Each problem is worth 10 points:

1. Suppose you are given a bipartite graph $G$ with $n$ vertices on each side, and a matching $M$ with $n - 1$ edges that is contained in the graph. Give an $O(n^2)$ time algorithm to check whether or not the graph has a matching of size $n$.

**Solution.** From class, we know that matching problem can be solved using the max-flow algorithm.

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**Input:** A bipartite graph $G$, and a matching of size $n - 1$

**Result:** True if $G$ has a matching of size $n$, false otherwise

Build the network-flow graph:

Suppose we call the bi-partitions $X$ and $Y$;

Create a source node $s$, and sink node $t$;

for all nodes $u$ in $X$ do
  create a forward edge($s$, $u$);

for all nodes $v$ in $Y$ do
  create a forward edge($v$, $t$);

for Edge($u$, $v$) in $G$ where $u$ in $X$, $v$ in $Y$ do
  create a forward edge from $u$ to $v$;

Build the residual graph:

for $(u$, $v$) in given $n - 1$ matching, where $u$ in $X$, $v$ in $Y$ do
  reverse the edge($s$, $u$), edge($u$, $v$), and edge($v$, $t$);

Set all forward and backward edges with capacity one;

Use the graph traverse (BFS) starting from node $s$ to determine if there is a path from $s$ to $t$ in the residual graph. Return true if there is one, false otherwise;

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**Runtime:** Each step in setting up the residual graph is bounded by $O(m) = O(n^2)$ where $m$ is the total number of edges in the graph. And the graph traverse takes $O(n + m) = O(n^2)$. So the total run time for this algorithm is $O(n^2)$

**Correctness:** In the original network-flow graph, all the max-flow can not exceed $n$ because there exists a cut that only contains $s$ and its capacity if $n$. In addition, if the max-flow is $n$, there exists a matching with size $n$, because the network-flow graph ensures all $n$ nodes in both $X$ and $Y$ has a flow with capacity one going through themselves. It indicates all nodes in $X$ has a flow with capacity one going to a unique node in $Y$, and therefore indicates we have a matching with size $n$. Here we are given a matching with size $n - 1$, so if we can find a
path in the residual graph, then we can increment the flow by at least one and at most one. And that will tell us if there exits a matching with size one more than size \( n - 1 \) and that is size \( n \).

2. Write down the dual of the following linear program:

\[
\begin{align*}
\text{maximize} & \quad a - b + c \\
\text{subject to} & \quad 5a + 2b \leq 3 \\
& \quad c - a \leq -2 \\
& \quad b + c \leq 0 \\
& \quad a, b, c \geq 0 \\
\end{align*}
\]

*Solution:*

\[
\begin{align*}
\text{minimize} & \quad 3x - 2y \\
\text{subject to} & \quad 5x - y \geq 1 \\
& \quad 2x + z \geq -1 \\
& \quad y + z \geq 1 \\
& \quad x, y, z \geq 0 \\
\end{align*}
\]

3. You are given the following points in the plane: \((1, 3), (2, 5), (3, 7), (5, 11), (7, 14), (8, 15), (10, 19)\). You want to find a line \( y = ax + b \) that approximately passes through these points (no line is a perfect fit). Write a linear program (you do not need to solve it) to find the line that minimizes the maximum absolute error,

\[
\max_{1 \leq i \leq 7} |y_i - ax_i - b|
\]

*Solution:*

\[
\begin{align*}
\text{minimize} & \quad c \\
\text{subject to} & \quad \text{for all } i = 1, \ldots, 7, \\
& \quad c \geq y_i - ax_i - b \\
& \quad c \geq -y_i + ax_i + b \\
\end{align*}
\]

In this linear program, \( c \) will be set to \( \max_{1 \leq i \leq 7} |y_i - ax_i - b| \), since after fixing \( a, b, c \) the constraints force to be larger than all of the errors. So, the value of \( a, b \) in the above program gives the description of the best line.
4. You are running a truck business and need to fill a truck that can carry a total weight of 100 tons and volume 30 cubic meters. You can put three types of materials into the truck.

(a) Item 1 has density 2 tons per cubic meter, maximum available amount is 40 cubic meters and the revenue associated with it is 1000 dollars per cubic meter.

(b) Item 2 has density 5 tons per cubic meter, maximum available amount is 20 cubic meters and the revenue associated with it is 2000 dollars per cubic meter.

(c) Item 3 has density 7 tons per cubic meter, maximum available amount is 15 cubic meters and the revenue associated with it is 1500 dollars per cubic meter.

Write a linear program to calculate how much of each amount the truck should carry to maximize profits (no need to solve it).

Solution: In the following linear program, $a, b, c$ denote the volume of each of the materials that can be carried.

\[
\begin{align*}
\text{maximize} & \quad 1000a + 2000b + 1500c \\
\text{subject to} & \quad 2a + 5b + 7c \leq 100 \\
& \quad a \leq 40 \\
& \quad b \leq 20 \\
& \quad c \leq 15 \\
& \quad a + b + c \leq 30 \\
& \quad a, b, c \geq 0
\end{align*}
\]