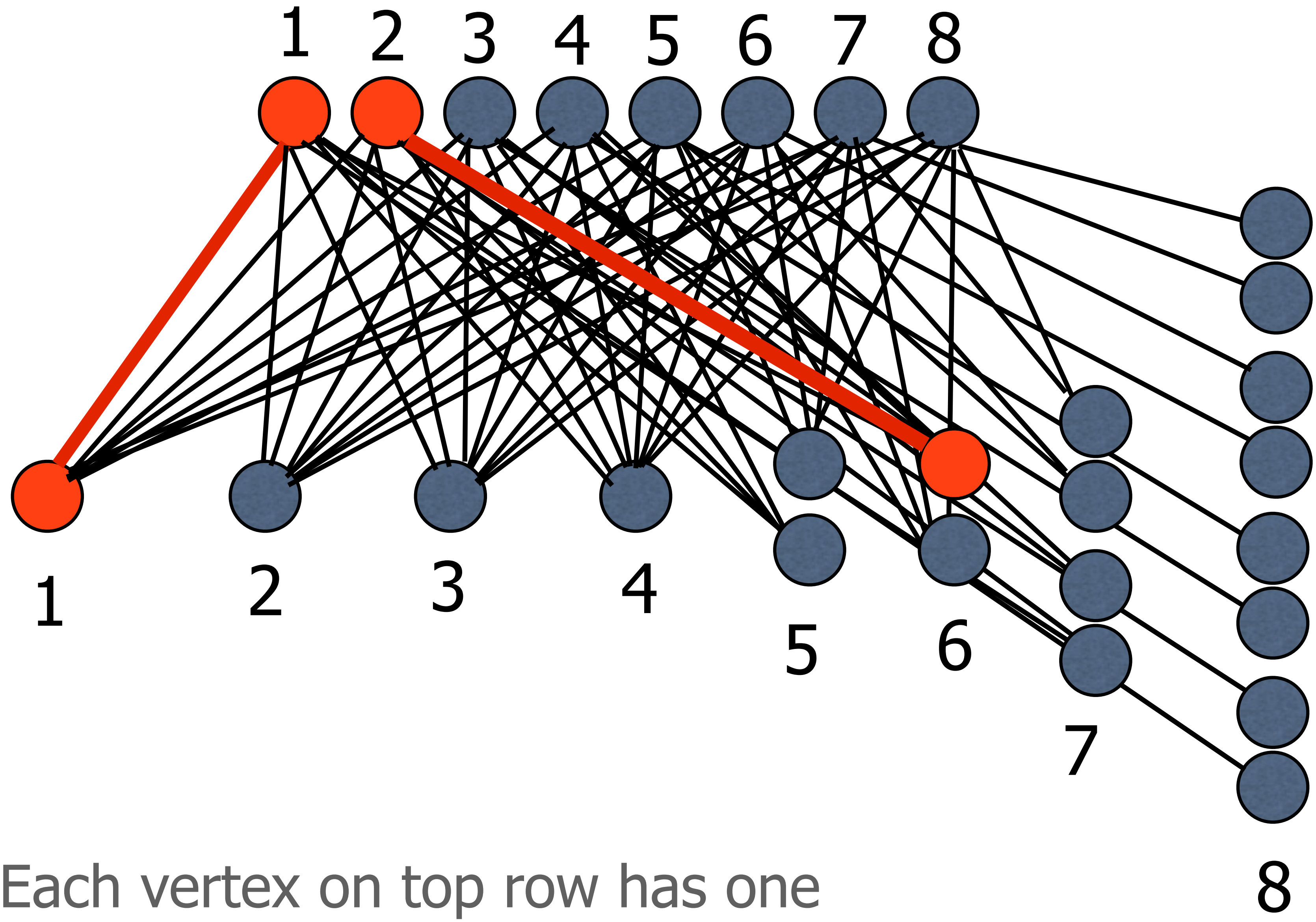
Each vertex on top row has one edge into each of the groups below.

Greedy Rule:
Pick uncovered edge, add its end points



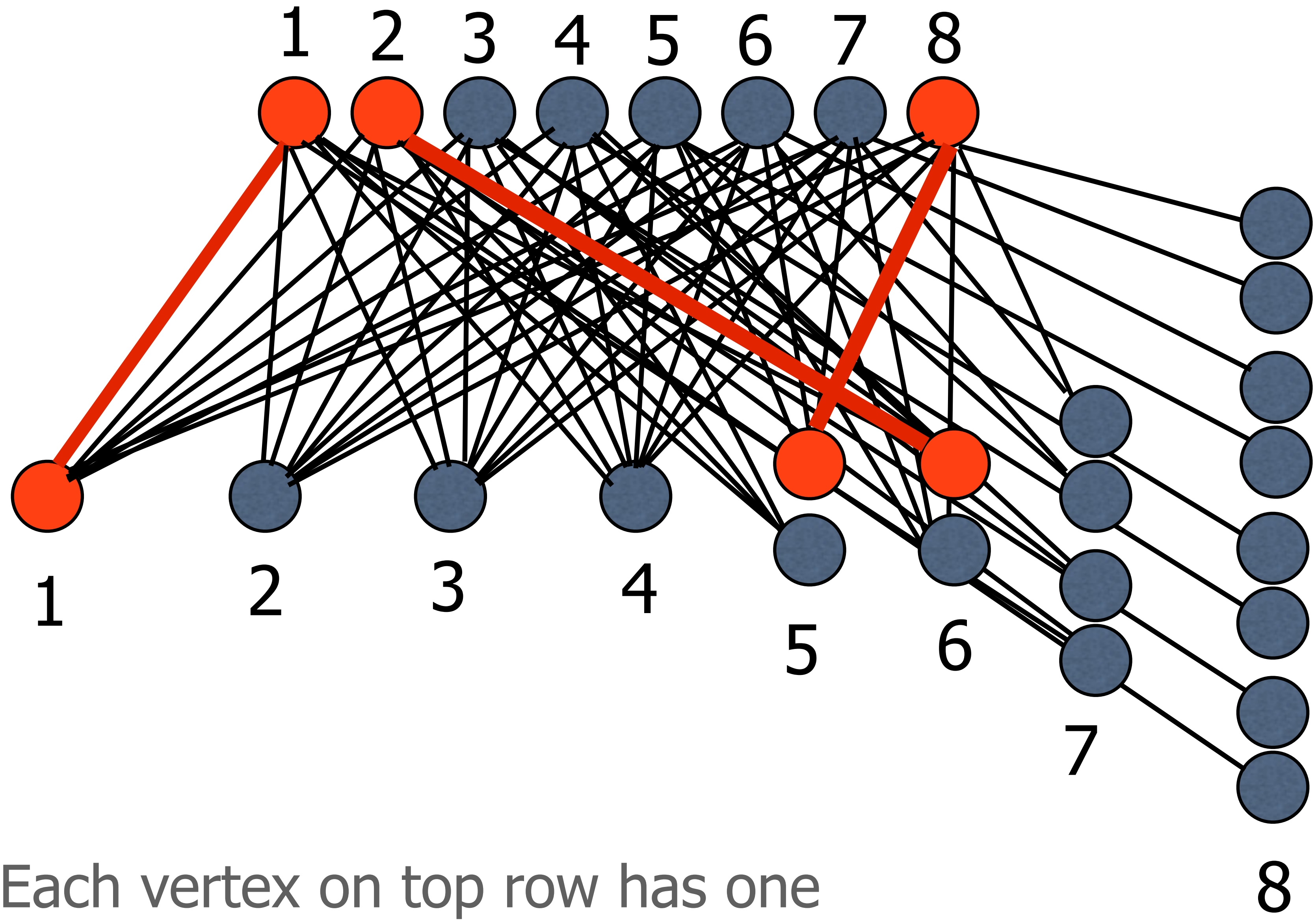
Each vertex on top row has one edge into each of the groups below.

Greedy Rule:
Pick uncovered edge, add its end points



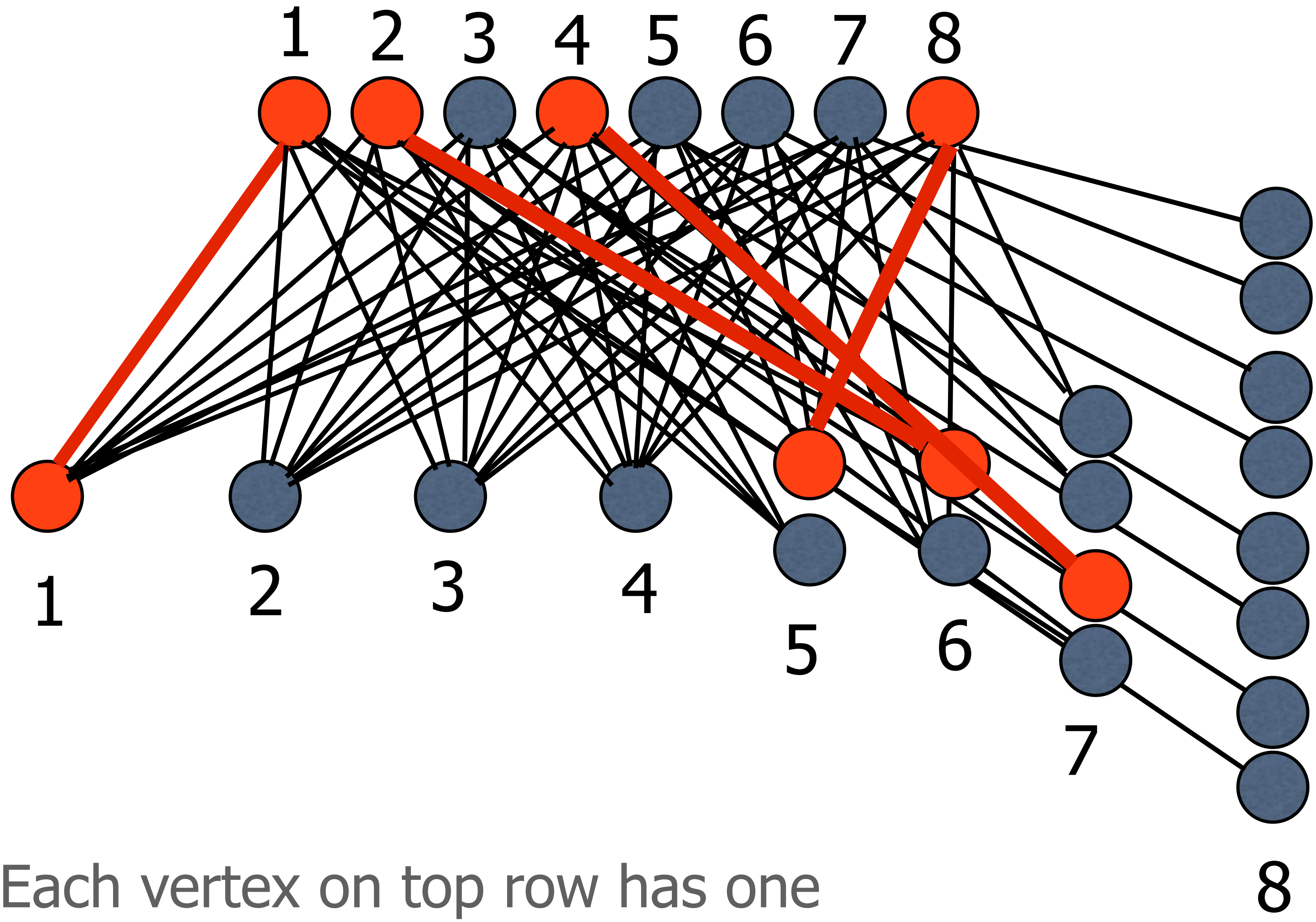
Each vertex on top row has one edge into each of the groups below.

