NP-completeness

- Many many problems are NP-complete
- If you solve one of them efficiently, you solve all of them efficiently
- We don't know how to solve any of them efficiently

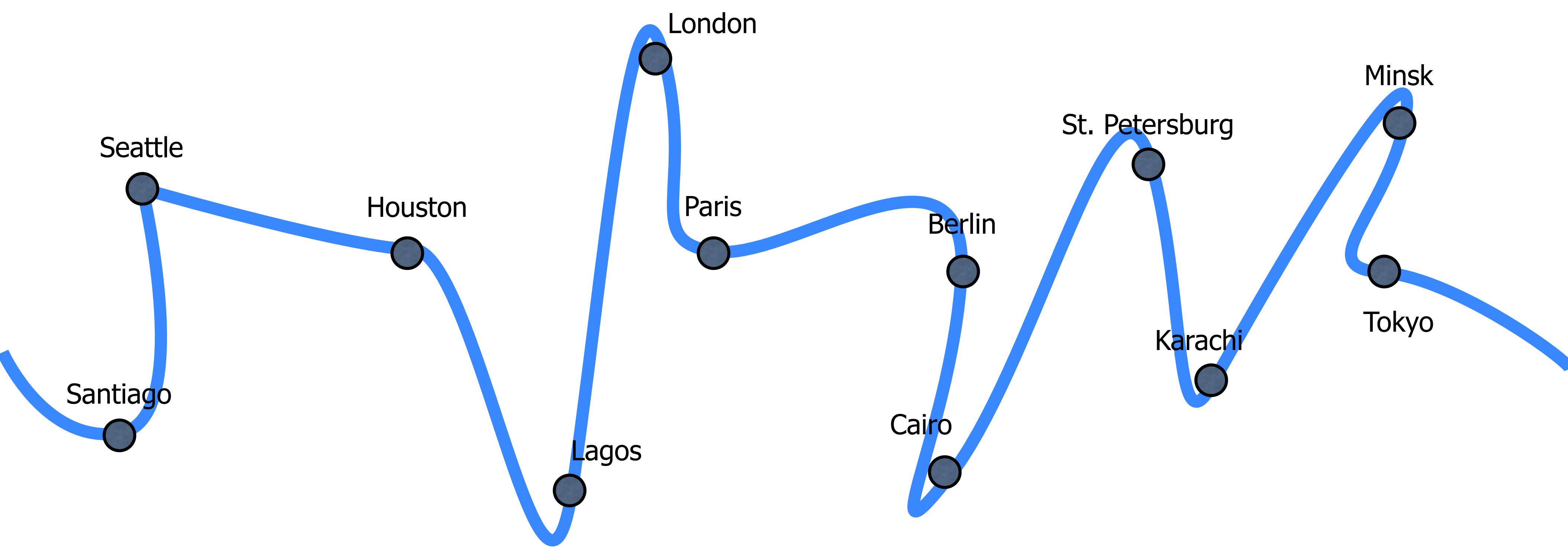
Approximation Algorithms

- So it's unlikely we'll solve one of these soon :(
- Instead of finding the best solution, we'll find a solution that is close :)

Traveling Salesman

Given: n cities with distances

Goal: Compute shortest tour to visit them



Traveling Salesman

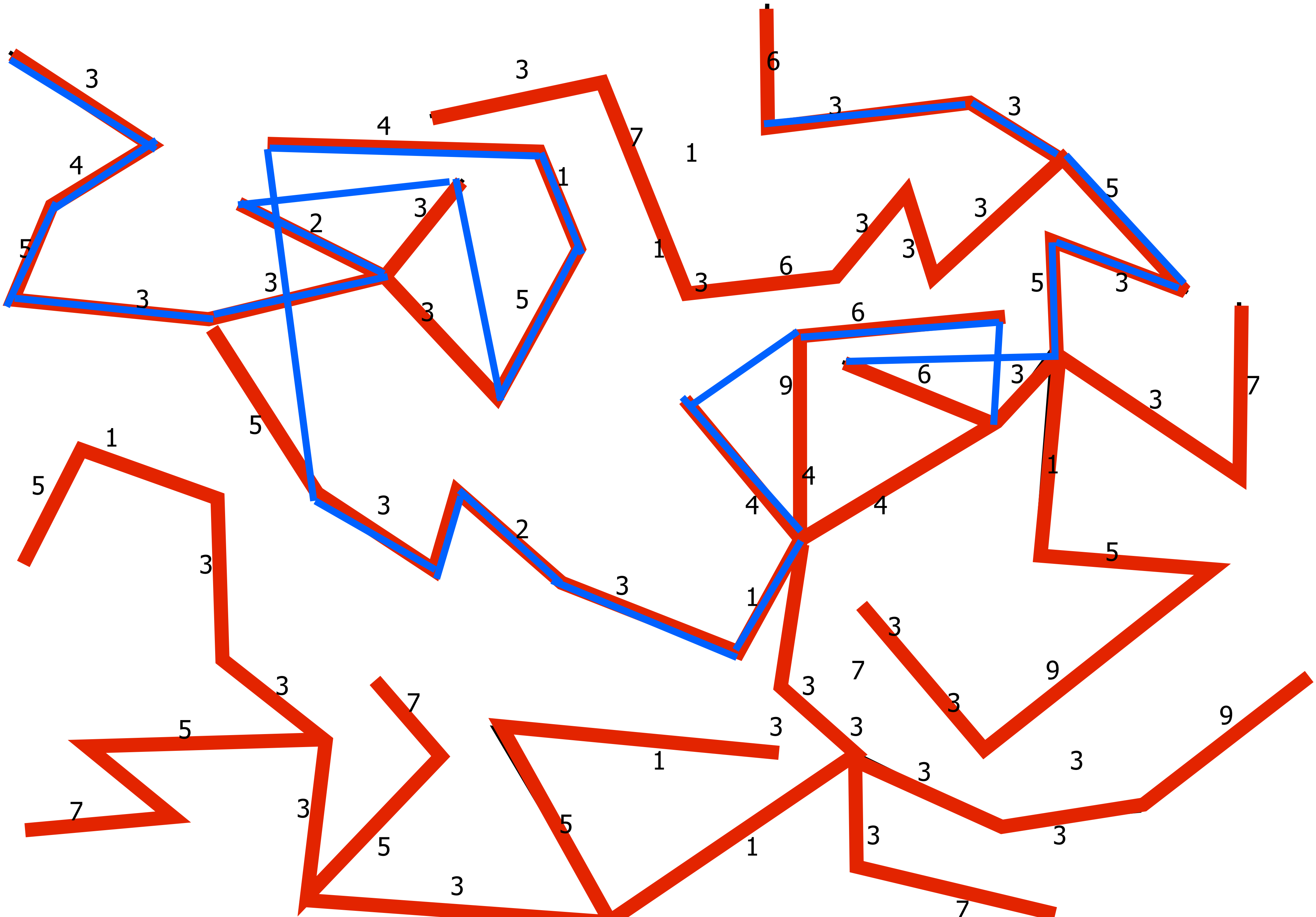
Given: n cities with distances
Goal: Compute shortest tour to visit them

