CSE431: Complexity Theory	February 10, 2023
Homework 3	
Anup Rao	Due: February 19, 2023

Read the fine print[1]. Each problem is worth 10 points:

1. Recall the following function:

$$2\mathbf{COL}(G) = \begin{cases} 1, & \text{if graph } G \text{ has a coloring with two colors} \\ 0, & \text{otherwise,} \end{cases}$$

   where a coloring of G with c colors is an assignment of a number in $[c]$ to each vertex such that no adjacent vertices get the same number.

   Prove that $2\mathbf{COL} \in \mathbf{NL}$. You can use the following fact: A graph G can be colored with two colors if and only if it contains no cycle of odd length.

2. Consider the following game between two players: Given a directed graph $G = (V, E)$, and a start vertex s, the players (starting with Player 1) alternately choose an outgoing edge incident to the current vertex to reach a vertex that was not previously visited. If one of the players cannot choose a next vertex, he loses. Let $\mathbf{GAME}(G)$ be the function that is 1 if and only if Player 1 has a strategy that ensures that she always wins no matter what Player 2 does.

   Show that **GAME** is in **PSPACE**.

3. Let **PATH** be the following function:

$$\mathbf{PATH}(G, s, t, k) = \begin{cases} 1, & \text{if the shortest path between vertices } s \text{ and } t \text{ has length exactly } k \\ 0, & \text{otherwise.} \end{cases}$$

   Prove that $\mathbf{PATH} \in \mathbf{NL}$.

4. Suppose **TQBF** is also **PSPACE**-complete under log-space reductions—meaning that for every $f \in \mathbf{PSPACE}$, there is a logspace computable function h such that $f(x) = \mathbf{TQBF}(h(x))$. Prove that this implies that $\mathbf{TQBF} \notin \mathbf{NL}$. Hint: Use Savitch's theorem and one of the hierarchy theorems.

[1] In solving the problem sets, you are allowed to collaborate with fellow students taking the class, but **each submission can have at most one author**. If you do collaborate in any way, you must acknowledge, for each problem, the people you worked with on that problem. The problems have been carefully chosen for their pedagogical value, and hence might be similar to those given in past offerings of this course at UW, or similar to other courses at other schools. Using any pre-existing solutions from these sources, for from the web, constitutes a violation of the academic integrity you are expected to exemplify, and is strictly prohibited. Most of the problems only require one or two key ideas for their solution. It will help you a lot to spell out these main ideas so that you can get most of the credit for a problem even if you err on the finer details. Please justify all answers. Some other guidelines for writing good solutions are here: `http://www.cs.washington.edu/education/courses/cse421/08wi/guidelines.pdf`.