Greedy Rule:
Pick uncovered edge, add its end points



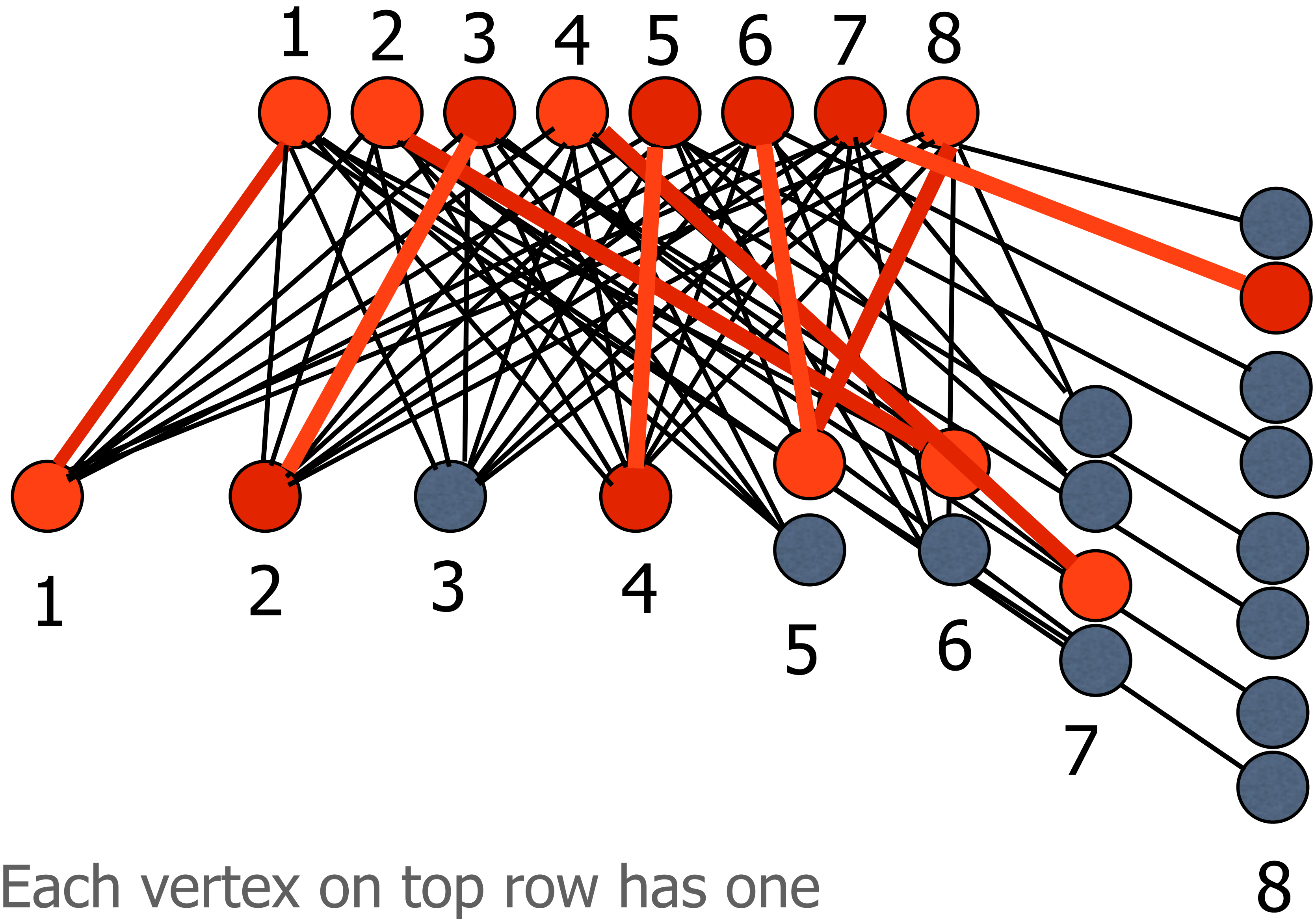
Each vertex on top row has one edge into each of the groups below.

Greedy Rule:
Pick uncovered edge, add its end points



Each vertex on top row has one edge into each of the groups below.

Greedy Rule:
Pick uncovered edge, add its end points

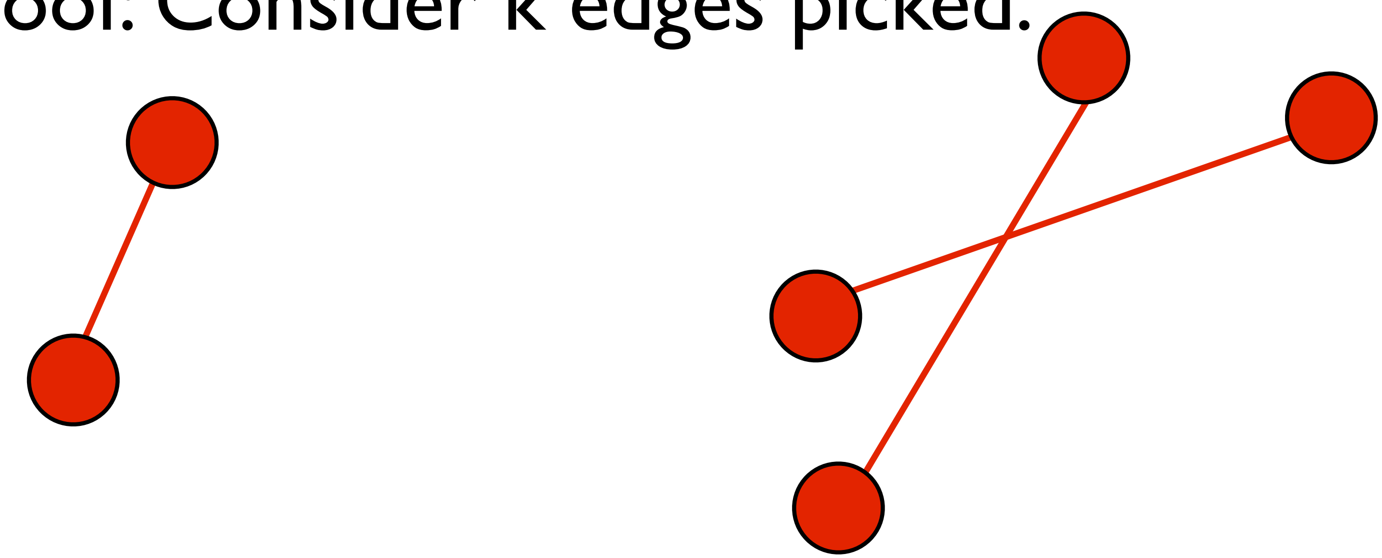


Each vertex on top row has one edge into each of the groups below.

Vertex Cover size 16

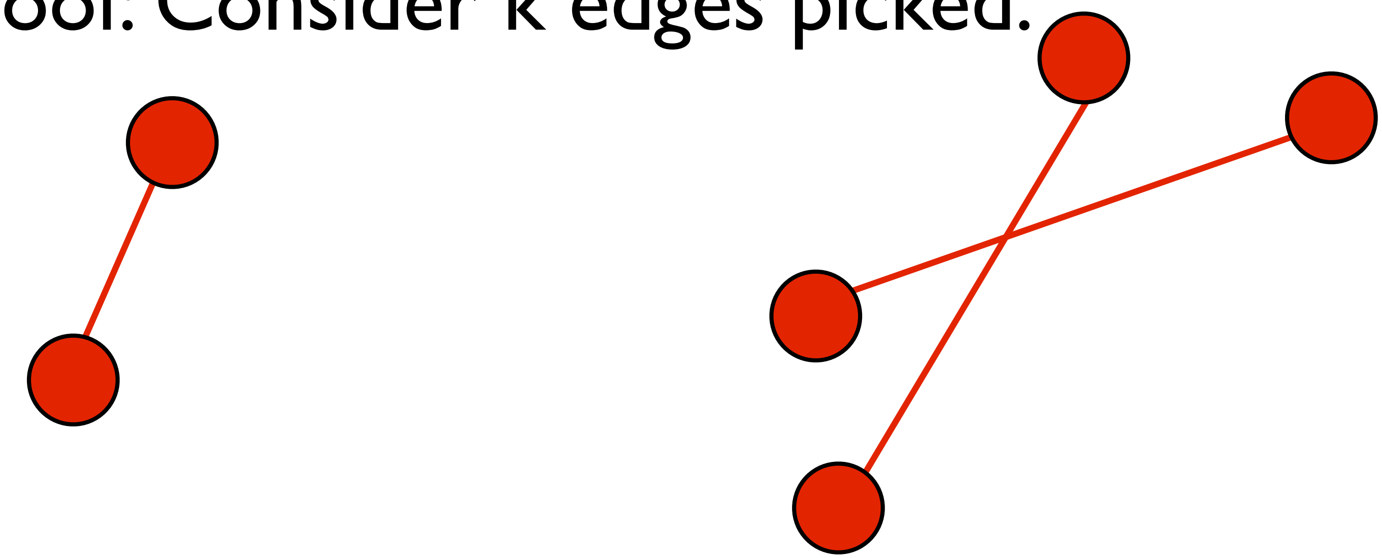
Theorem: Size of greedy vertex cover is at most twice as big as size of optimal cover

Proof: Consider k edges picked.



