

Hierarchy Theorems

Def: $D_{TIME}(t(n)) = \{ f: \{0,1\}^* \rightarrow \{0,1\} \mid f \text{ is computable in time } O(t(n)) \}$.

Def: $D_{SPACE}(s(n)) = \{ f: \{0,1\}^* \rightarrow \{0,1\} \mid f \text{ is computable in space } O(s(n)) \}$.

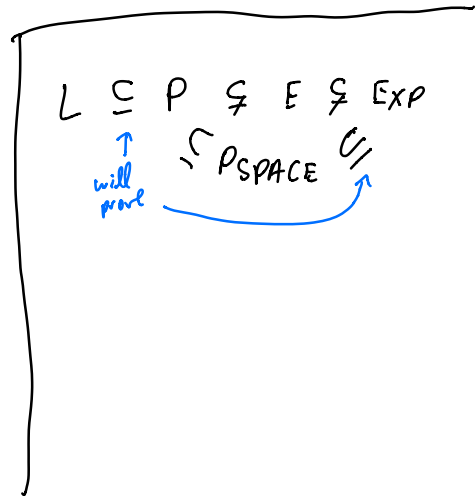
Def: $P = \bigcup_{c \geq 1} D_{TIME}(n^c) = D_{TIME}(n) \cup D_{TIME}(n^2) \cup \dots$

Def: $EXP = \bigcup_{c \geq 1} D_{TIME}(2^{n^c})$

Def: $E = \bigcup_{c \geq 1} D_{TIME}(2^{cn})$

Def: $L = D_{SPACE}(\log n)$.

Def: $PSPACE = \bigcup_{c \geq 1} D_{SPACE}(n^c)$.



Thm: (Time Hierarchy) If $r(n) \log r(n) \leq o(t(n))$, then $D_{TIME}(r(n)) \neq D_{TIME}(t(n))$. (and $r(n), t(n)$ are time-constructible)

$\left(\lim_{n \rightarrow \infty} \frac{r(n) \log r(n)}{t(n)} = 0 \right)$
 of the simulator

Pf: Consider:

$$f(x) = \begin{cases} 1 & \text{if } M_x(x) \text{ halts in } \leq t(x) \text{ steps} \\ & \text{and outputs } 0. \\ 0 & \text{o.w.} \end{cases}$$

Suppose β is the code of a machine computing f in time $c \cdot r(n)$. Let C_β be s.t. M_β can be sim. in time $C_\beta \cdot r \log r$ steps. $\exists \beta'$ s.t. β, β' are equiv. yet $t(|\beta'|) > C_\beta \cdot c \cdot r(|\beta'|) \cdot \log r(|\beta'|)$.

$$M_\beta(\beta') = M_{\beta'}(\beta') \neq f(\beta').$$

Thm: There is a T.M. M s.t. given the code α and input x ,
if M_α takes T steps to compute $M_\alpha(x)$ then
 $M(\alpha, x)$ computes $M_\alpha(x)$ in time $O(CT \lg T)$, where
 C depends only on α .

Def: Say $t: \mathbb{N} \rightarrow \mathbb{N}$ is time constructible if
 $t(n) \geq n$ and on input x , \exists machine M
that computes $t(|x|)$ in time $O(t(|x|))$. \square

Thm: There is a machine M s.t. given code α and input x , if
 $M_\alpha(x)$ takes space $S \geq \lg |x|$ then M computes the same output
in space $O(CS)$, where C depends only on code α .

Def: Say $s: \mathbb{N} \rightarrow \mathbb{N}$ is space constructible if $s(n) \geq \lg n$ and $\exists M$ s.t.
on input x $M(x)$ computes $s(|x|)$ in space $O(s(|x|))$.

Thm: If a, s are space constructible with $q(n) = O(s(n))$
then $DSPACE(a(n)) \neq DSPACE(s(n))$.