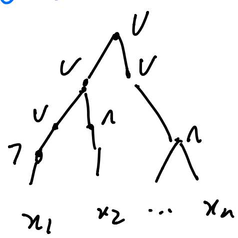
Parity is not in AC_0 → polynomial sized circuits with $O(1)$ alternations.

$x_1 \oplus x_2 \oplus \cdots \oplus x_n$



alternations

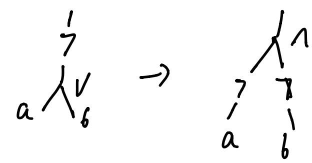
= # of switches between $\wedge$ and $\vee$ on any input-output path.

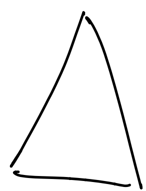
1 alternation

2 alternations

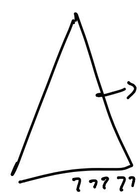
$$\neg(a \vee b) = \neg a \wedge \neg b$$

$$\neg(a \wedge b) = \neg a \vee \neg b$$





Size S →



→ no negations

size (2S).

$$x \wedge y = xy$$

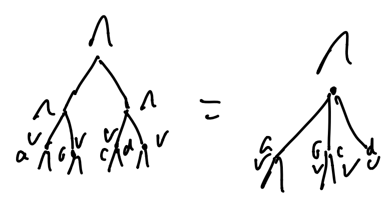
$$x \vee y = 1 - (1-x)(1-y)$$

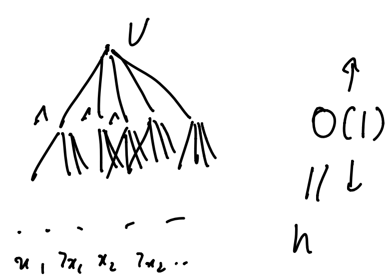
$$\neg x = 1 - x$$

$$x_1 \wedge x_2 \wedge \cdots \wedge x_n = x_1 x_2 x_3 \cdots x_n \approx 0$$

$$x_1 \vee x_2 \cdots \vee x_n = 1 - (1-x_1)(1-x_2)\cdots(1-x_n) \approx 1$$

$$g_1 \wedge g_2 \wedge \cdots \wedge g_n = g_1 g_2 g_3 \cdots g_n \not\approx 0$$





Idea

Suppose $g_1, \dots, g_k$ are all degree d polynomials. Pick $S \subseteq \{1, 2, \dots, k\}$ uniformly at random

$$g_1 \vee g_2 \vee \dots \vee g_k = \underbrace{1-(1-g_1)\cdots(1-g_k)}_{\text{deg } dk} \approx \underbrace{\sum_{i \in S} g_i}_{\text{deg } d}$$

Claim: If $\exists j$ s.t. $g_j \neq 0$
$$\Pr_S\left[\sum_{i \in S} g_i = 0\right] \leq 1/2 .$$

Pf: Suppose $g_1 = 1$.

$$\sum_{i \in S} g_i = \begin{cases} g_1 + \sum_{\substack{i \in S \\ i > 1}} g_i & 1/2 \text{ prob} \\ \sum_{\substack{i \in S \\ i > 1}} g_i & 1/2 \text{ prob} \end{cases}$$

$\mathbb{F}_3$ = field mod 3.

$4 = 2^2 = 1$

$1^2 = 1$

$0^2 = 0$

Pick sets $S_1, S_2, \dots, S_\ell \in \{1, \dots, k\}$ unif. at random

$$\underbrace{g_1 \vee g_2 \vee \dots \vee g_k}_{dk} \approx 1 - \underbrace{\left(1-\left(\sum_{i \in S_1} g_i\right)^2\right)\left(1-\left(\sum_{i \in S_2} g_i\right)^2\right)\cdots\left(1-\left(\sum_{i \in S_\ell} g_i\right)^2\right)}_{\text{deg } 2d\ell}$$

$\downarrow$

$\Pr[\ = \ ] \geq 1 - 2^{-\ell}$

$$g_1 \wedge g_2 \wedge \dots \wedge g_k = \neg(\neg g_1 \vee \neg g_2 \cdots \vee \neg g_k) \approx \text{deg } 2d\ell$$

Claim: repeat for every gate

Circuit of depth h $\approx$ poly of deg $(2\ell)^h$

$$\Pr[\text{circuit} = \text{poly}] \geq 1 - s \cdot 2^{-\ell}$$

Set $\ell = \log^2 n$ we get

$$\Pr[\text{circuit} = \text{poly of deg } \log^{2h} n] \geq 1 - \text{poly}(n) \cdot 2^{-\log^2 n}$$

$$\geq 99\% \quad (\text{for large } n).$$

$$\Pr_{\substack{x_1, \dots, x_n \\ \text{random sets}}} [\text{circuit}(x_1, \dots, x_n) = \text{poly}] \geq 99\%$$

Suppose circuit computes f

$$g(y_1, \dots, y_n) = f(y_1 - 1, y_2 - 1, \dots, y_n - 1) + 1$$

$y_1, \dots, y_n \in \{\pm 1\}$

$$y_1 y_2 \cdots y_n = (y_1 - 1) \oplus (y_2 - 1) \oplus \cdots (y_n - 1) + 1$$

$$1 - 1 = 0 \mod 3$$

$$-1 - 1 = 1 \mod 3$$

By approximation

$\exists$ poly g of deg $\log^{2h}(n)$

and a set $T \subseteq \{\pm 1\}^n$

of size $(0.99) \cdot 2^n$

s.t $y_1 \cdots y_n = g(y_1, \ldots, y_n)$