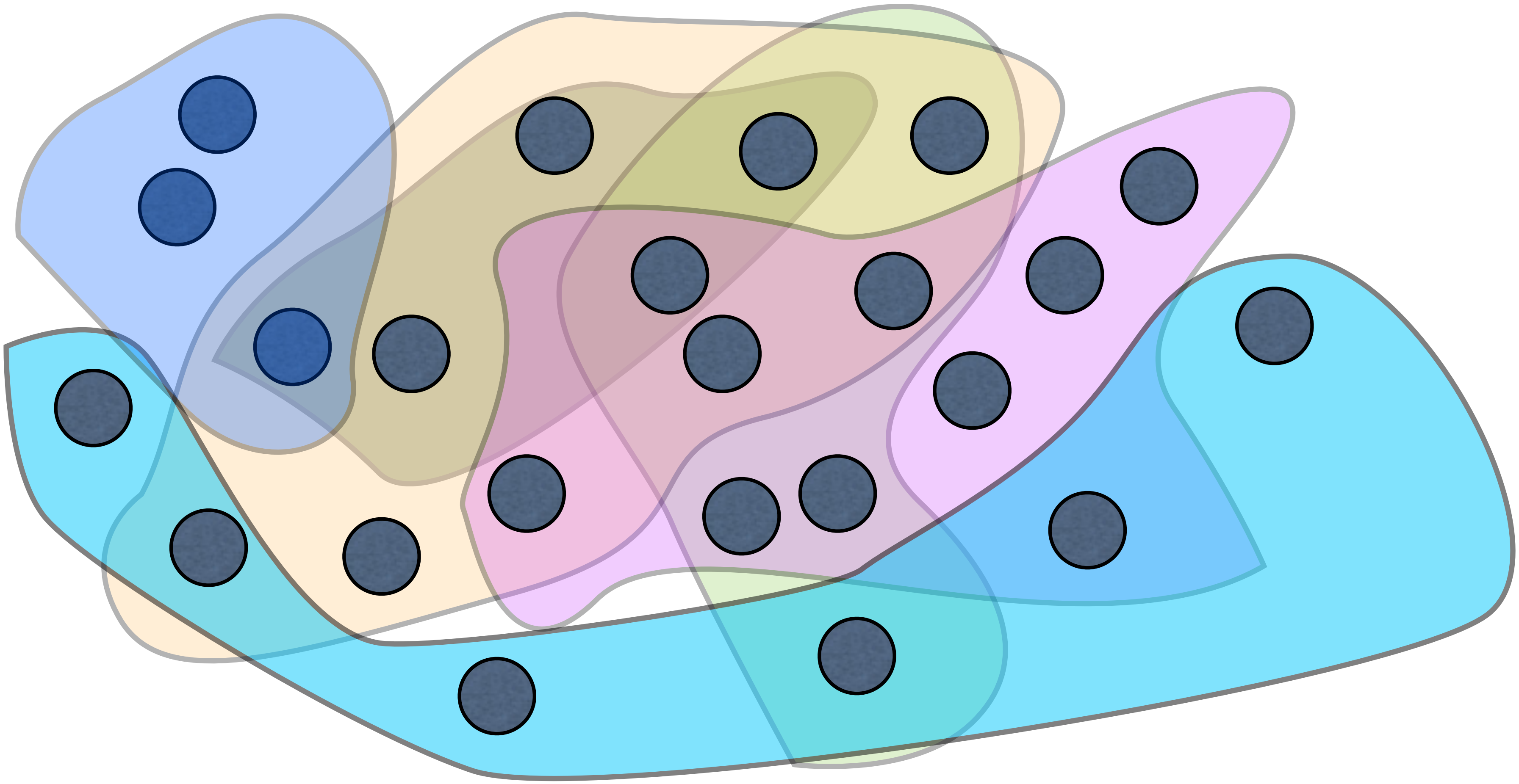
Theorem: Size of greedy vertex cover is at most twice as big as size of optimal cover

Proof: Consider k edges picked.



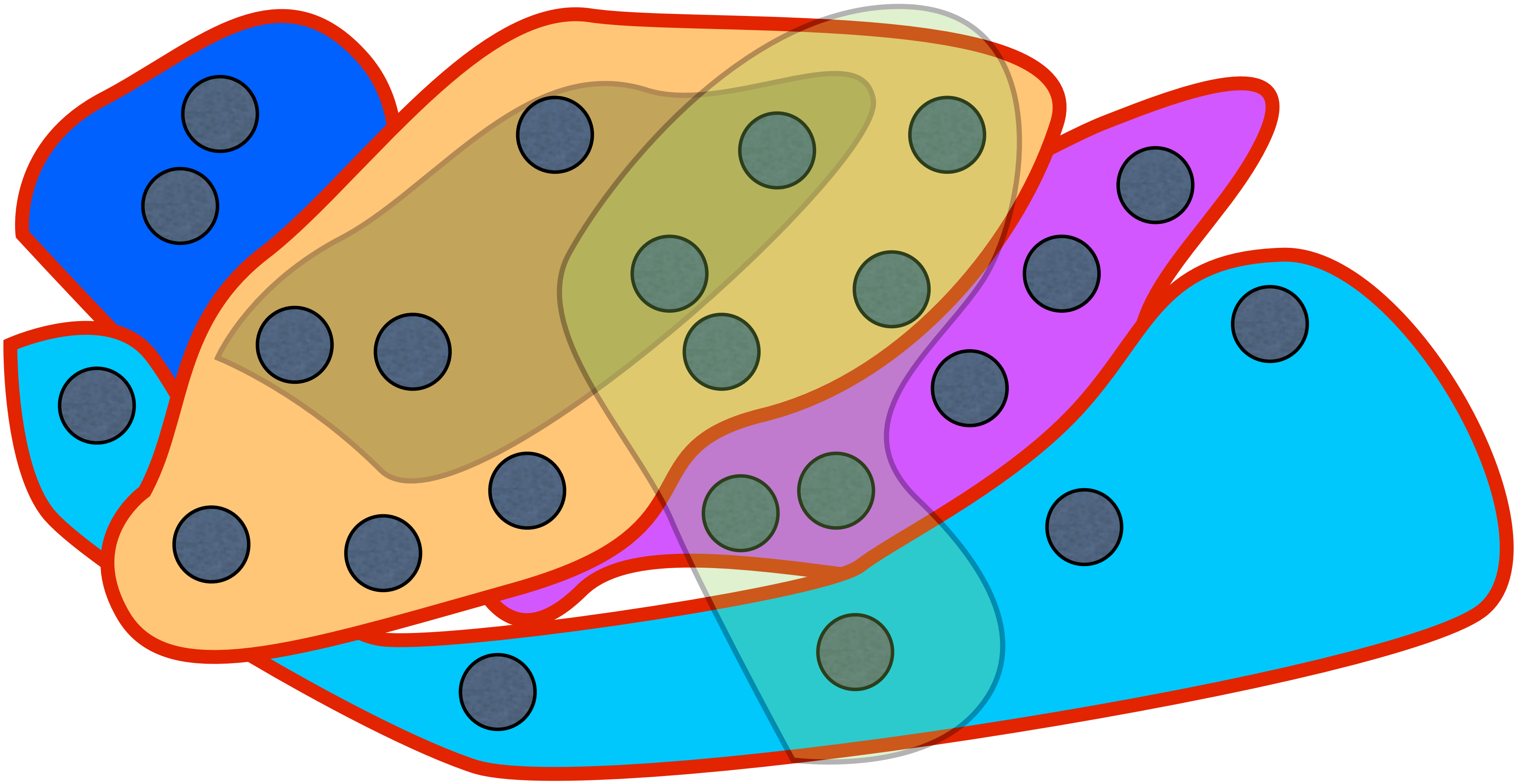
Edges do not touch: every cover must pick one vertex per edge! Thus every vertex cover has k vertices.

