CSE421: Design and Analysis of Algorithms

Homework 7

Due:

Each problem is worth 10 points:

1. Suppose you are given a bipartite graph G with n vertices on each side, and a matching M with $n-1$ edges that is contained in the graph. Give an $O(n^2)$ time algorithm to check whether or not the graph has a matching of size n.

   Solution. From class, we know that matching problem can be solved using the max-flow algorithm.

   Input: A bipartite graph G, and a matching of size n - 1
   Result: True if G has a matching of size n, false otherwise
   Build the network-flow graph:;
   Suppose we call the bi-partitions X and Y;
   Create a source node s, and sink node t. ;
   for *all nodes u in X* **do**
   ⌊ create a forward edge(s, u);
   for *all nodes v in Y* **do**
   ⌊ create a forward edge(v, t);
   for *Edge(u, v) in G where u in X, v in Y* **do**
   ⌊ create a forward edge from u to v;
   Build the residual graph:;
   for *(u, v) in given n - 1 matching, where u in X, v in Y* **do**
   ⌊ reverse the edge(s, u), edge(u, v), and edge(v, t);
   Set all forward and backward edges with capacity one;
   Use the graph traverse (BFS) starting from node s to determine if there is a path from s to t in the residual graph. Return true if there is one, false otherwise;

   Runtime: Each step in setting up the residual graph is bounded by $O(m) = O(n^2)$ where m is the total number of edges in the graph. And the graph traverse takes $O(n+m) = O(n^2)$. So the total rum time for this algorithm is $O(n^2)$

   Correctness: In the original network-flow graph, all the max-flow can not exceed n because there exists a cut that only contains s and its capacity if n. In addition, if the max-flow is n, there exists a matching with size n, because the network-flow graph ensures all n nodes in both X and Y has a flow with capacity one going through themselves. It indicates all nodes in X has a flow with capacity one going to a unique node in Y, and therefore indicates we have a matching with size n. Here we are given a matching with size n - 1, so if we can find a path in the residual graph, then we can increment the flow by at least one and at most one. And that will tell us if there exits a matching with size one more than size n - 1 and that is size n.

2. Write down the dual of the following linear program:

$$\begin{aligned} &\text{maximize } a - b + c \\ &\text{subject to} \\ &\quad 5a + 2b \leq 3 \\ &\quad c - a \leq -2 \\ &\quad b + c \leq 0 \\ &\quad a, b, c \geq 0 \end{aligned}$$

Solution:

$$\begin{aligned} &\text{minimize } 3x - 2y \\ &\text{subject to} \\ &\quad 5x - y \geq 1 \\ &\quad 2x + z \geq -1 \\ &\quad y + z \geq 1 \\ &\quad x, y, z \geq 0 \end{aligned}$$

3. You are given the following points in the plane: $(1, 3), (2, 5), (3, 7), (5, 11), (7, 14), (8, 15), (10, 19)$. You want to find a line $y = ax + b$ that approximately passes through these points (no line is a perfect fit). Write a linear program (you do not need to solve it) to find the line that minimizes the maximum absolute error,

$$\max_{1 \leq i \leq 7} |y_i - ax_i - b|$$

Solution:

$$\begin{aligned} &\text{minimize } c \\ &\text{subject to} \\ &\text{for all } i = 1, \ldots, 7, \\ &\qquad c \geq y_i - ax_i - b \\ &\qquad c \geq -y_i + ax_i + b \end{aligned}$$

In this linear program, c will be set to $\max_{1 \leq i \leq 7} |y_i - ax_i - b|$, since after fixing a, b, c the constraints force to be larger than all of the errors. So, the value of a, b in the above program gives the description of the best line.

4. You are running a truck business and need to fill a truck that can carry a total weight of 100 tons and volume 30 cubic meters. You can put three types of materials into the truck.

   (a) Item 1 has density 2 tons per cubic meter, maximum available amount is 40 cubic meters and the revenue associated with it is 1000 dollars per cubic meter.

   (b) Item 2 has density 5 tons per cubic meter, maximum available amount is 20 cubic meters and the revenue associated with it is 2000 dollars per cubic meter.

   (c) Item 3 has density 7 tons per cubic meter, maximum available amount is 15 cubic meters and the revenue associated with it is 1500 dollars per cubic meter.

   Write a linear program to calculate how much of each amount the truck should carry to maximize profits (no need to solve it).

   Solution: In the following linear program, a, b, c denote the volume of each of the materials that can be carried.

$$
\begin{aligned}
&\text{maximize } 1000a + 2000b + 1500c \\
&\text{subject to} \\
2a + 5b + 7c &\leq 100 \\
a &\leq 40 \\
b &\leq 20 \\
c &\leq 15 \\
a + b + c &\leq 30 \\
a, b, c &\geq 0
\end{aligned}
$$