Metric TSP: distances satisfy triangle inequality:

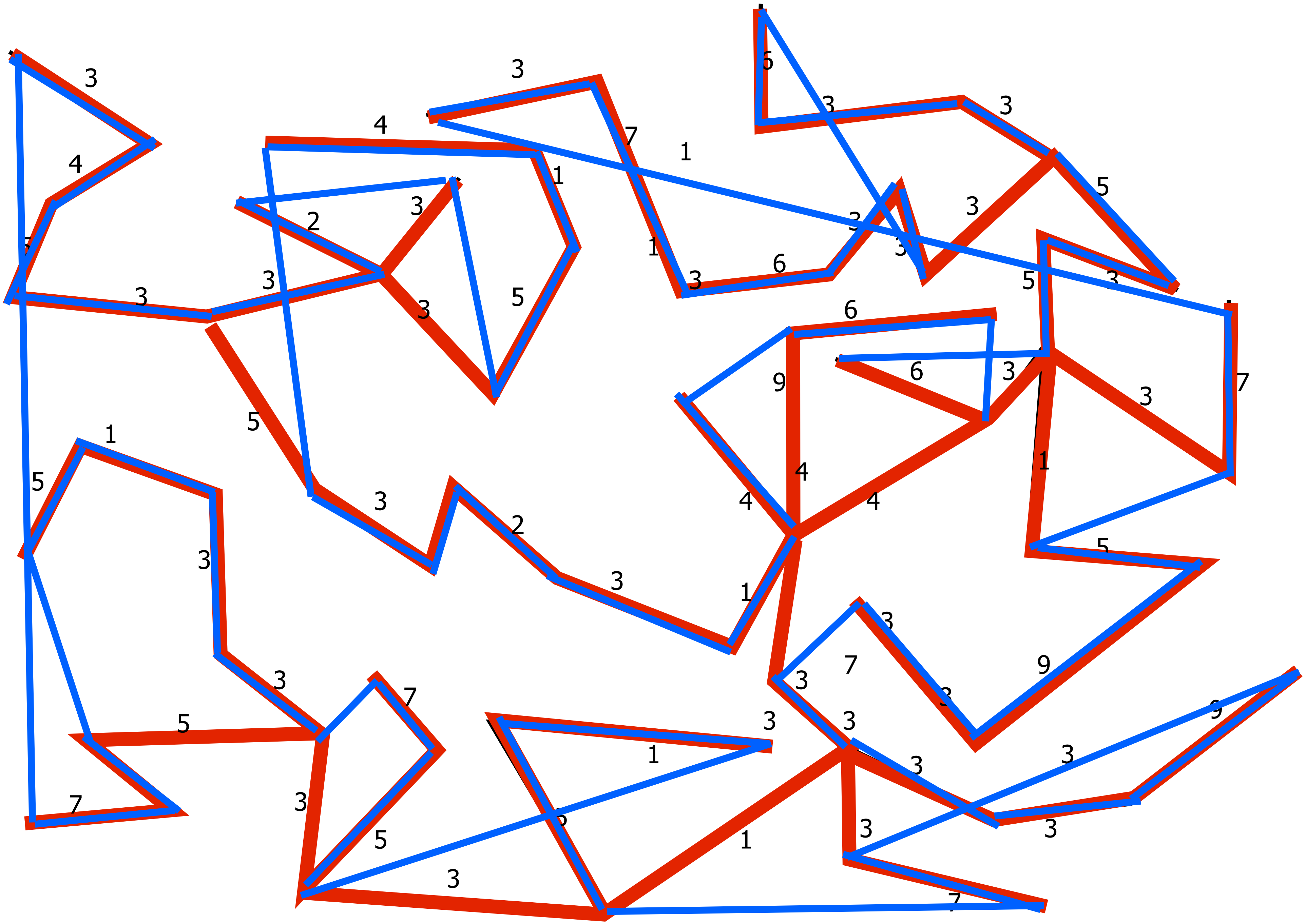
distance(a,c) ≤ distance(a,b) + distance(b,c)

Idea: use MST!
Prove: tour within factor 2 of best possible

MST tour: Show that it is within factor 2!