Set Cover



Find smallest
collection of sets
containing every point

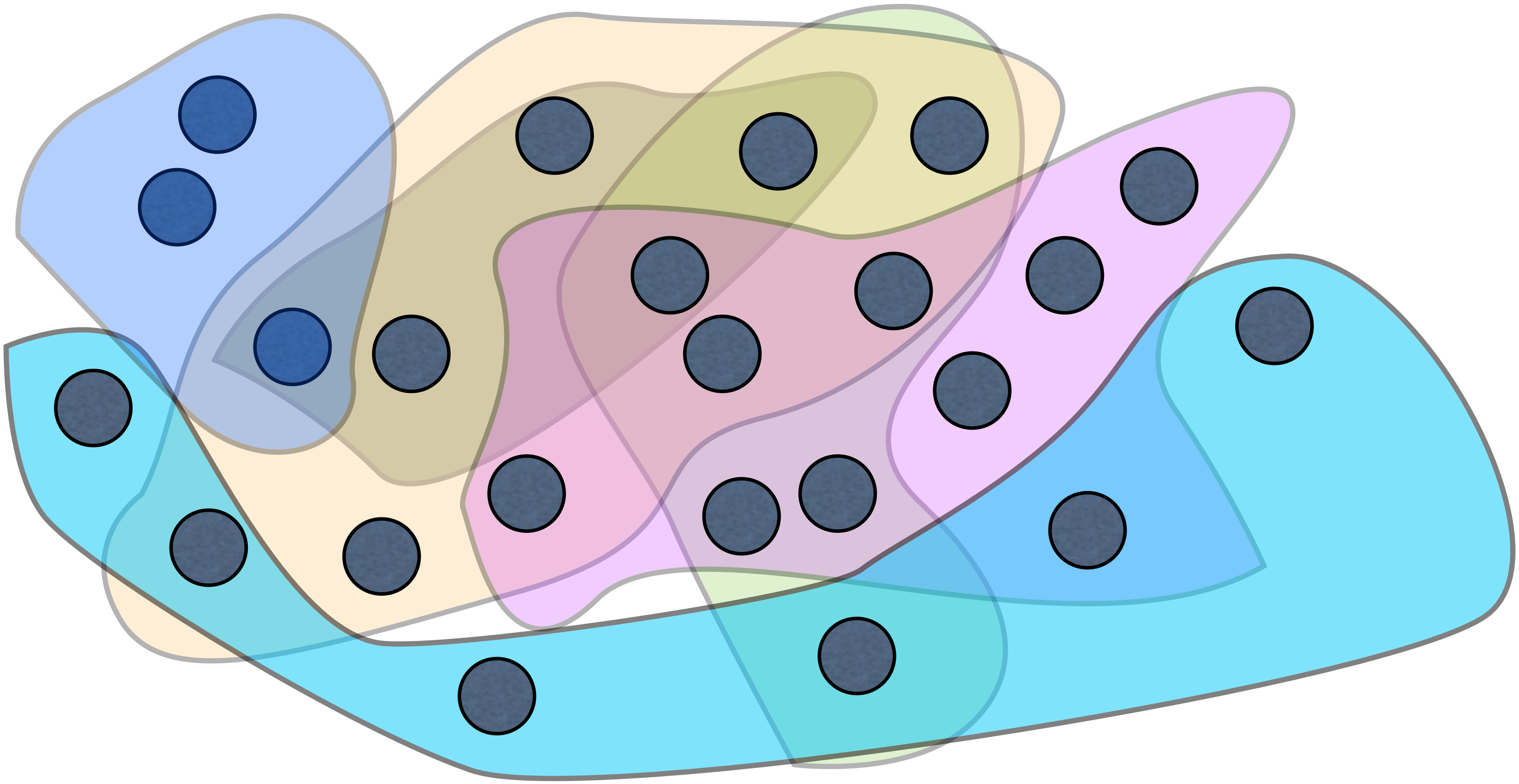
Set Cover



Find smallest collection of sets containing every point

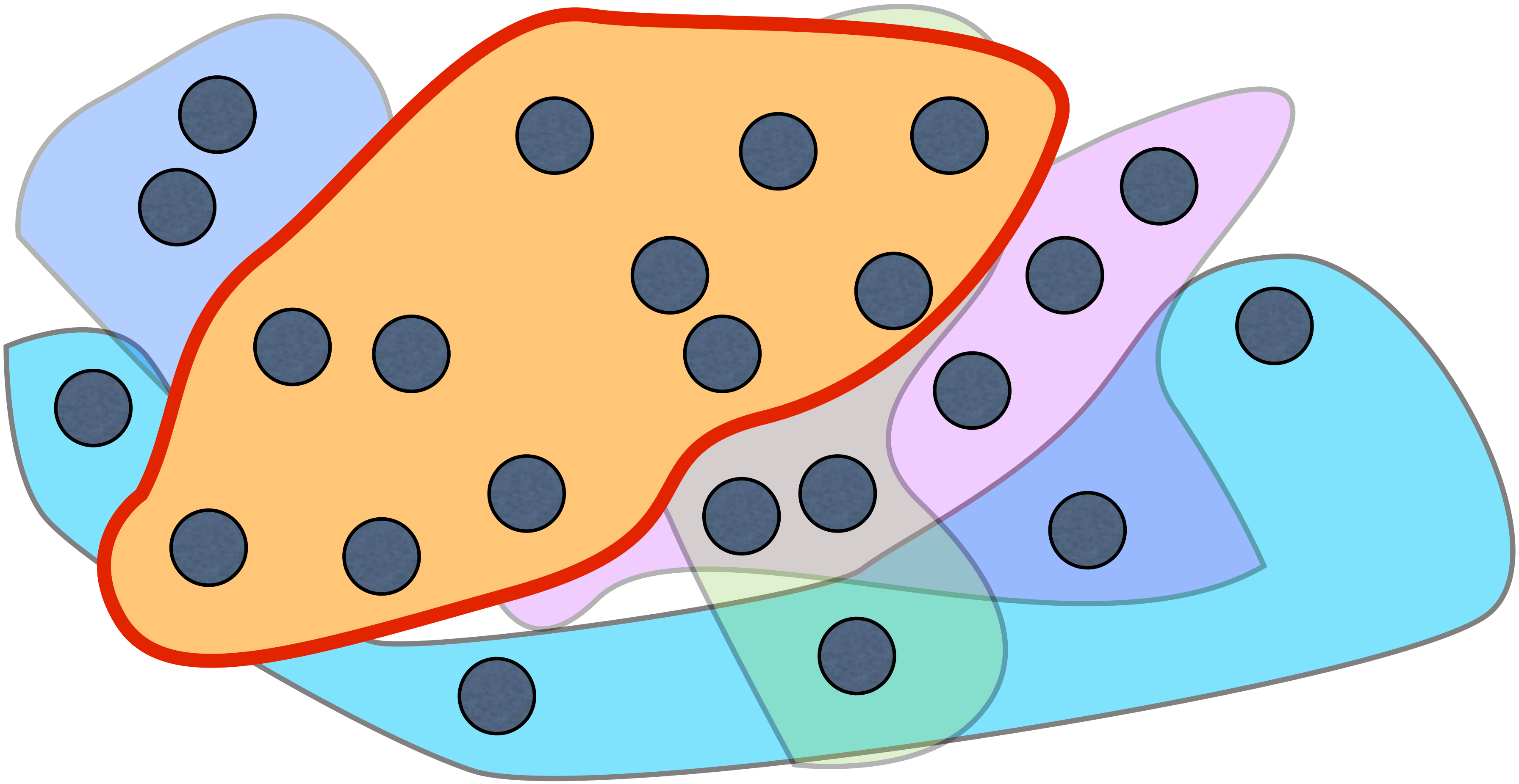
Set Cover size 4

Greedy Set Cover: Pick the set that maximizes # new elements covered



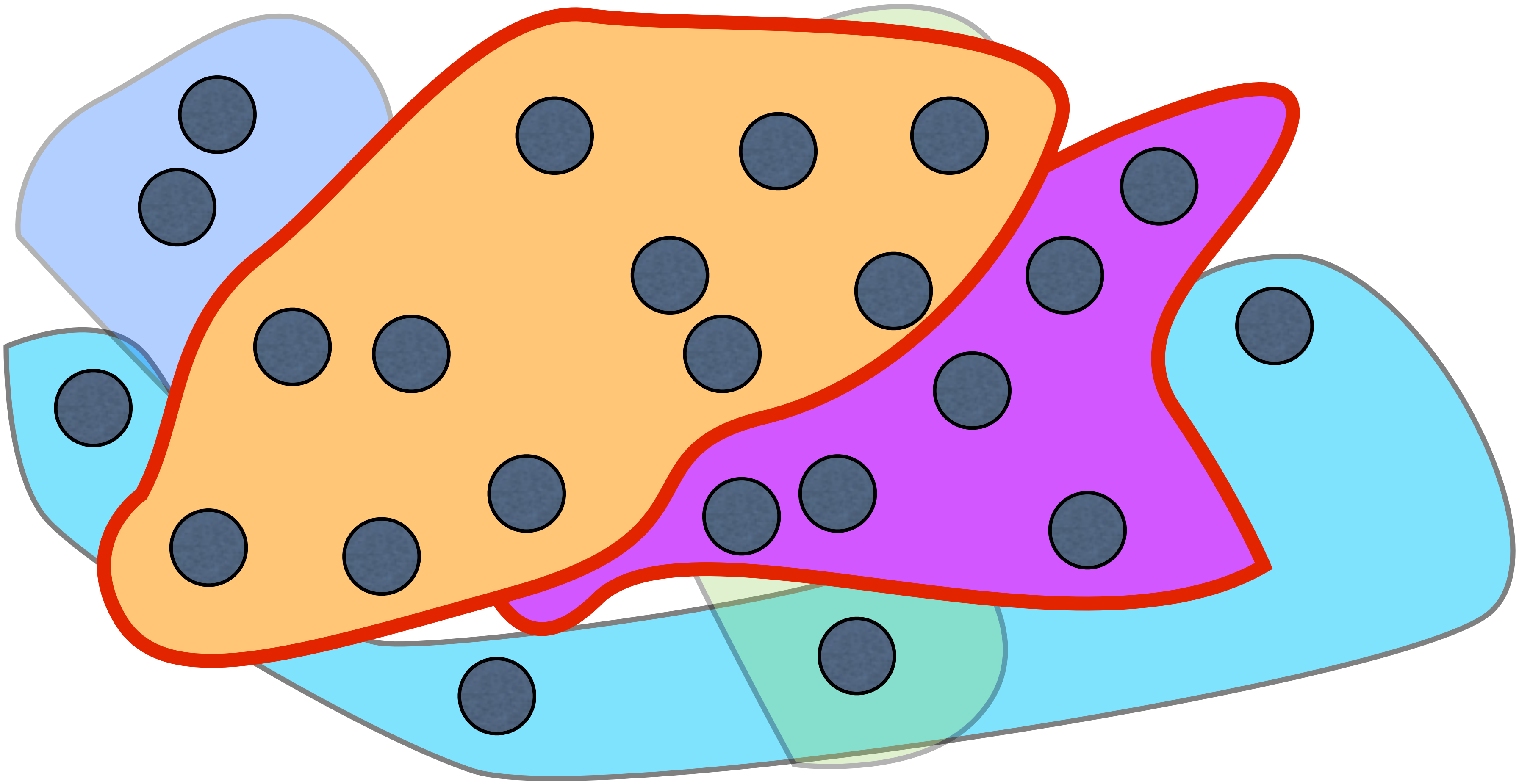
Find smallest collection of sets containing every point

Greedy Set Cover: Pick the set that maximizes # new elements covered



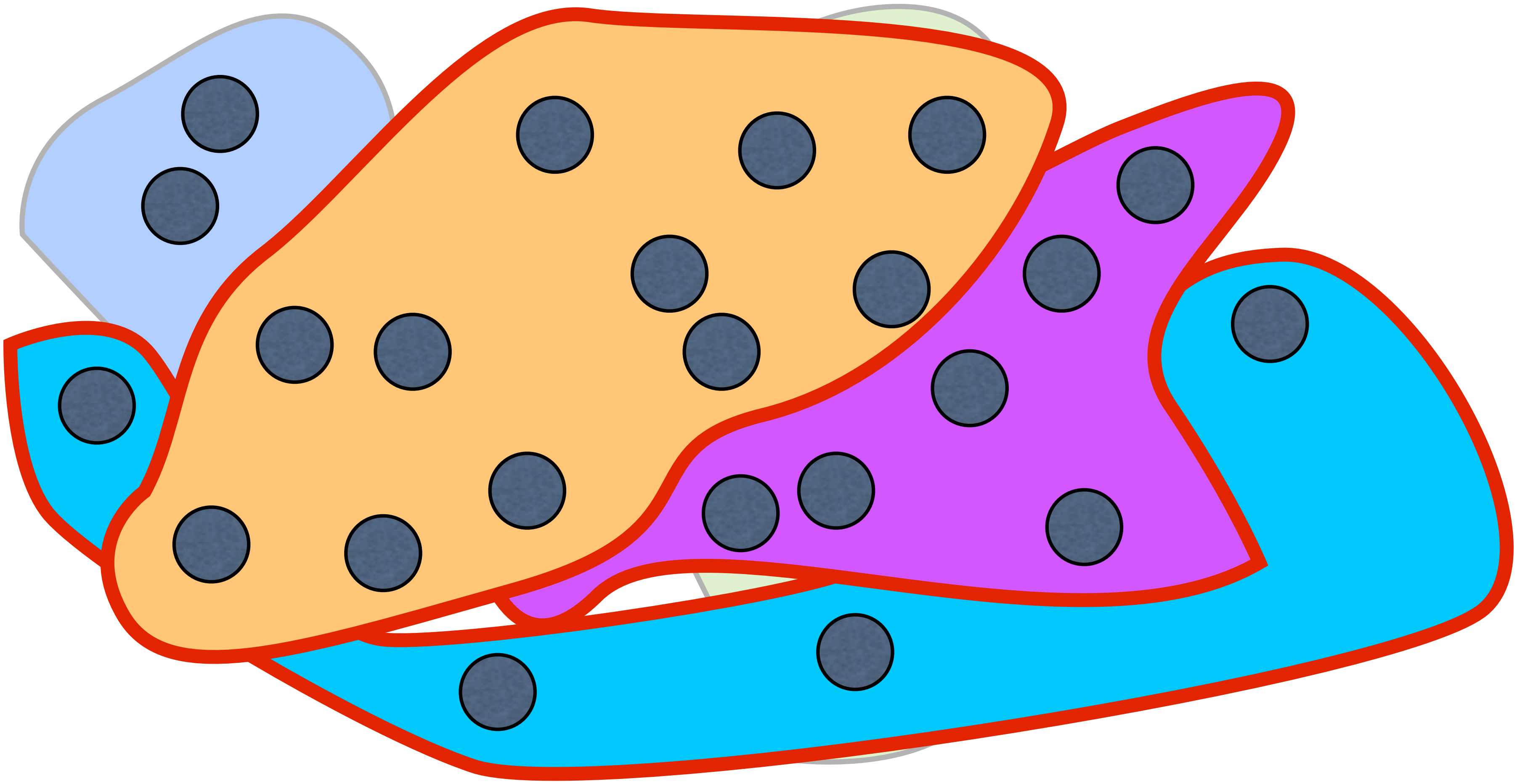
Find smallest collection of sets containing every point

