CSE431: Complexity Theory May 29, 2021

Homework 4

Anup Rao Due: June 4, 2021

Read the fine print[1]. Each problem is worth 10 points:

1. Prove that **NP** $\neq$ co-**NP** implies that **BPP** $\neq$ **NP**.

2. Consider the following game between two players: Given a directed graph $G = (V, E)$, and a start vertex s, the players (starting with Player 1) alternately choose an outgoing edge incident to the current vertex to reach a vertex that was not previously visited. If one of the players cannot choose a next vertex, he loses. Let **GAME**(G) be the function that is 1 if and only if Player 1 has a strategy that ensures that she always wins no matter what Player 2 does.

   Show that **GAME** is in **PSPACE**.

3. Consider the following algorithm for computing the permanent. Recall that the permanent is defined as:

$$\mathsf{perm}(M) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} M_{i\sigma(i)}.$$

   The formula for the permanent looks very similar to the formula for the determinant (for which we do have polynomial time algorithms):

$$\det(M) = \sum_{\sigma \in S_n} \mathsf{sign}(\sigma) \prod_{i=1}^{n} M_{i\sigma(i)}.$$

   Here $\mathsf{sign}(\sigma)$ is either 1 or -1. This motivates the following randomized algorithm. Given the matrix M, let us sample the matrix A randomly as follows:

$$A_{ij} = \begin{cases} -\sqrt{M_{ij}} & \text{with probability } 1/2, \\ \sqrt{M_{ij}} & \text{with probability } 1/2. \end{cases}$$

   All entries are sampled independently. (Note that A_{ij} may be a complex number if M_{ij} is negative).

   The algorithm is to just output $\det(A)^2$, which can be computed in polynomial time. Show that the expected value of the output of this algorithm is the same as the permanent. Hint:

[1] In solving the problem sets, you are allowed to collaborate with fellow students taking the class, but **each submission can have at most one author**. If you do collaborate in any way, you must acknowledge, for each problem, the people you worked with on that problem. The problems have been carefully chosen for their pedagogical value, and hence might be similar to those given in past offerings of this course at UW, or similar to other courses at other schools. Using any pre-existing solutions from these sources, for from the web, constitutes a violation of the academic integrity you are expected to exemplify, and is strictly prohibited. Most of the problems only require one or two key ideas for their solution. It will help you a lot to spell out these main ideas so that you can get most of the credit for a problem even if you err on the finer details. Please justify all answers. Some other guidelines for writing good solutions are here: `http://www.cs.washington.edu/education/courses/cse421/08wi/guidelines.pdf`.

Use linearity of expectation to expand the expression for the output of the algorithm. Argue that the only terms in the expansion that have non-zero expectation correspond to the monomials of the permanent.

Does this algorithm prove that computing whether or not $\mathsf{perm}(M) > 0$ is in **BPP**? Discuss the consequences of finding a randomized algorithm in **BPP** for determining whether $\mathsf{perm}(M) > 0$? What would that imply about the relationship between the complexity classes $\mathbf{BPP}, \mathbf{NP}$ and the functions computable by polynomial sized circuits?