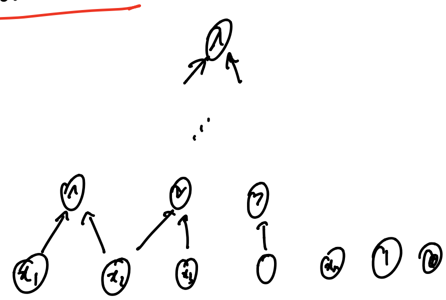
Circuit Complexity

Turing machines : $f: \{0,1\}^* \to \{0,1\}$ programs with loops

Boolean circuits : $f: \{0,1\}^n \to \{0,1\}$ programs without loops



size : # gates

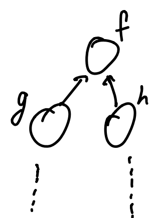
depth : length of longest input-output path.

Thm If $f:\{0,1\}^n \to \{0,1\}$ can be computed in depth d

then f can be computed in size $2^{d+1}-1$.

Pf: Base case : $d=0$ size $\leq 1 = 2^{0+1}-1$.

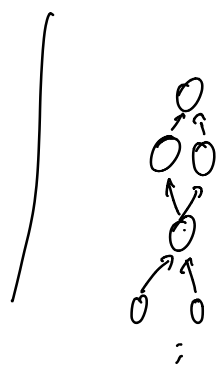
Ind step : $d>0$



By induction, f can be computed in size

$$\left(2^{(d-1)+1}-1\right) + \left(2^{(d-1)+1}-1\right) + 1$$

$$= 2^d - 1 + 2^d - 1 + 1 = 2^{d+1}-1.$$



Thm1: (almost all) $\exists\, f:\{0,1\}^n \to \{0,1\}$ s.t no circuits of size $< \dfrac{2^n}{3 \cdot n}$
(Shannon) can compute f.

Thm2: $\forall\, f:\{0,1\}^n \to \{0,1\}$ f can be computed in size $O(2^n/n)$.
(Lupanov)

Pf of Thm 1: # of circuits of size s $\quad (s > n+2)$

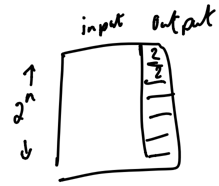
$$\leq (3s^2 + n+2)^s$$

$$\leq (4s^2)^s = 2^{s\log(4s^2)} = 2^{2s\log(s) + 2s}$$

$$= (2^{\log(4s^2)})^s \quad \leq 2^{3s\log s}.$$

# of functions $f: \{0,1\}^n \to \{0,1\}$

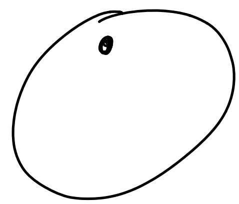
$$= 2^{(2^n)} = \underbrace{2 \times 2 \times 2 \cdots \times 2}_{2^n \text{ times}}$$



When $s = \frac{2^n}{3n}$

$$2^{3s\log s} = 2^{3 \cdot \frac{2^n}{3n} \cdot \log\left(\frac{2^n}{3n}\right)}$$

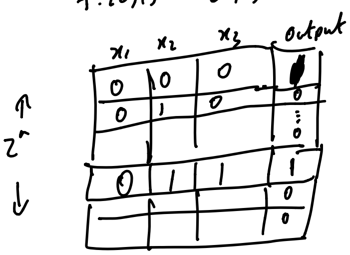
$$< 2^{\frac{2^n}{n} \cdot n} = 2^{2^n}$$



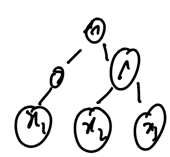
Pf of Thm 2

Attempt #1

$f: \{0,1\}^3 \to \{0,1\}$



$$(\bar{x}_1 \wedge x_2 \wedge x_3) \vee (\bar{x}_1 \wedge \bar{x}_2 \wedge \bar{x}_3)$$



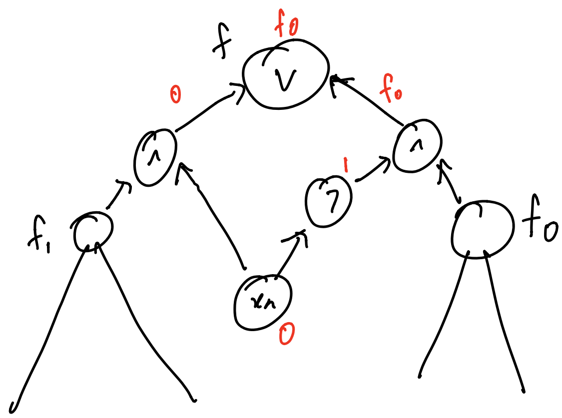
$f: \{0,1\}^n \to \{0,1\}$ $\qquad$ size $O(n \cdot 2^n)$

Attempt #2

want $f(x_1, \dots, x_n)$. First recursively compute $f(x_1, \dots, x_{n-1}, 0) = f_0$

$f(x_1, \dots, x_{n-1}, 1) = f_1$

Combine.



Claim: size $\le 5 \cdot (2^{n+1} - 1)$

Pf: Base case: $n = 0$

size $= 1 \le 5 \cdot (2-1)$

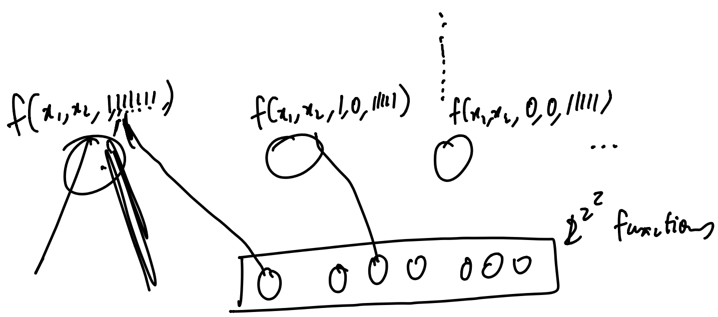
Ind step

size

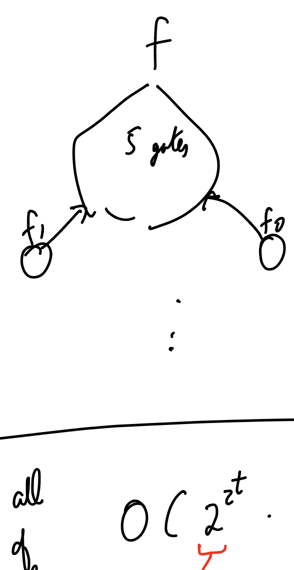
$\le 2 \cdot 5 \cdot (2^n - 1) + 5$

$= 5(2^{n+1} - 2 + 1)$

$= 5(2^{n+1} - 1)$.



Final Attempt



t

— size $O(2^{n-t})$

stop — t variables left.

Compute all functions of t variables $O(2^{2^t} \cdot 2^t)$

2^{2^t}: # of functions of first t vars

2^t: size using attempt #2.

Overall size

$\propto 2^{n-t} + 2^{2^t} + t$

(set $t \sim \log n - 1$, $2^t = n/2$)

$2^{n - \log n + 1} + 2^{n/2 + \log n - 1}$

$= \frac{2^{n+1}}{n} + \frac{2^{n/2} n}{2}$

$= O\left(\frac{2^{n+1}}{n}\right)$.