Greedy Set Cover: Pick the set that maximizes # new elements covered



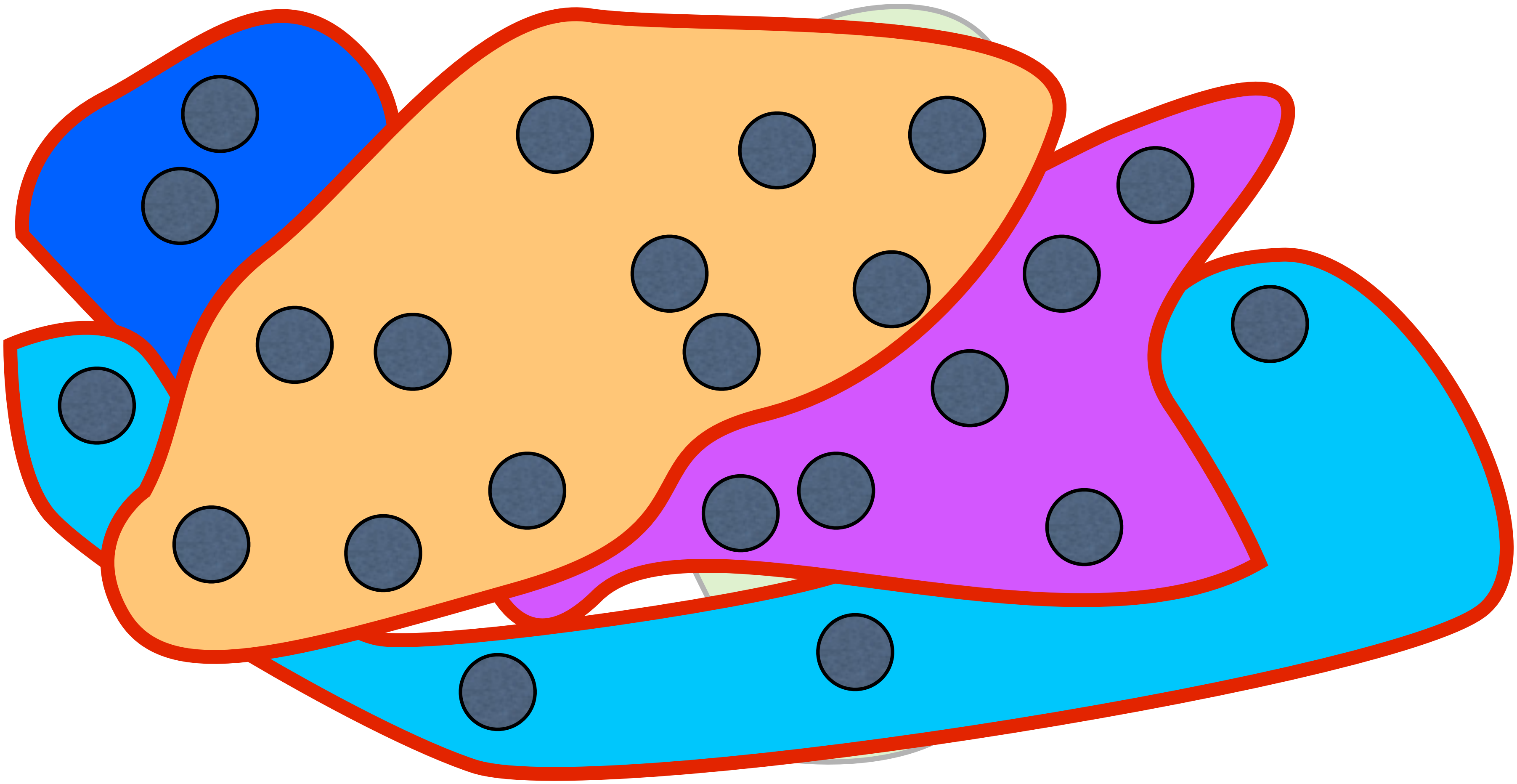
Find smallest collection of sets containing every point

Greedy Set Cover: Pick the set that maximizes # new elements covered



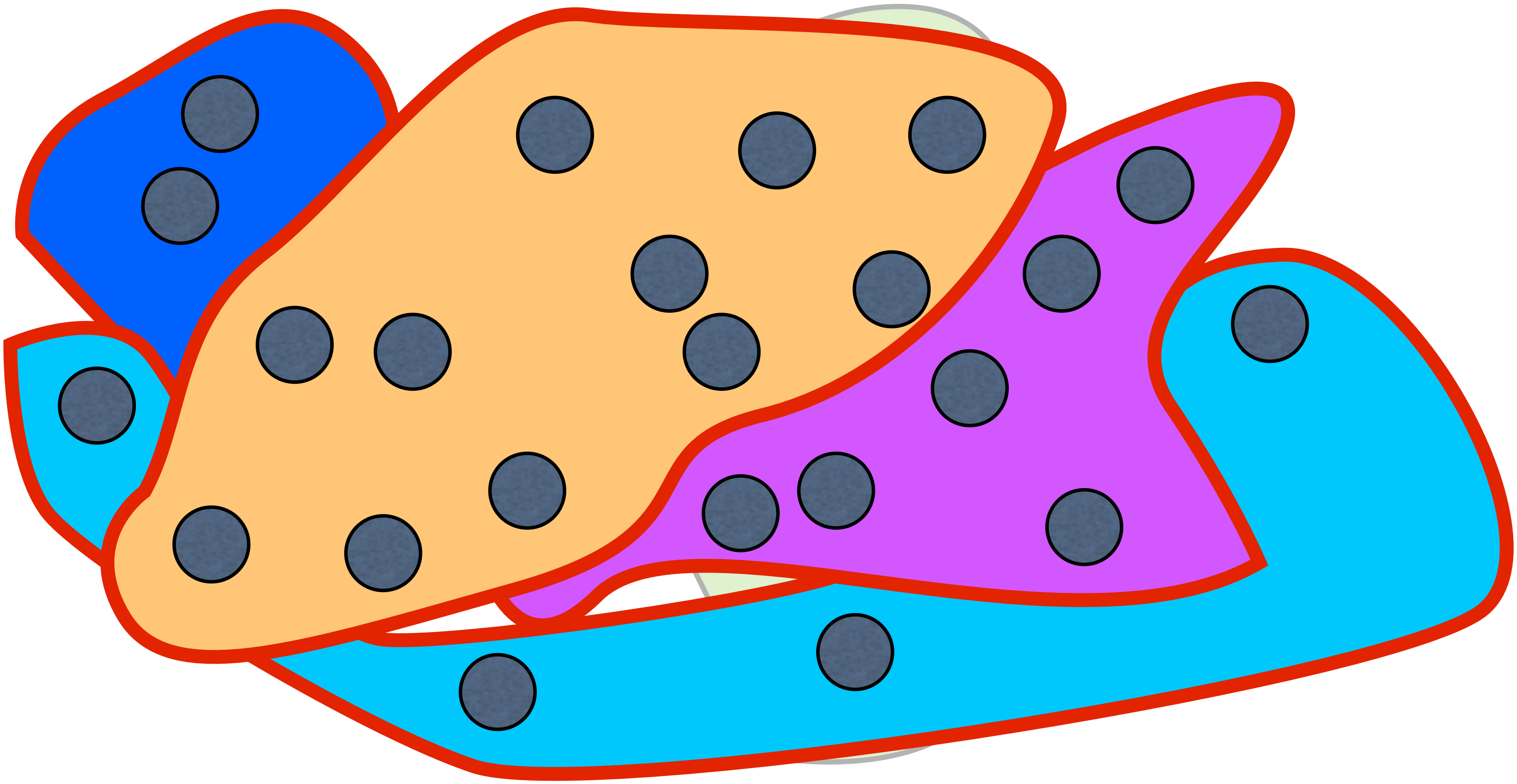
Find smallest collection of sets containing every point