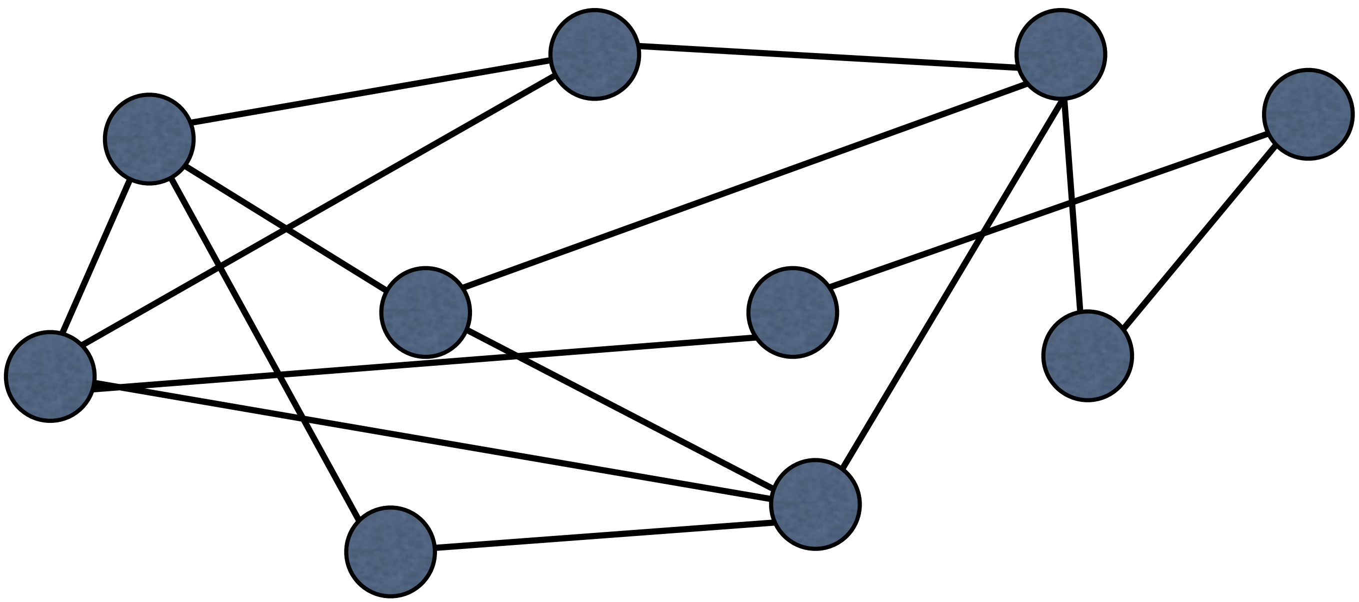
MST tour: Take the Euler tour of tree.



Claim: Every tour costs at least as much as MST.
Pf: Every tour contains a spanning tree

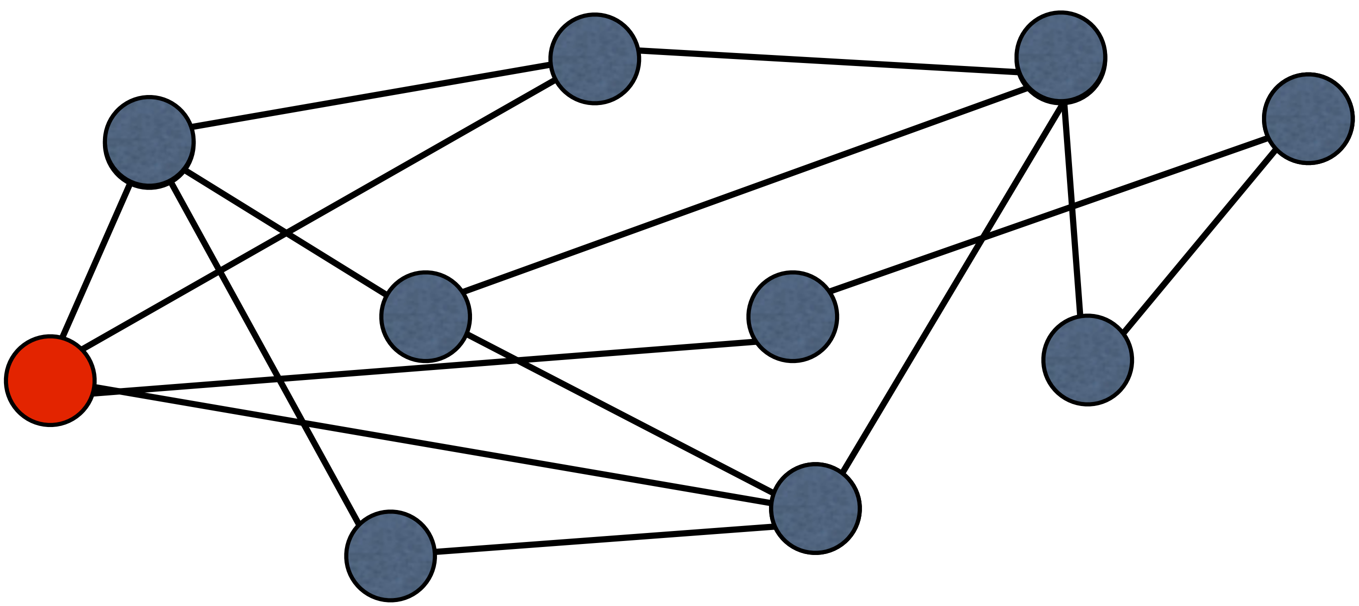
Claim: Euler tour costs at most 2 MST.
Pf: Can carry out Euler tour using each edge at most 2 times.

