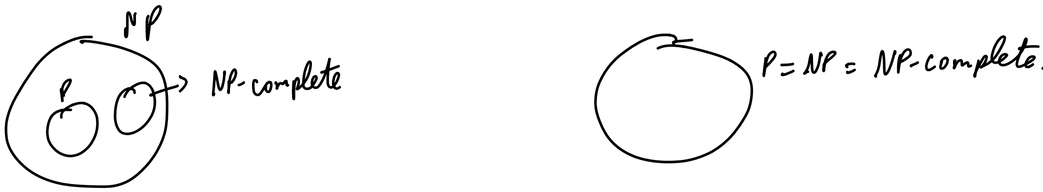
The Problem with Diagonalization and P vs NP

$P \stackrel{?}{=} NP$



Diagonalization cannot help!

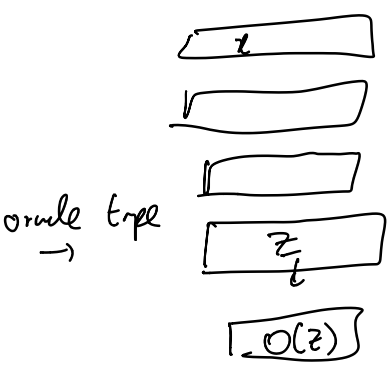
$$f(\alpha) = \begin{cases} 1 & \text{if } M_\alpha^O(\alpha) \text{ halts with } 0 \\ 0 & \text{o.w.} \end{cases}$$

Oracle

$O : \{0,1\}^* \to \{0,1\}$

Oracle machine:

Turing m. M^O has oracle tape



- Like a normal machine
- Can compute $O(z)$ in a single step.

P^O : poly time computable functions with oracle access to O.

Exp^O

$$P^{3\text{-}SAT} \supseteq NP \cup coNP$$

$coNP = \{f : \exists g \text{ s.t. } g(x) = 1 - f(x),\ g \in NP\}$

$f \in NP$. .....

$\Leftrightarrow \quad f(x) = 1 \Leftrightarrow \exists w \quad V(x,w) = 1$

$g \in coNP \quad g(x) = 0 \Leftrightarrow \exists w \quad V(x,w) = 1$

Thm: $\exists$ oracles A, B s.t

$$P^A = NP^A$$

$$P^B \neq NP^B.$$

Pf: $A(\alpha, x)$ simulate execution of $M_\alpha(x)$ for $2^{|x|}$ steps.
If $M_\alpha(x)$ halts, return output of $M_\alpha(x)$.
otherwise return 0.

Claim: $P^A = EXP$.

$P^A \subseteq EXP$: $A \in EXP$ so every $f \in P^A$ can be computed in EXP.

$EXP \subseteq P^A$: Suppose $M_\alpha(x)$ computes f in EXP. 2^{n^c}

$M_\beta(\alpha, x, y)$
Outputs $M_\alpha(x)$.

Run oracle on $(\beta, x, 1^{n^c})$

Claim: $NP^A = EXP$

$NP^A \subseteq EXP$

$EXP \subseteq P^A \subseteq NP^A$

B

$$f(x) = \begin{cases} 1 & \text{if } \exists y,\ |y| = |x|,\ B(y) = 1 \\ 0 & \text{o.w.} \end{cases}$$

$f \in EXP^B$, $f \in NP^B$, $\boxed{f \notin P^B}$ hard part

Define $B: \{0,1\}^* \to \{0,1\}$ in phases.

Initially set $B(x) = *$ for every x.

Let $\alpha_1, \alpha_2, \ldots$ be a enumeration of all machines M^B.

In phase $i = 1, 2, 3, \ldots$

Let n_i be large enough so that

$B(x) = *$ for every input x with $|x| = n_i$.

Run $M^B_{\alpha_i}(1^{n_i})$ for $2^{n_i/2}$ steps.

whenever $M^B_{\alpha_i}(1^{n_i})$ queries $B(x)$, $|x| = n_i$

set $B(x) = 0$, return 0.

if $M^B_{\alpha_i}(1^{n_i})$ halts in $2^{n_i/2}$ steps with output 1

set $B(x) = 0 \ \forall x$, $|x| = n_i$

if $M^B_{\alpha_i}(1^{n_i})$ halts in $2^{n_i/2}$ step with output 0

let $x \in \{0,1\}^{n_i}$ be s.t. x was not queried,

set $B(x) = 1$.

Suppose for contradiction that $f(x) = M^B_{\alpha_i}(x)$ and α_i is poly time $10^6 \cdot n^c$

WANT: $f(1^{n_i}) \neq M^B_{\alpha_i}(1^{n_i})$ $\quad 10^6 \cdot n_i^c \geq 2^{n_i/2}$

$\exists$ some large j s.t

$M^B_{\alpha_i}$ and $M^B_{\alpha_j}$ are equivalent.

yet $10^6 n_j^c \leq 2^{n_j/2}$

$M^B_{\alpha_i}(1^{n_j}) = M^B_{\alpha_j}(1^{n_j}) \neq f(1^{n_j})$ ☒