Greedy Set Cover: Pick the set that maximizes # new elements covered



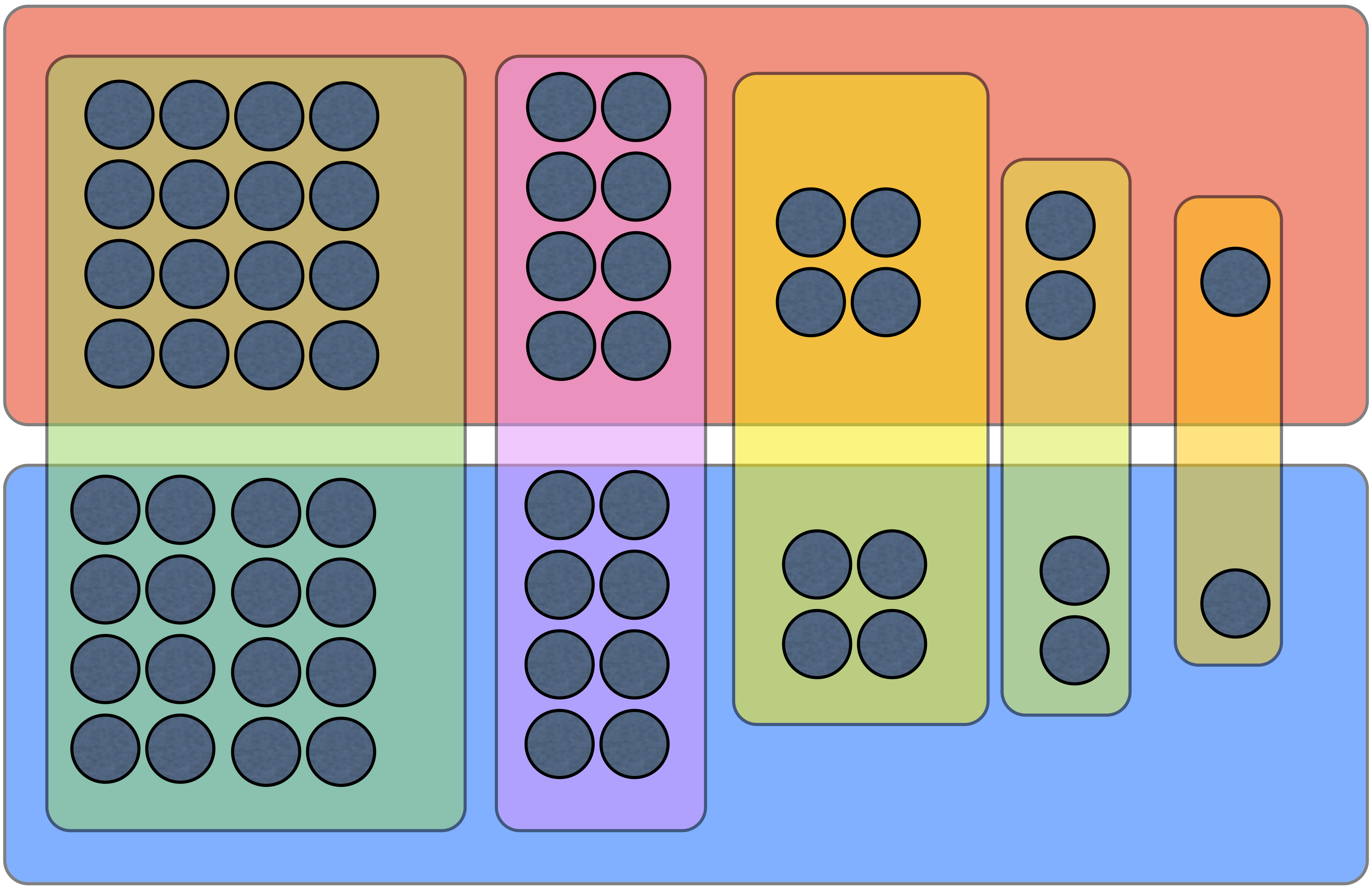
Find smallest collection of sets containing every point

Greedy Set Cover: Pick the set that maximizes # new elements covered

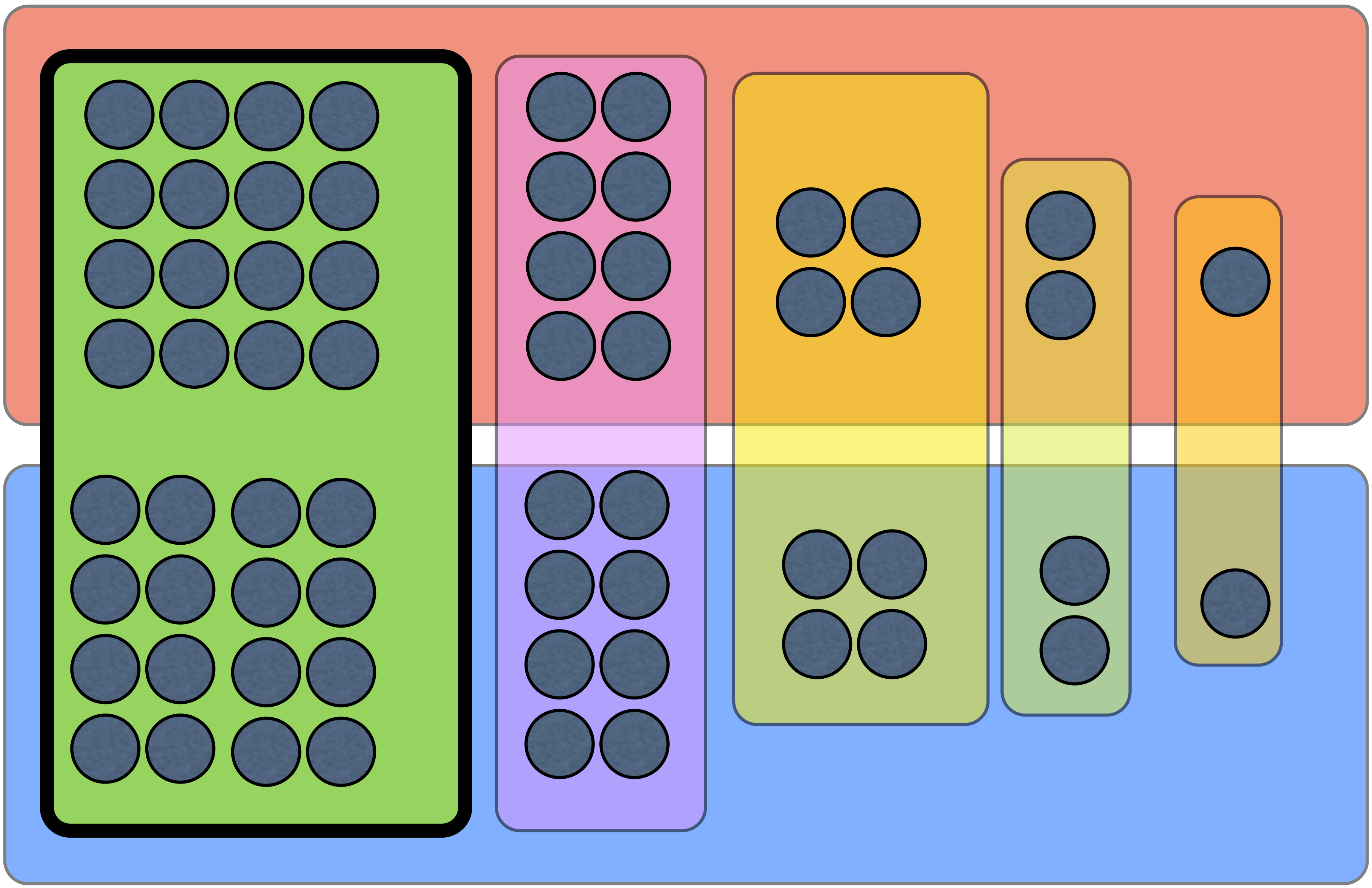


Theorem: Greedy finds best cover upto a factor of ln(n).

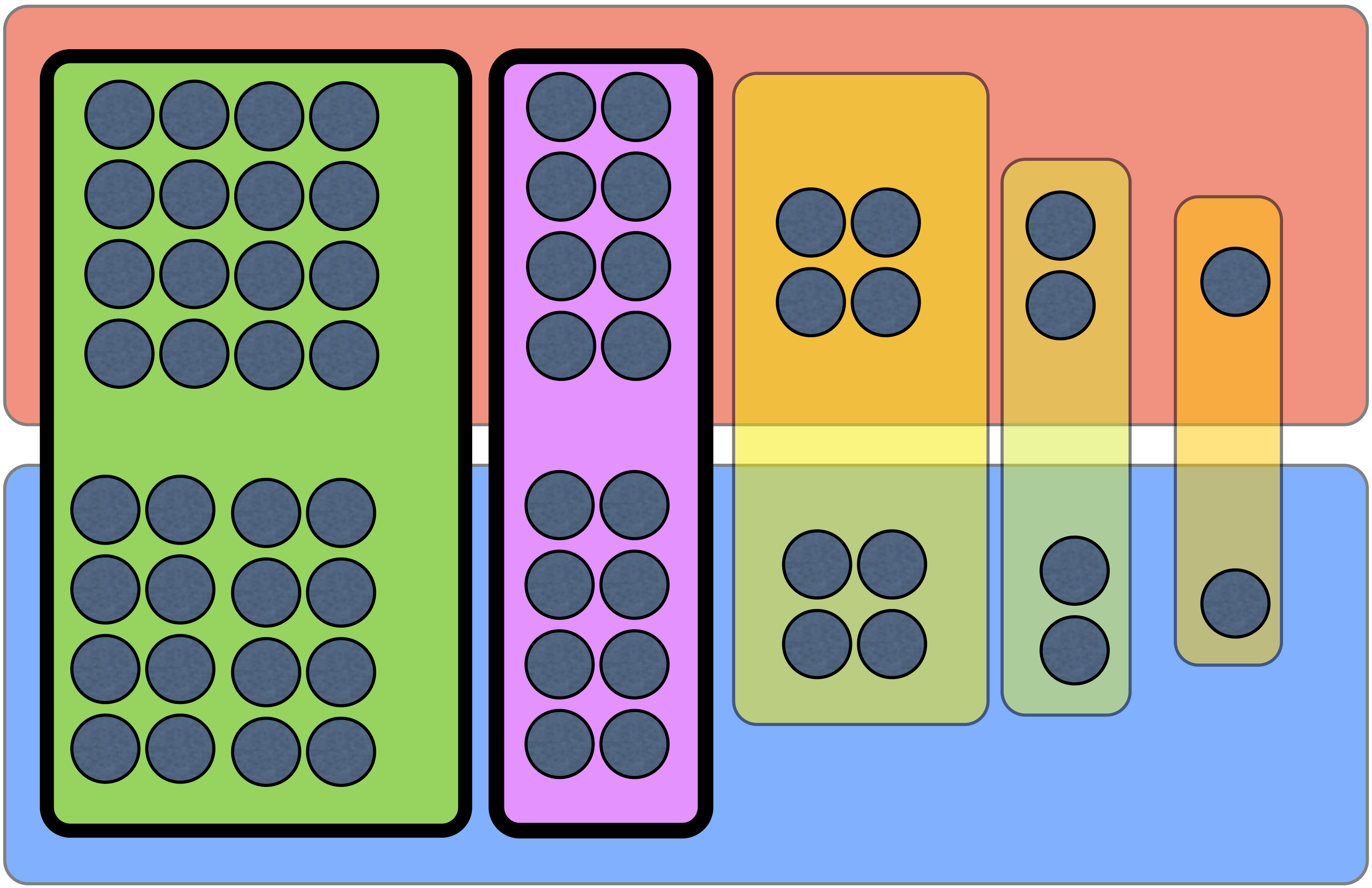
Greedy Set Cover: Pick the set that maximizes # new elements covered