Vertex Cover



Find smallest set of
vertices touching
every edge

Vertex Cover



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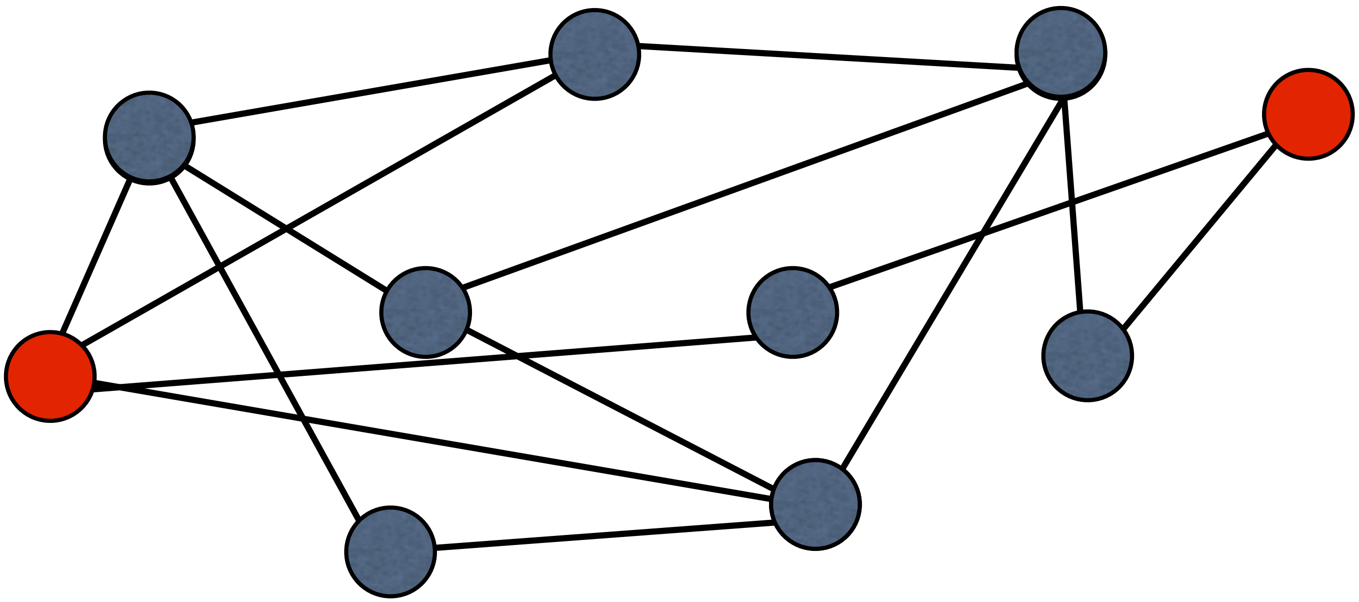
Vertex Cover



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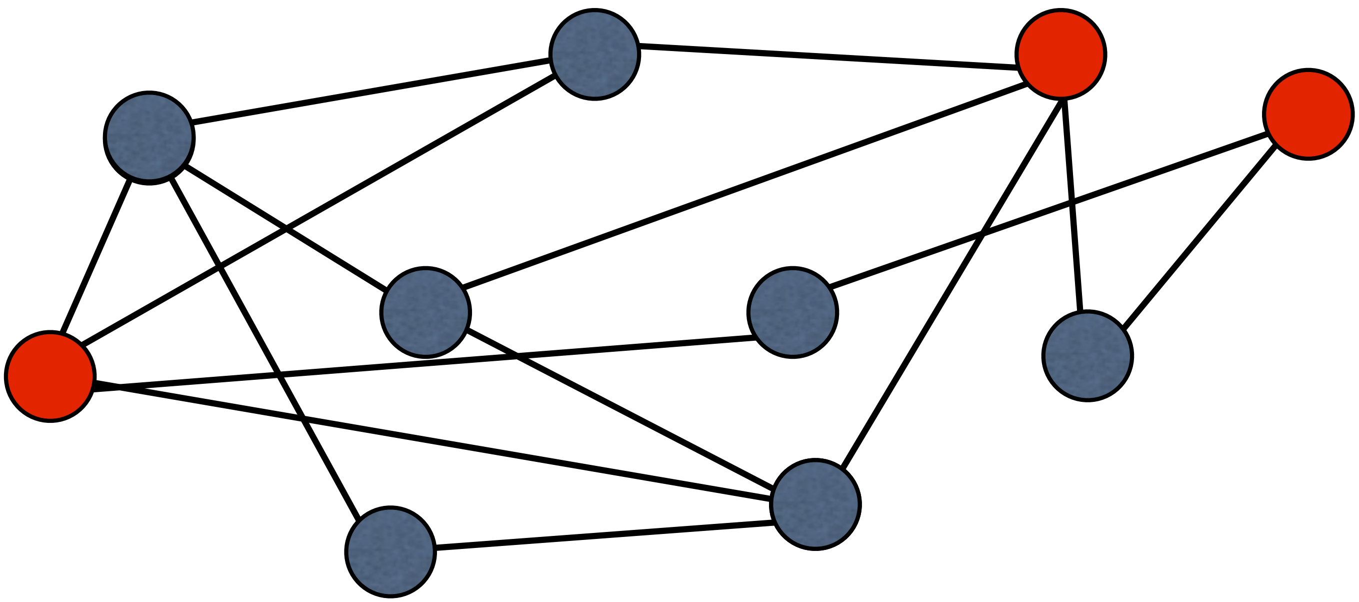

