

## Lecture 16: Permanent and Interactive Proofs

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### Interactive proofs

One way to define **NP** is via the idea of a proof system. **NP** is the set of functions  $f$  for which there is a polynomial time verifier algorithm  $V$  such that given any  $x$  with  $f(x) = 1$ , there exists a prover  $P$  that can prove to the verifier that  $f(x) = 1$  by providing a polynomial sized witness  $w$  for which  $V(x, w) = 1$ , yet if  $f(x) = 0$ , no such prover exists.

What happens if we allow the verifier to have a longer *interactive* conversation? Presumably, giving the verifier the ability to adaptively ask the prover questions based on his previous responses should give the verifier more power, and so allow the verifier to verify the correctness of the value for a larger set of functions. In fact, this does *not* give the verifier additional power: for if there is such an interactive verifier  $V^I$  for verifying that  $f(x) = 1$ , we can design a non-interactive verifier that does the same job. The new verifier will demand that the prover provide the entire transcript of interactions between  $V^I$  and a convincing prover. The new verifier can then verify that the transcript is correct, and would have convinced  $V^I$ . Thus, if  $f$  has an interactive verifier, then  $f \in \mathbf{NP}$ .

The story is more interesting if we allow the verifier to be randomized. We say that  $f \in \mathbf{IP}$  if there is a polynomial time randomized verifier  $V$  such that

*Completeness* For all  $x$ , if  $f(x) = 1$ , there is an oracle  $P$  such that  $\Pr_r[V^P(x, r) = 1] \geq 2/3$ .

*Soundness* For all  $x$ , if  $f(x) = 0$ , for every oracle  $P$ ,  $\Pr_r[V^P(x, r) = 1] \leq 1/3$ .

Since any prover can be simulated in polynomial space, if  $f \in \mathbf{IP}$ , then  $f \in \mathbf{PSPACE}$ . The algorithm for  $f$  can just try all possible sequences of messages from the prover until it finds a sequence of messages that convinces the verifier, if such a sequence exists.

**Theorem 1.**  $\mathbf{IP} \subseteq \mathbf{PSPACE}$ .

It is easy to check that allowing the prover to be randomized does not change the model.

We shall eventually prove that  $\mathbf{IP} = \mathbf{PSPACE}$  (and so  $\mathbf{IP}$  is potentially much more powerful than  $\mathbf{NP}$ ).

*Example: Graph non-Isomorphism*

Two graphs on  $n$  vertices are said to be *isomorphic* if the vertices of one of the graphs can be permuted to make the two equal.

Consider the problem of testing whether two graphs are *not* isomorphic: the boolean function  $f$  such that  $f(G_1, G_2)$  is 1 if and only if  $G_1$  is not isomorphic to  $G_2$ .  $f \in \text{coNP}$ , since the prover can just send the verifier the permutation that proves that they are isomorphic. We do not know if  $f \in \text{NP}$ , but it is easy to prove that  $f \in \text{IP}$ .

Here is the simple interactive protocol:

1. The verifier picks a random  $i \in \{1, 2\}$ .
2. The verifier randomly permutes the vertices of  $G_i$  and sends the resulting graph to the prover.
3. The prover responds with  $b \in \{1, 2\}$ .
4. The verifier accepts if  $i = b$ .

If  $G_1, G_2$  are not isomorphic, then any permutation of  $G_i$  determines  $i$ , so the prover can determine  $i$  and send it back. However, if  $G_1, G_2$  are isomorphic, then the graph that the prover receives has the same distribution whether  $i = 1$  or  $i = 2$ , thus the prover can guess the value of  $i$  with probability at most  $1/2$ . Repeating the protocol several times, the verifier can make the probability of being duped by a lying prover exponentially small.