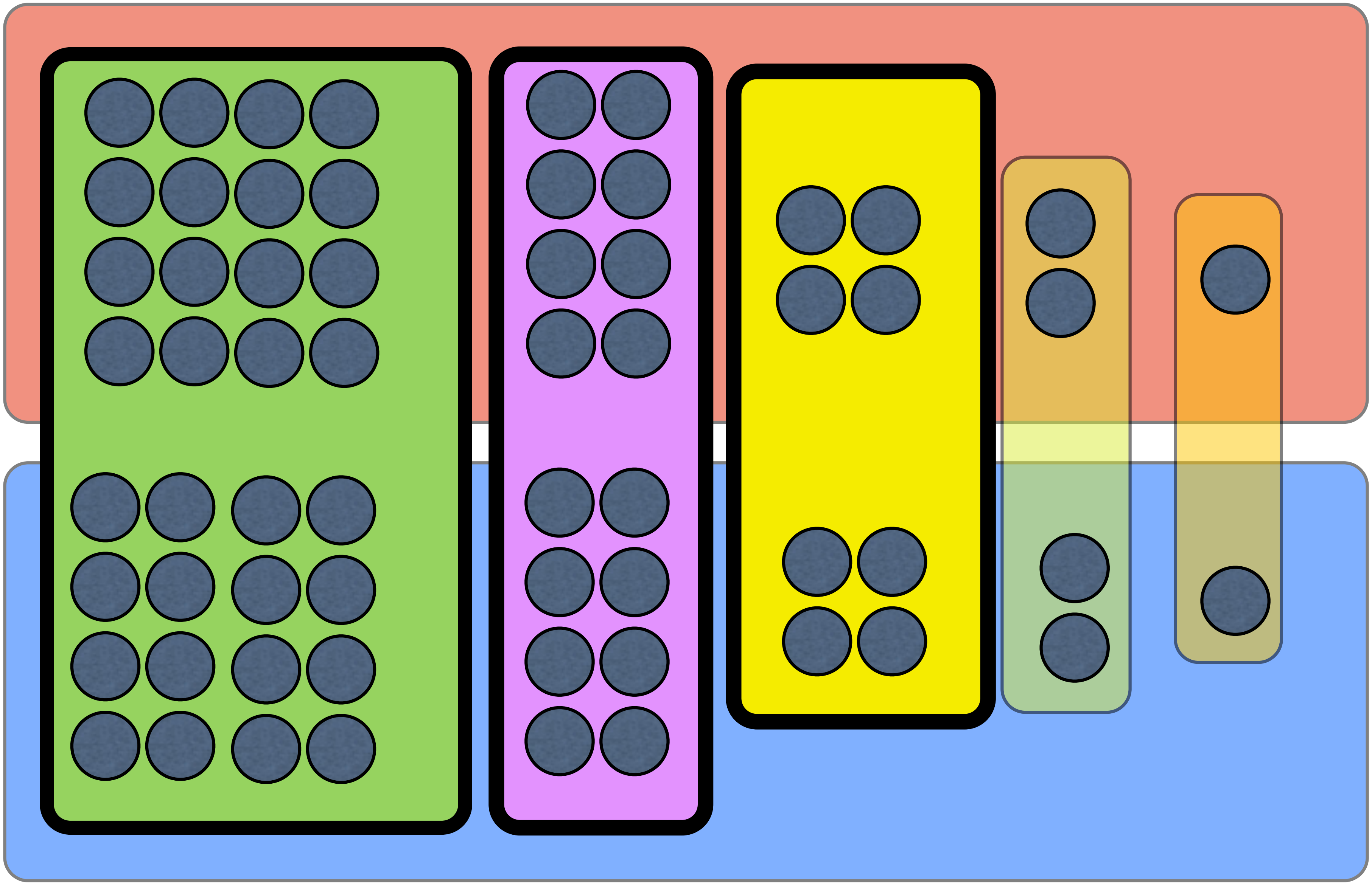
Greedy Set Cover: Pick the set that maximizes # new elements covered



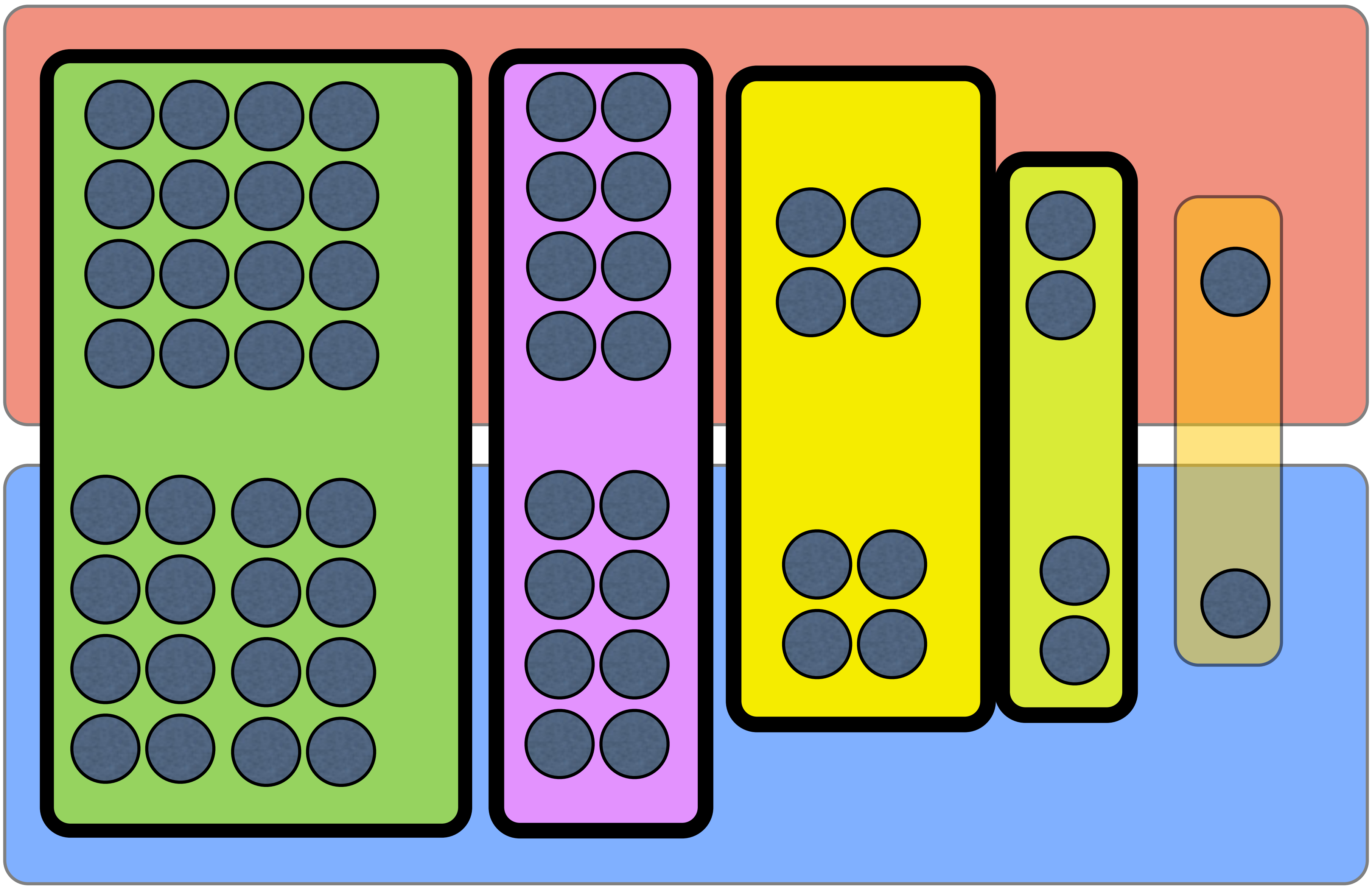
Greedy Set Cover: Pick the set that maximizes # new elements covered