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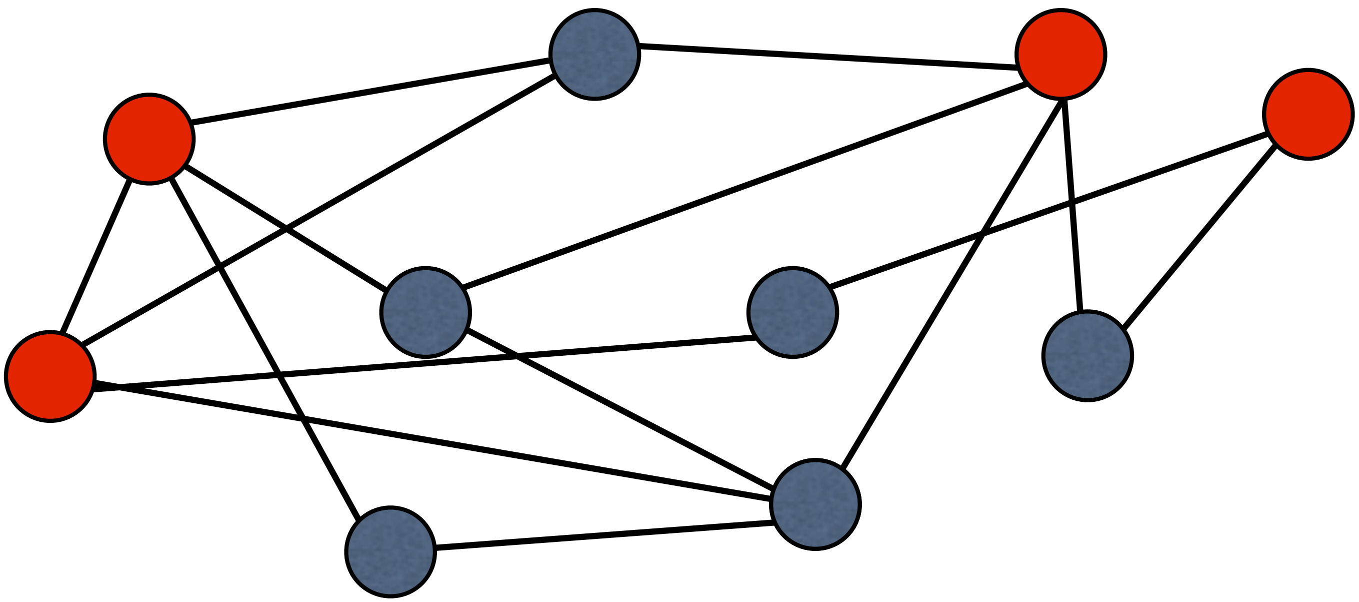

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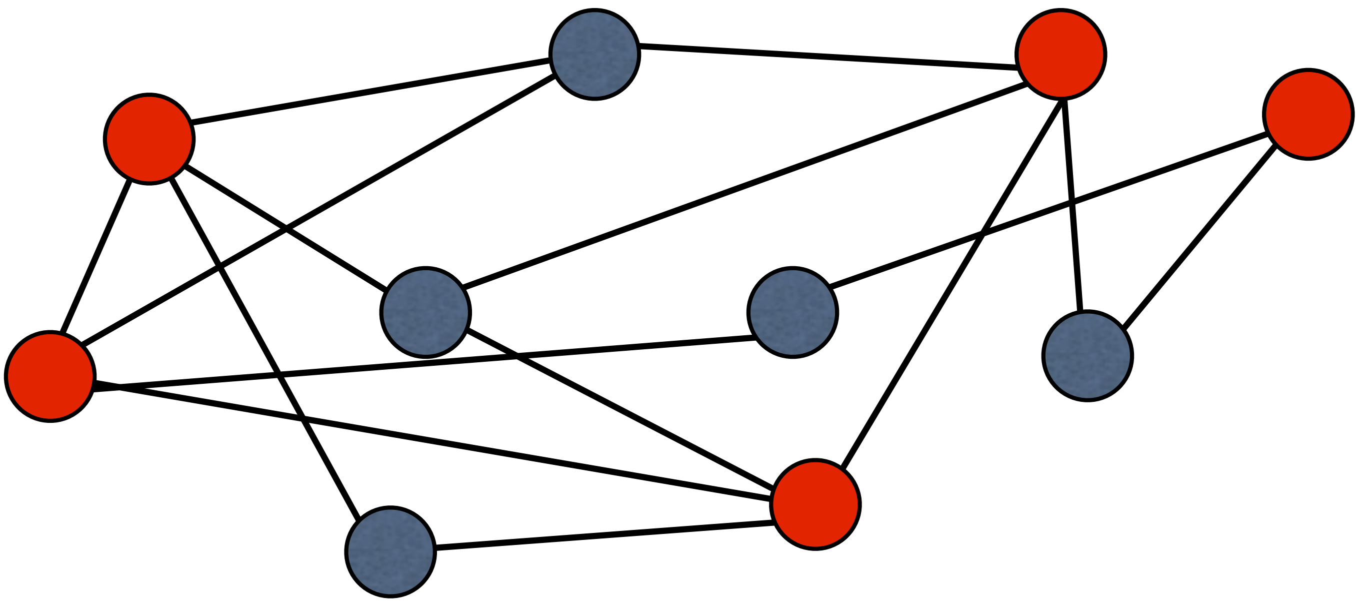
Vertex Cover



