$f(x,y)$

Alice		Bob
x	$\xrightarrow{m_1(x)}$	y
	$\xleftarrow{m_2(y,m_1)}$	
	$\vdots$	

$$\text{Distinctness}(x_1,\dots,x_n,x_{n+1}) = \begin{cases} 1 & \text{if } x_1,\dots,x_n,x_{n+1} \text{ are distinct} \\ 0 & \text{o.w.} \end{cases}$$

<u>Thm</u>: Every formula computing Distinctness has size $\Omega(n^2)$.

Alice	Is $z \in X$?	Bob
$X \subseteq \{1,\dots,2n\}$	$\xrightarrow{m}$	$z \in \{1,\dots,2n\}$
$\lvert X\rvert = n$		

<u>Claim</u>: # bits sent $\geq \log\binom{2n}{n} \geq \log\left(\frac{2^{2n}}{(2n+1)}\right) \geq 2n - O(\log n)$.

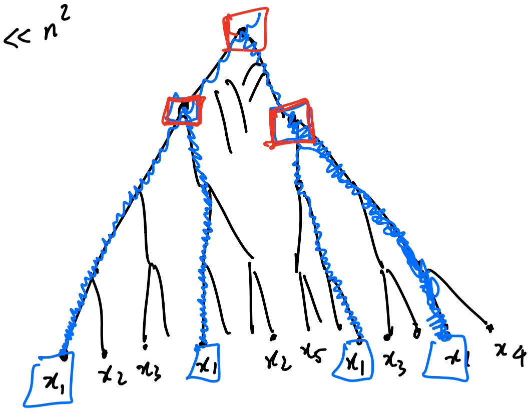
<u>Pf</u>: If message k bits, # of messages $\leq 2^k$

$$\binom{2n}{n} \leq 2^k$$

or else two sets get mapped to same message.
(by pigeonhole).

$$\Rightarrow \quad k \geq \log\binom{2n}{n}$$

Size $\ll n^2$
$\shortparallel$
S



<u>Pick</u> $i \in \{1, \dots, n, n+1\}$ uniformly at random.

Let T be tree induced by x_i.

<u>Claim</u>: Expected size of $|T|$ $\leq O\left(\frac{S}{(n+1)}\right)$.

<u>FACT</u>: In any formula tree # vertices $= \Theta(\#\text{ leaves})$.

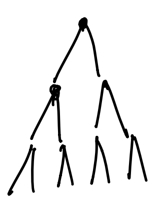
Distinctness $\{x_1, \dots, x_{n+1}\}$

<u>Alice</u>		<u>Bob</u>
$X = \{x_2, \dots, x_{n+1}\}$	# bits $O(S/n)$ $\longrightarrow$	x_1

$$\Rightarrow \quad S/n \geq \Omega(n)$$

$$\Rightarrow \quad S \geq \Omega(n^2).$$

X ——— X



monotone circuit : no negation gates.

monotone function:

$$x \geq y \Rightarrow f(x) \geq f(y).$$

$\forall i \ x_i \geq y_i$

$$\mathrm{Match}_k(G) = \begin{cases} 1 & \text{if } G \text{ has matching of size } k \\ 0 & \text{o.w.} \end{cases}$$

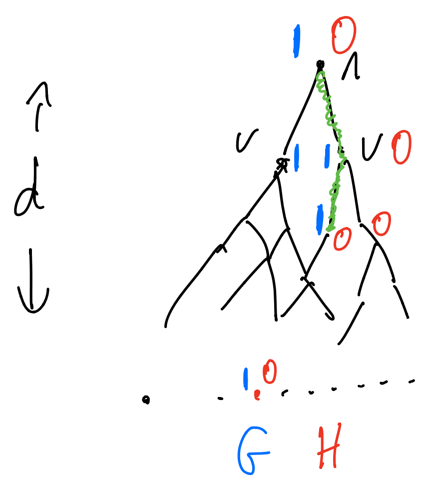
Match_k is monotone.

<u>Thm</u>: Any monotone circuit computing $\mathrm{Match}_{n/3}(G)$ must have depth $\Omega(n)$.

1

<u>Alice</u>	<u>Bob</u>
G: $\mathrm{Match}_k(G) = 1$	H: $\mathrm{Match}_k(H) = 0$

Goal: find $e \in G$
$e \notin H$.



Protocol communicating
d bits

(2) **Alice** $|X \cap Y| \stackrel{?}{=} 0$ **Bob**

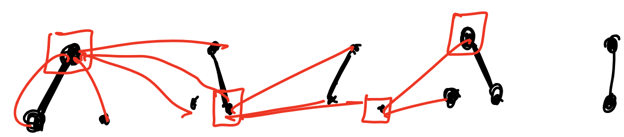
$X \subseteq \{1,\dots,m\}$ $Y \subseteq \{1,\dots,m\}$

Thm: $\Omega(m)$ bits reqd, even with a randomized protocol.

Lemma: (1) requires $\Omega(n)$ bits.

Alice : $\{1, \quad 3\}$

$n = 3m+2$
$k = m+1$

$G =$ 

Bob : $\{\quad 2, \quad 3\}$

H : all potential edges touching $\Box$

H has no matching of size $m+1$.

$$\det(A) = \sum_{\pi} \operatorname{sign}(\pi) \cdot \prod_{i=1}^{n} A_{i\pi(i)} \quad \Big| \quad \pi : \{1,\dots,n\} \to \{1,\dots,n\}$$

A_{ij}

$$A = \begin{array}{c|c} & j \\ \hline i & \begin{bmatrix} & & \\ & A_{ij} & \\ & & \end{bmatrix} \end{array}$$

$$\operatorname{perm}(M) = \sum_{\pi} \prod_{i=1}^{n} M_{i\pi(i)}.$$

$$\pi : \{1,2\} \to \{1,2\} \qquad \begin{array}{ll} \pi_1(1) = 1 & \pi_1(2) = 2 \\ \pi_2(1) = 2 & \pi_2(2) = 1 \end{array}$$

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$$

$$\det(M) = M_{11} M_{22} - M_{12} M_{21}$$

$$= M_{1\pi_1(1)} M_{2\pi_1(2)} - M_{1\pi_2(1)} M_{2\pi_2(2)}$$

$$A_{ij} = \begin{cases} +\sqrt{M_{ij}} & 1/2 \\ -\sqrt{M_{ij}} & 1/2 \end{cases}$$

$$A_{ij} = B_{ij} \cdot \sqrt{M_{ij}} \qquad B_{ij} \in \{\pm 1\}.$$

$$\det(A)^2 = \left(\sum_{\pi} \operatorname{sign}(\pi) \prod_{i=1}^{n} A_{i\pi(i)} \right)^2$$

$$= \left(\sum_{\pi} \operatorname{sign}(\pi) \left(\prod_{i=1}^{n} B_{i\pi(i)} \right) \sqrt{\prod_{i=1}^{n} M_{i\pi(i)}} \right)^2$$

$$= \left(\sum_{\pi} (\cdots) \right) \left(\sum_{\pi} (\cdots) \right)$$

$$= \underbrace{\sum_{\pi} (\cdots)^2}_{} + \sum_{\substack{\pi, \pi' \\ \pi \neq \pi'}} (\cdots)(\cdots)$$

$$= \mathrm{perm}(M) + \underbrace{0}_{\text{in expectation}}$$