Greedy Set Cover: Pick the set that maximizes # new elements covered

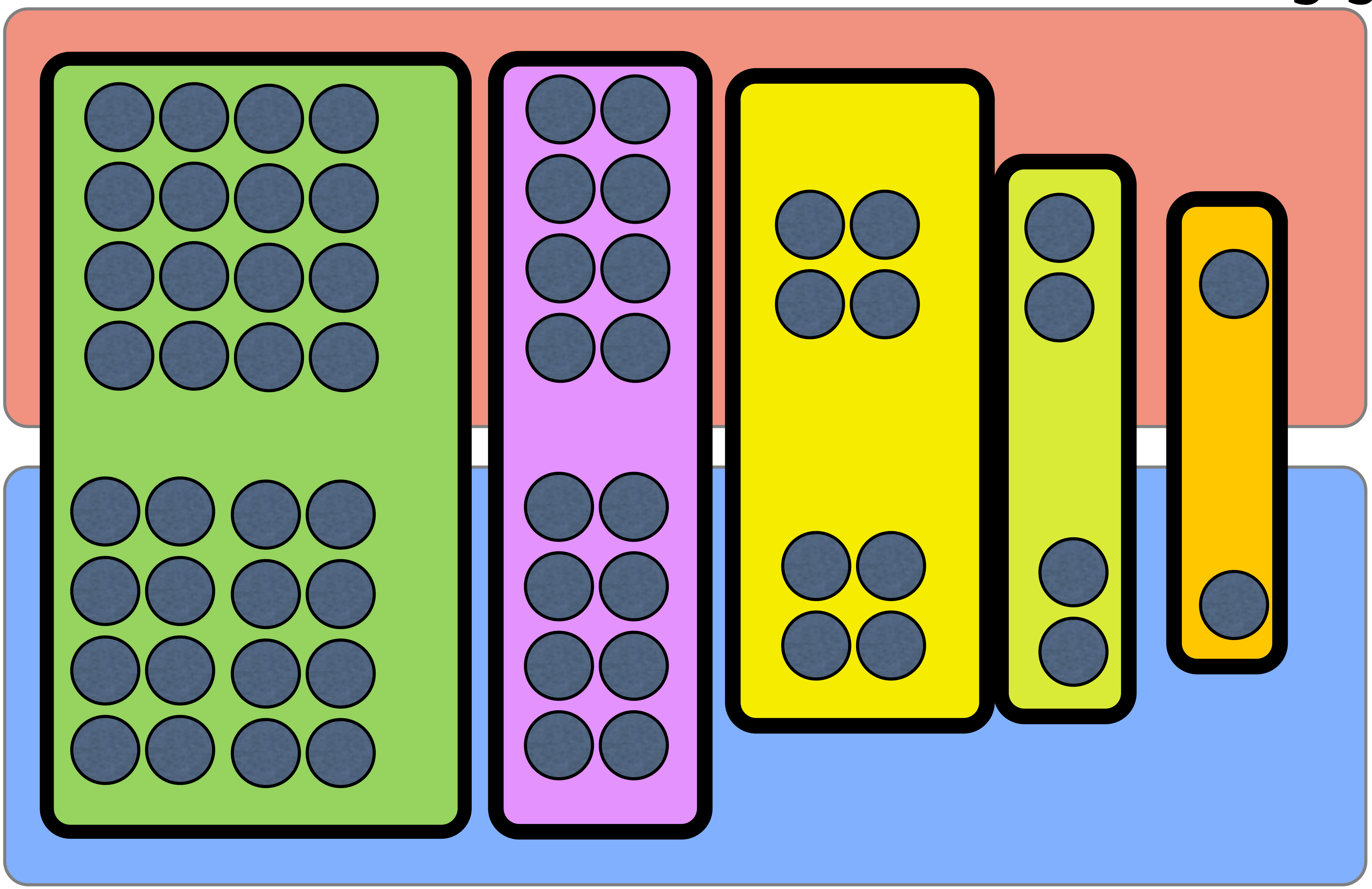


Greedy Set Cover: Pick the set that maximizes # new elements covered



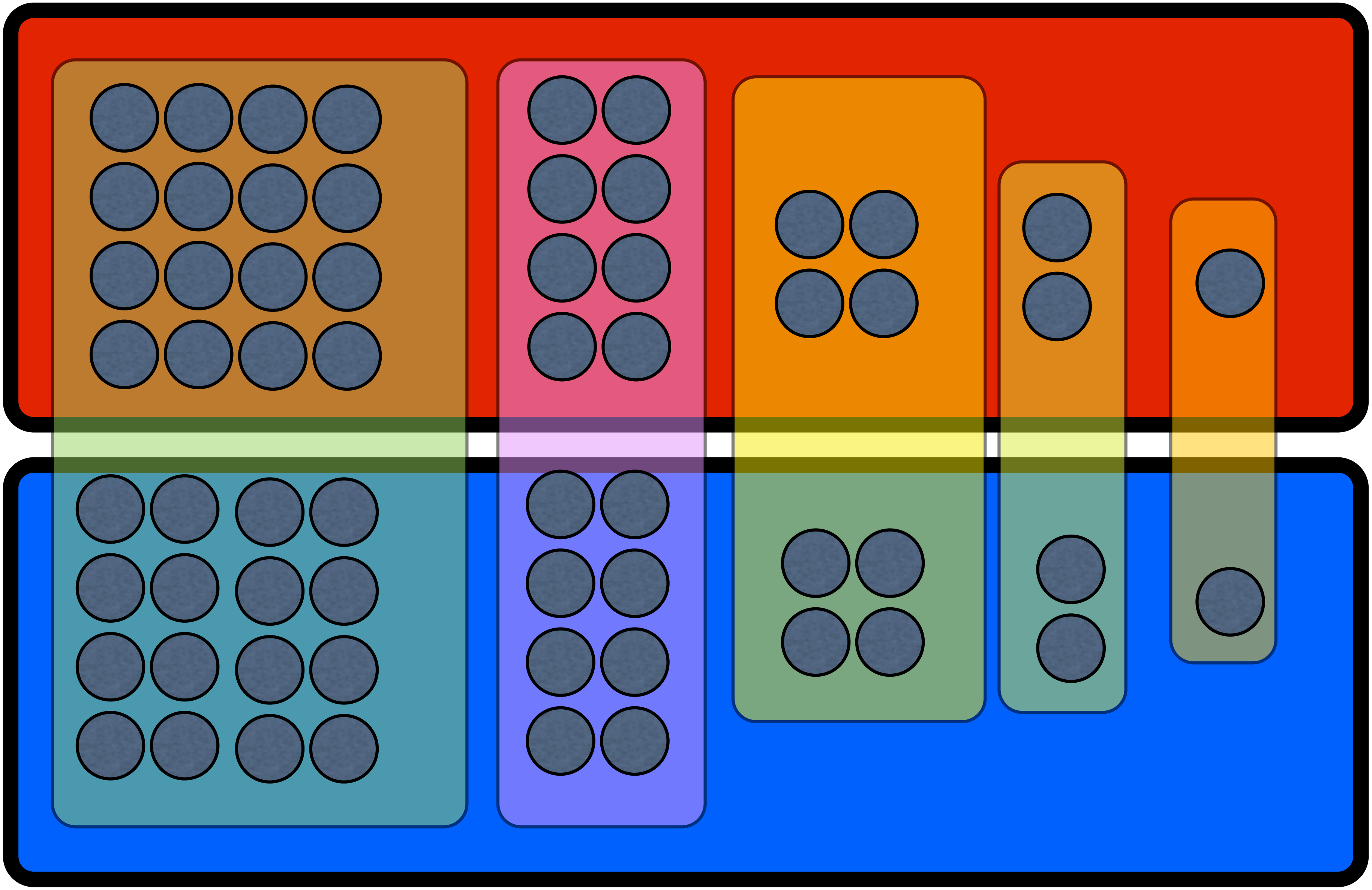
Greedy Set Cover: Pick the set that maximizes # new elements covered

greedy solution: 5 sets



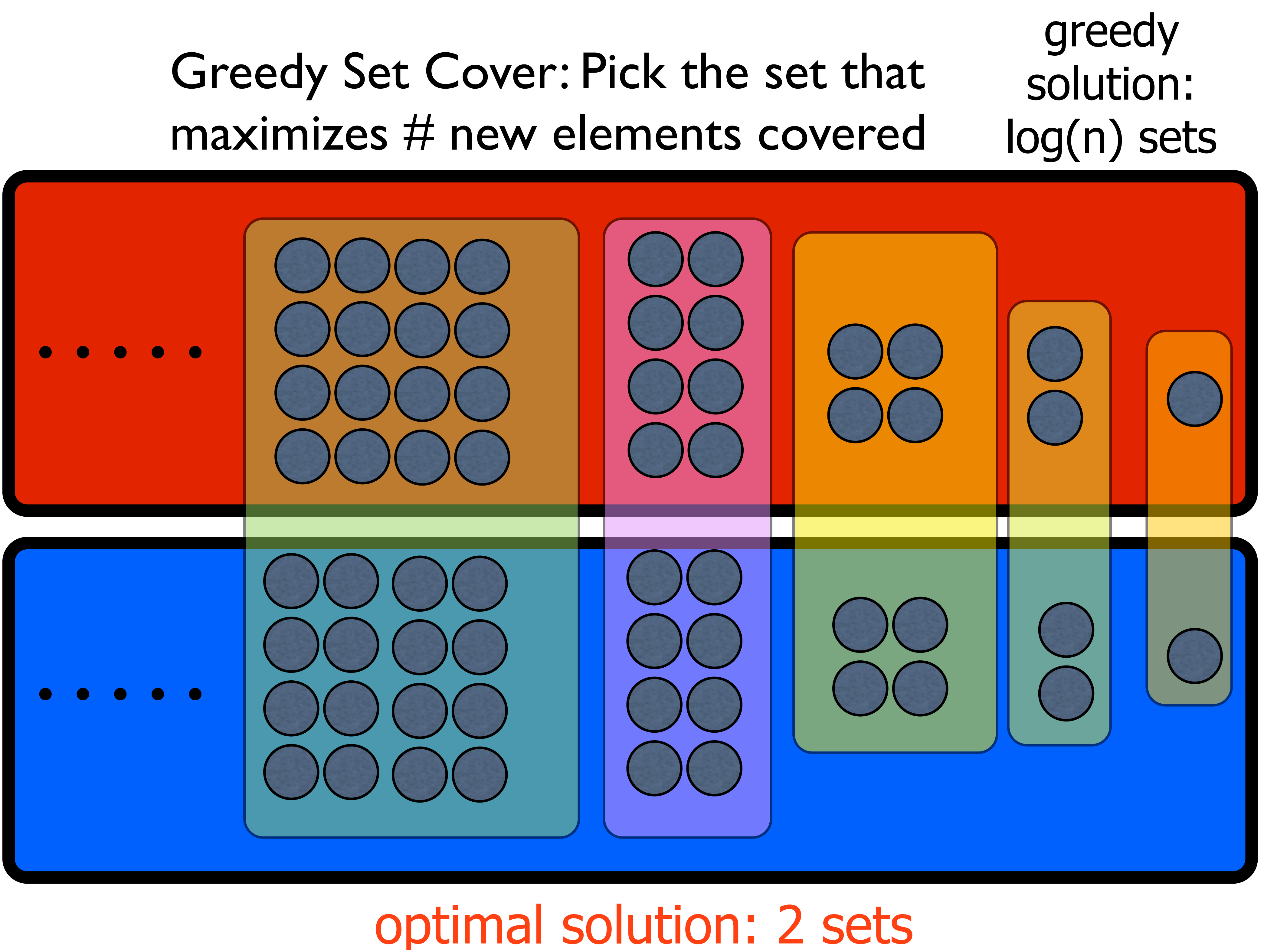
Greedy Set Cover: Pick the set that maximizes # new elements covered

greedy solution: 5 sets



optimal solution: 2 sets

Greedy Set Cover: Pick the set that maximizes # new elements covered



Greedy Set Cover: Pick the set that maximizes # new elements covered

Theorem: If the best solution has k sets, greedy finds at most k ln(n) sets.

Pf: Suppose there is a set cover of size k.

Greedy Set Cover: Pick the set that maximizes # new elements covered

Theorem: If the best solution has k sets, greedy finds at most k ln(n) sets.

Pf:

Suppose there is a set cover of size k.

There is set that covers 1/k fraction of remaining elements, since there are k sets that cover all remaining elements. So in each step, algorithm will cover 1/k fraction of remaining elements.

Greedy Set Cover: Pick the set that maximizes # new elements covered

Theorem: If the best solution has k sets, greedy finds at most k ln(n) sets.

Pf:

Suppose there is a set cover of size k.

There is set that covers 1/k fraction of remaining elements, since there are k sets that cover all remaining elements. So in each step, algorithm will cover 1/k fraction of remaining elements.

#elements uncovered after t steps $\leq n(1-1/k)^t < ne^{-t/k}$. So after t = k ln (n) steps, number of uncovered elements < 1.