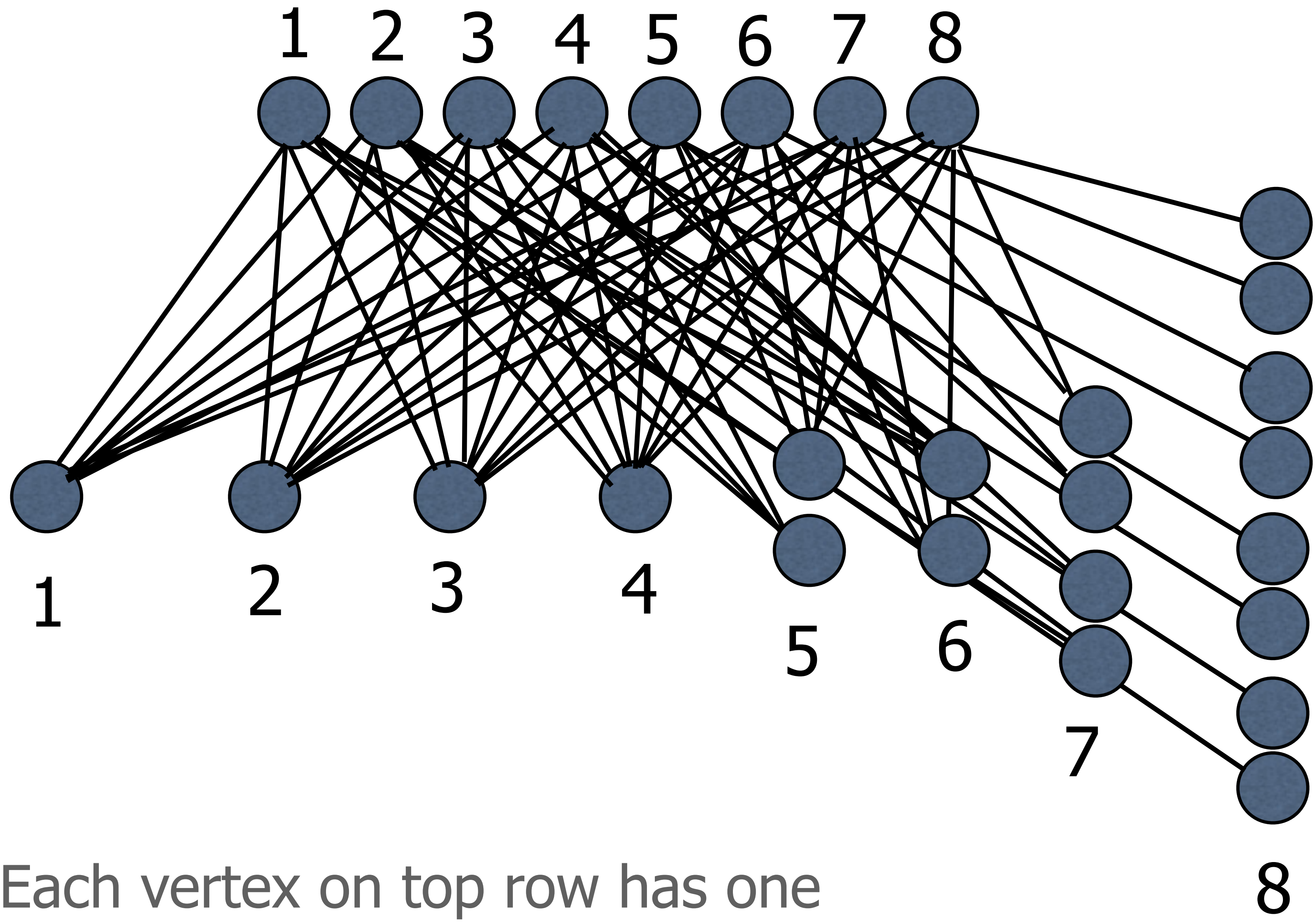
Find smallest set of vertices touching every edge

Vertex Cover size 5

Greedy algorithms?

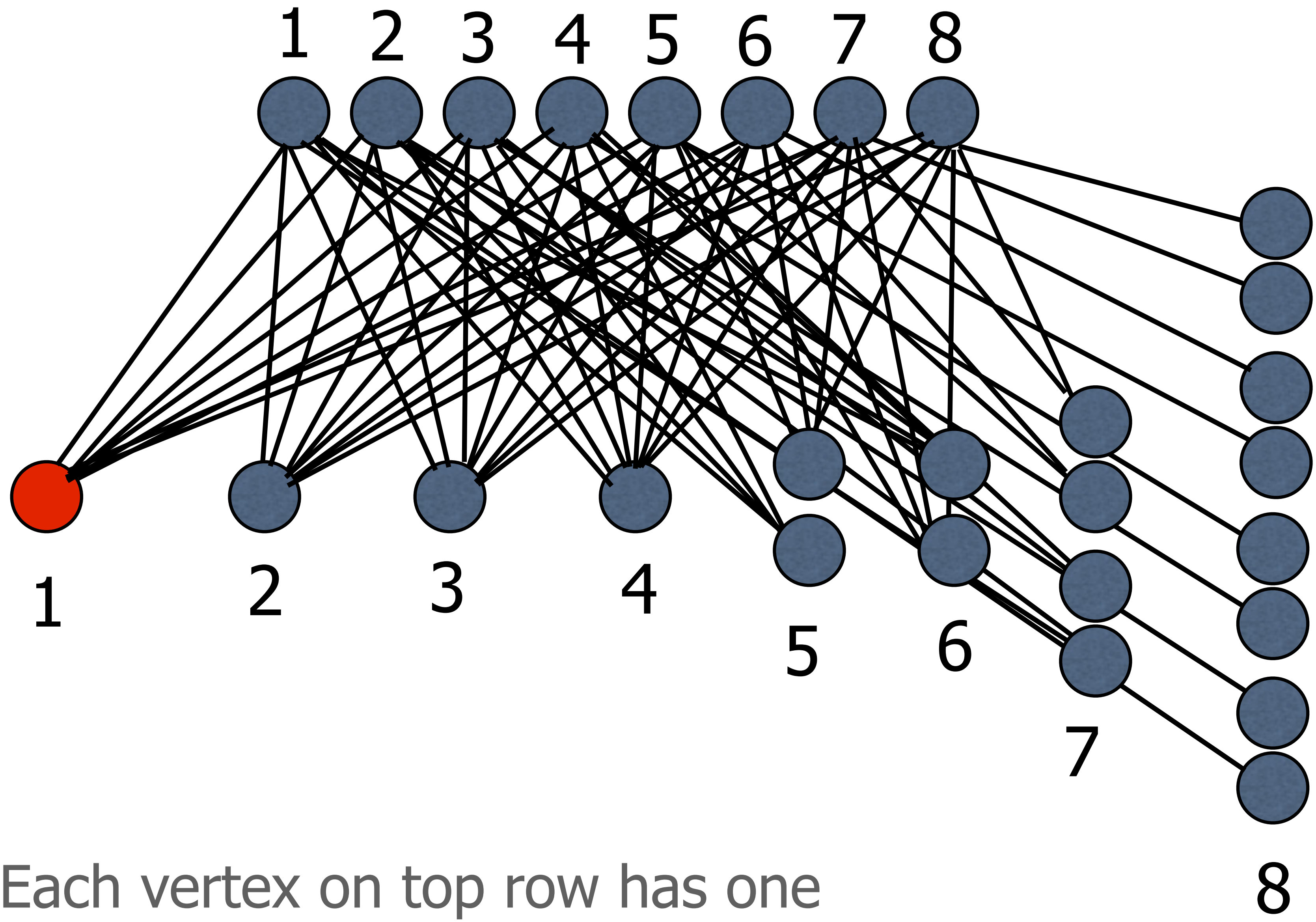
- Include vertex that covers most new edges?

