**Vertex cover**

**VERTEX-COVER.** Given a graph $G = (V, E)$ and an integer $k$, is there a subset of vertices $S \subseteq V$ such that $|S| \leq k$, and for each edge, at least one of its endpoints is in $S$?

**Ex.** Is there a vertex cover of size $\leq 4$?

**Ex.** Is there a vertex cover of size $\leq 3$?

![Graph and vertex cover diagram]

- **Independent set of size 6**
- **Vertex cover of size 4**
3-satisfiability reduces to vertex cover

**Theorem.** $3$-SAT $\leq_P$ VERTEX-COVER.

**Pf.** Given an instance $\Phi$ of $3$-SAT, we construct an instance $(G, k)$ of VERTEX-COVER that has a vertex cover of size $2k$ iff $\Phi$ is satisfiable.

**Construction.**

- $G$ contains 3 nodes for each clause, one for each literal.
- Connect 3 literals in a clause in a triangle.

\[
\Phi = \left( \overline{x_1} \lor x_2 \lor x_3 \right) \land \left( x_1 \lor \overline{x_2} \lor x_3 \right) \land \left( x_1 \lor x_2 \lor x_4 \right)
\]
3-satisfiability reduces to vertex cover

**Theorem.** $3$-Sat $\leq_P$ Vertex-Cover.

**Pf.** Given an instance $\Phi$ of $3$-Sat, we construct an instance $(G, k)$ of Vertex-Cover that has a vertex cover of size $2k$ iff $\Phi$ is satisfiable.

**Construction.**
- $G$ contains 3 nodes for each clause, one for each literal.
- Connect 3 literals in a clause in a triangle.
- Connect literal to each of its negations.

$$\Phi = \left( \overline{x_1} \lor x_2 \lor x_3 \right) \land \left( x_1 \lor \overline{x_2} \lor x_3 \right) \land \left( \overline{x_1} \lor x_2 \lor x_4 \right)$$
Lemma. $G$ contains vertex cover of size $2k$ iff $\Phi$ is satisfiable.

Pf. $\Rightarrow$ Let $S$ be a vertex cover of size $2k$.

- $S$ must contain exactly two nodes in each triangle.
- Set the excluded literal to $true$ (and remaining variables consistently).
- Truth assignment is consistent and all clauses are satisfied.

Pf $\Leftarrow$ Given satisfying assignment, select one true literal from each triangle, and exclude that one. This is a vertex cover of size $2k$.  

$$
\begin{aligned}
\Phi &= \left( \overline{x_1} \lor x_2 \lor x_3 \right) \land \left( x_1 \lor \overline{x_2} \lor x_3 \right) \land \left( \overline{x_1} \lor x_2 \lor x_4 \right)
\end{aligned}
$$
**Directed hamilton cycle reduces to hamilton cycle**

**DIR-HAM-CYCLE:** Given a digraph $G = (V, E)$, does there exist a simple directed cycle $\Gamma$ that contains every node in $V$?
Theorem. 3-SAT \leq_p \text{DIR-HAM-CYCLE}.

Pf. Given an instance \( \Phi \) of 3-SAT, we construct an instance of \text{DIR-HAM-CYCLE} that has a Hamilton cycle iff \( \Phi \) is satisfiable.

Construction. First, create graph that has \( 2^n \) Hamilton cycles which correspond in a natural way to \( 2^n \) possible truth assignments.
3-satisfiability reduces to directed hamilton cycle

**Construction.** Given 3-SAT instance $\Phi$ with $n$ variables $x_i$ and $k$ clauses
3-satisfiability reduces to directed hamilton cycle

Construction. Given 3-SAT instance $\Phi$ with $n$ variables $x_i$ and $k$ clauses.
- Intuition: traverse path $i$ from left to right $\iff$ set variable $x_i = true$. 

\[
\begin{align*}
3k + 3
\end{align*}
\]
3-satisfiability reduces to directed hamilton cycle

**Construction.** Given 3-SAT instance $\Phi$ with $n$ variables $x_i$ and $k$ clauses.

- For each clause, add a node and 6 edges.

\[
C_1 = x_1 \lor \overline{x_2} \lor x_3 \quad \text{clause node 1}
\]
\[
C_2 = \overline{x_1} \lor \overline{x_2} \lor \overline{x_3} \quad \text{clause node 2}
\]
3-satisfiability reduces to directed hamilton cycle

Lemma.  $\Phi$ is satisfiable iff $G$ has a Hamilton cycle.

