Nondeterministic Polynomial Time

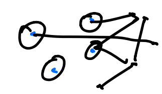
$L \subseteq P \subseteq E \subseteq Exp$, $P \subseteq NP \subseteq E$, $P \subseteq PSPACE \subseteq Exp$

Independent set:

$$ISet(G, k) = \begin{cases} 1 & \text{if } G \text{ has an independent set of size } k \\ 0 & \text{o.w.} \end{cases}$$

G: graph, k: number

independent set → a set of vertices with no induced edges.



Subset sum:

$$Subsum(a_1, \ldots, a_\ell, t) = \begin{cases} 1 & \text{if } \exists S \text{ s.t. } \sum_{i \in S} a_i = t \\ 0 & \text{o.w.} \end{cases}$$

Composit num:

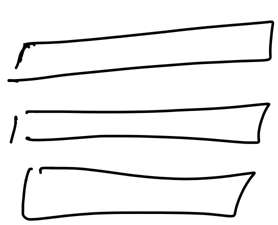
$$Comp(N) = \begin{cases} 1 & \text{if } N = A \cdot B,\ A, B \neq 1 \\ 0 & \text{o.w.} \end{cases}$$

Def: $f : \{0,1\}^* \to \{0,1\}$ is in NP if $\exists$ poly p and a polytime machine V s.t. $\forall x \in \{0,1\}^*$,

$$f(x) = 1 \iff \exists w \in \{0,1\}^{p(|x|)},\ V(x, w) = 1.$$

OPEN: $P \overset{?}{=} NP$.

Original Definition



Program

1. read x_i, y_i ... goto → line 5 / line 7
2.

⋮

$NTIME(t(n))$

$$NP = \bigcup_{c \geq 1} NTIME(n^c).$$

Thm: If r, t are time constructible functions with $r(n+1) \le o(t(n))$ then

$$\mathrm{NTIME}(r(n)) \neq \mathrm{NTIME}(t(n)).$$

Polytime Reductions

Def: f is polytime reducible to g if $\exists$ poly time computable h s.t $f(x) = g(h(x))$. $\boxed{f \le_p g}$.

Def: f is NP-hard if $g \le_p f$ for every $g \in NP$.

Def f is NP-complete if f is NP-hard and $f \in NP$.

Circuit-SAT

$$\text{Circuit-SAT}(\underbrace{C}_{\text{a Boolean circuit}}) = \begin{cases} 1 & \text{if } \exists w \text{ s.t } C(w) = 1 \\ 0 & \text{o.w} \end{cases}$$

Thm: Circuit-Sat is NP-hard.

Pf: Suppose $f \in NP$. WANT: $f \le_p$ Circuit-SAT

$\Leftrightarrow \exists h$ s.t $f(x) = \text{Circuit-SAT}(h(x))$.

We know $\exists V$ s.t

$f(x) = 1 \iff \exists w$ s.t $V(x,w) = 1 \Leftrightarrow \exists w$ s.t $C_x(w) = 1$

$h(x)$: output $C_x(w)$ s.t $C_x(w) = V(x,w)$.

$\blacksquare$

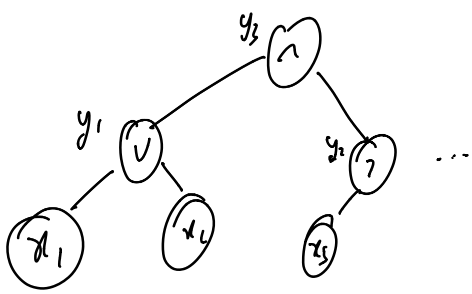
$f \leq_p$ Circuit-SAT $\leq_p$ 3-SAT $\leq_p$ Iset
$\leq_p$ Subset Sum

3-SAT

$$\phi := (x_1 \vee \bar{x}_2 \vee x_3) \wedge (x_5 \vee x_2 \vee \bar{x}_1) \wedge \dots$$

$$\text{3-SAT}(\phi) = \begin{cases} 1 & \text{if } \exists x \text{ s.t. } \phi(x) = 1 \\ 0 & \text{o.w.} \end{cases}$$

Thm: Circuit-SAT $\leq_p$ 3-SAT



$y_1 = x_1 \vee x_2$

$y_2 = \neg x_3$

$y_3 = y_1 \wedge y_2$

$\vdots$