Algorithm: Pick vertex that covers most new edges



Each vertex on top row has one edge into each of the groups below.

Algorithm: Pick vertex that covers most new edges



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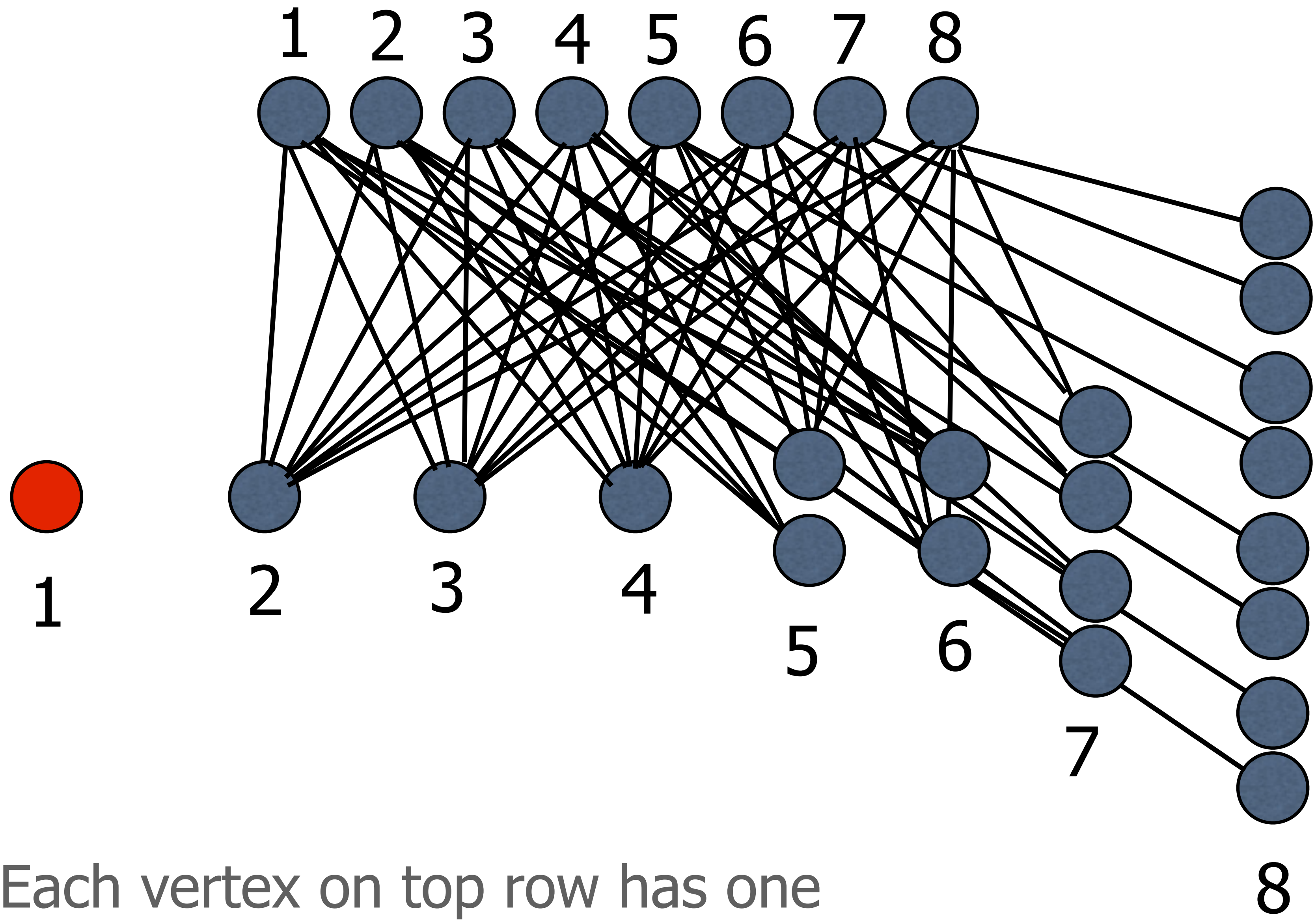

