Hierarchy Theorems

Def: $\mathrm{DTIME}(t(n)) = \{ f : \{0,1\}^* \to \{0,1\} \mid f \text{ is computable in time } O(t(n)) \}$.

Def: $\mathrm{DSPACE}(s(n)) = \{ f : \{0,1\}^* \to \{0,1\} \mid f \text{ is computable in space } O(s(n)) \}$.

Def: $P = \bigcup_{c \geq 1} \mathrm{DTIME}(n^c) = \mathrm{DTIME}(n) \cup \mathrm{DTIME}(n^2) \cup \cdots$

Def: $\mathrm{EXP} = \bigcup_{c \geq 1} \mathrm{DTIME}(2^{n^c})$

Def: $E = \bigcup_{c \geq 1} \mathrm{DTIME}(2^{cn})$

Def: $L = \mathrm{DSPACE}(\log n)$.

Def: $\mathrm{PSPACE} = \bigcup_{c \geq 1} \mathrm{DSPACE}(n^c)$.

$L \subseteq P \subsetneq E \subsetneq \mathrm{EXP}$

$\subseteq \mathrm{PSPACE} \subseteq$

will prove

Thm: (Time Hierarchy) If $r(n) \log r(n) \leq o(t(n))$, $\left(\lim_{n \to \infty} \frac{r(n) \log r(n)}{t(n)} = 0 \right)$

then $\mathrm{DTIME}(r(n)) \neq \mathrm{DTIME}(t(n))$. (and $r(n)$, $t(n)$ are time-constructible)

Pf: Consider:

$$f(\alpha) = \begin{cases} 1 & \text{if } M_\alpha(\alpha) \text{ halts in } \leq t(|\alpha|) \text{ steps of the simulator and outputs } 0. \\ 0 & \text{o.w.} \end{cases}$$

Suppose β is the code of a machine computing f in time $c \cdot r(n)$. Let C_β be s.t. M_β can be sim. in time $C_\beta \cdot r \log r$ steps. $\exists\ \beta'$ s.t. β, β' are equiv. get $t(|\beta'|) > C_\beta \cdot c \cdot r(|\beta'|) \cdot \log r(|\beta'|)$.

$$M_\beta(\beta') = M_{\beta'}(\beta') \neq f(\beta').$$

Thm: There is a T.M. M s.t given the code α and input x, if M_α takes T steps to compute $M_\alpha(x)$ then $M(\alpha, x)$ computes $M_\alpha(x)$ in time $O(C\,T \log T)$, where C depends only on α.

Def: Say $t: \mathbb{N} \to \mathbb{N}$ is time constructible if $t(n) \geq n$ and on input x, $\exists$ machine M that computes $t(|x|)$ in time $O(t(|x|))$

Thm: There is a machine M s.t given code α and input x, if $M_\alpha(x)$ takes space $S \geq \log |x|$ then M computes the same output in space $O(C S)$, where C depends only on code α.

Def: Say $s: \mathbb{N} \to \mathbb{N}$ is space constructible if $s(n) \geq \log n$ and $\exists$ M s.t on input x $M(x)$ computes $S(|x|)$ in space $O(s(|x|))$.

Thm: If q, s are space constructible with $q(n) = o(s(n))$ then $DSPACE(q(n)) \neq DSPACE(s(n))$.