Pf.  $\Rightarrow$

- Suppose 3-SAT instance has satisfying assignment $x^*$.
- Then, define Hamilton cycle in $G$ as follows:
  - if $x^*_i = \text{true}$, traverse row $i$ from left to right
  - if $x^*_i = \text{false}$, traverse row $i$ from right to left
  - for each clause $C_j$, there will be at least one row $i$ in which we are going in "correct" direction to splice clause node $C_j$ into cycle
    (and we splice in $C_j$ exactly once)
Lemma. $\Phi$ is satisfiable iff $G$ has a Hamilton cycle.

Pf. $\iff$

- Suppose $G$ has a Hamilton cycle $\Gamma$.
- If $\Gamma$ enters clause node $C_j$, it must depart on mate edge.
  - nodes immediately before and after $C_j$ are connected by an edge $e \in E$
  - removing $C_j$ from cycle, and replacing it with edge $e$ yields Hamilton cycle on $G - \{C_j\}$
- Continuing in this way, we are left with a Hamilton cycle $\Gamma'$ in $G - \{C_1, C_2, \ldots, C_k\}$.
- Set $x^*_i = true$ iff $\Gamma'$ traverses row $i$ left to right.
- Since $\Gamma$ visits each clause node $C_j$, at least one of the paths is traversed in "correct" direction, and each clause is satisfied. $\blacksquare$
3-colorability

**3-COLOR.** Given an undirected graph $G$, can the nodes be colored red, green, and blue so that no adjacent nodes have the same color?
Application: register allocation

Register allocation. Assign program variables to machine register so that no more than \( k \) registers are used and no two program variables that are needed at the same time are assigned to the same register.

Interference graph. Nodes are program variables names; edge between \( u \) and \( v \) if there exists an operation where both \( u \) and \( v \) are "live" at the same time.

Observation. [Chaitin 1982] Can solve register allocation problem iff interference graph is \( k \)-colorable.

Fact. \( 3\text{-COLOR} \leq_p \text{K-REGISTER-ALLOCATION} \) for any constant \( k \geq 3 \).
3-satisfiability reduces to 3-colorability

Theorem. 3-SAT \leq_p 3-COLOR.

Pf. Given 3-SAT instance \( \Phi \), we construct an instance of 3-COLOR that is 3-colorable iff \( \Phi \) is satisfiable.
3-satisfiability reduces to 3-colorability

Construction.

(i) Create a graph $G$ with a node for each literal.
(ii) Connect each literal to its negation.
(iii) Create 3 new nodes $T$, $F$, and $B$; connect them in a triangle.
(iv) Connect each literal to $B$.
(v) For each clause $C_j$, add a gadget of 6 nodes and 13 edges.

![Diagram](image-url)

- **true**
- **false**
- **base**

`T` and `F` are connected to the `true` and `false` nodes respectively. The `base` node is connected to all the literals and clauses.

`to be described later`
3-satisfiability reduces to 3-colorability

**Lemma.** Graph $G$ is 3-colorable iff $\Phi$ is satisfiable.

**Pf.** $\Rightarrow$ Suppose graph $G$ is 3-colorable.

- Consider assignment that sets all $T$ literals to true.
- (iv) ensures each literal is $T$ or $F$.
- (ii) ensures a literal and its negation are opposites.
Lemma. Graph $G$ is 3-colorable iff $\Phi$ is satisfiable.

Pf. $\Rightarrow$ Suppose graph $G$ is 3-colorable.

- Consider assignment that sets all $T$ literals to true.
- (iv) ensures each literal is $T$ or $F$.
- (ii) ensures a literal and its negation are opposites.
- (v) ensures at least one literal in each clause is $T$.

$$C_j = x_1 \lor \overline{x_2} \lor x_3$$

6-node gadget
Lemma. Graph $G$ is 3-colorable iff $\Phi$ is satisfiable.

Pf. $\Rightarrow$ Suppose graph $G$ is 3-colorable.

- Consider assignment that sets all $T$ literals to true.
- (iv) ensures each literal is $T$ or $F$.
- (ii) ensures a literal and its negation are opposites.
- (v) ensures at least one literal in each clause is $T$.

$3$-satisfiability reduces to $3$-colorability.
Lemma. Graph $G$ is 3-colorable iff $\Phi$ is satisfiable.

Pf. $\iff$ Suppose 3-SAT instance $\Phi$ is satisfiable.

- Color all true literals $T$.
- Color node below green node $F$, and node below that $B$.
- Color remaining middle row nodes $B$.
- Color remaining bottom nodes $T$ or $F$ as forced. □

3-satisfiability reduces to 3-colorability

$a literal set to true$ in 3-SAT assignment

$C_j = x_1 \lor \overline{x_2} \lor x_3$