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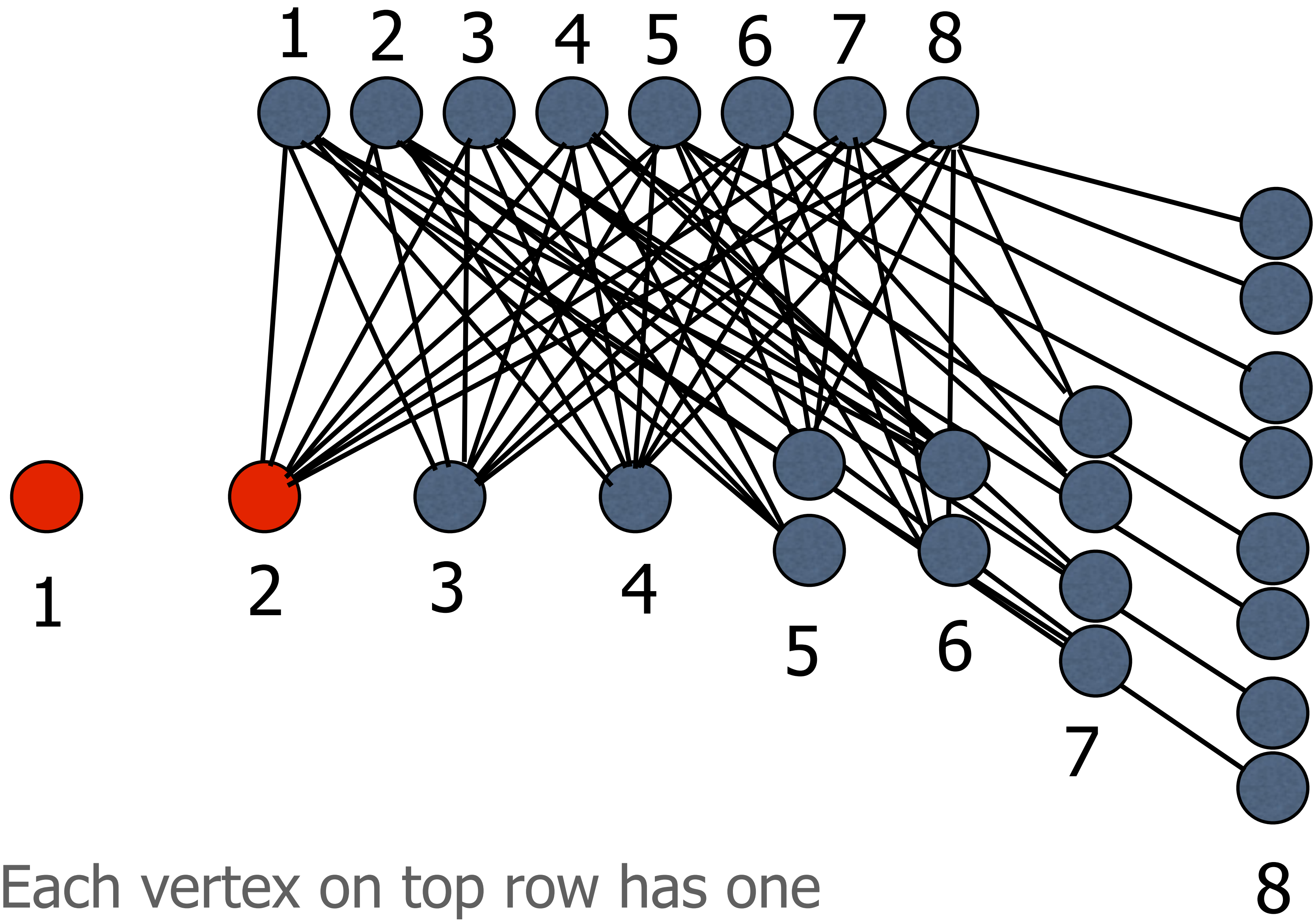

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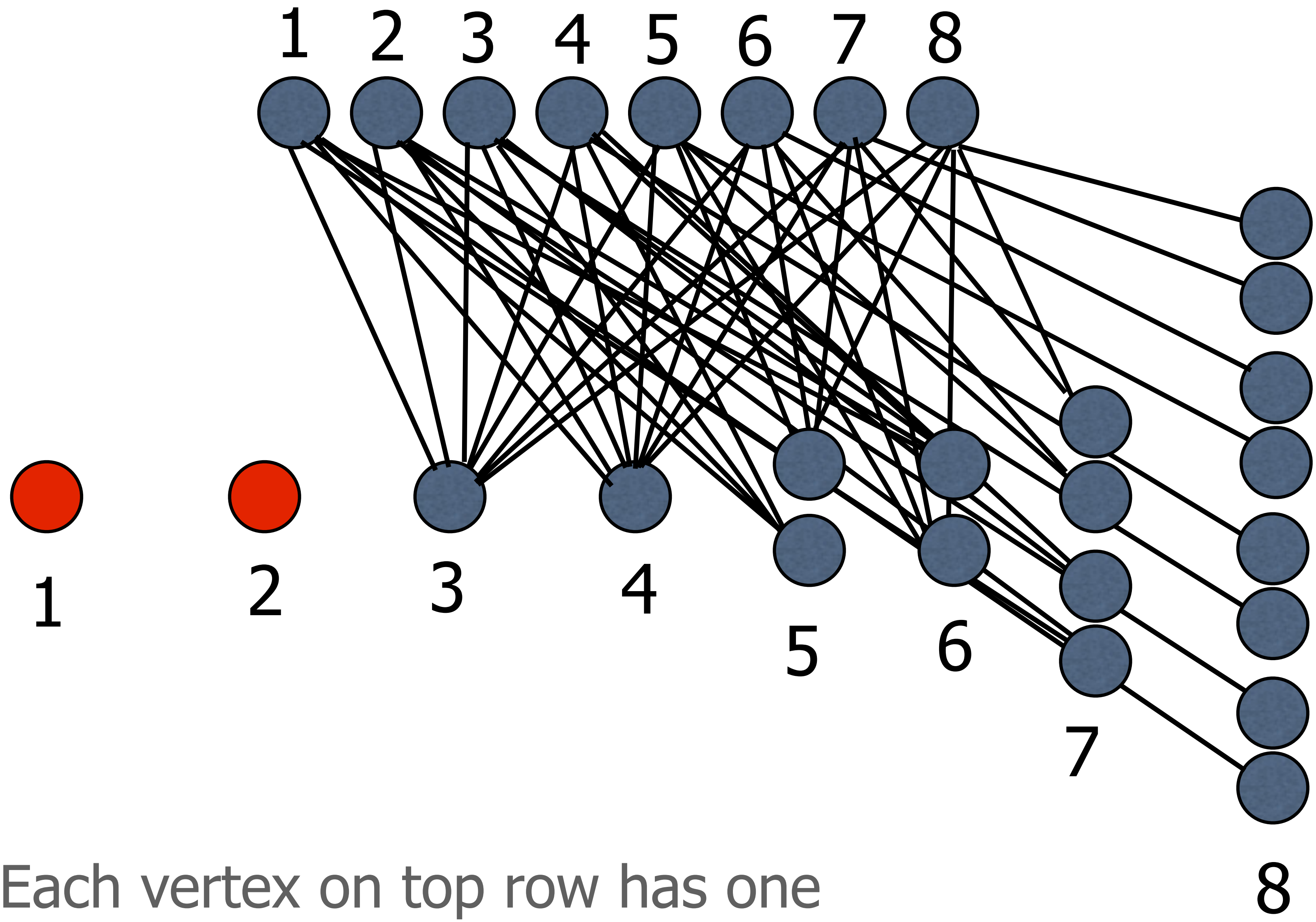

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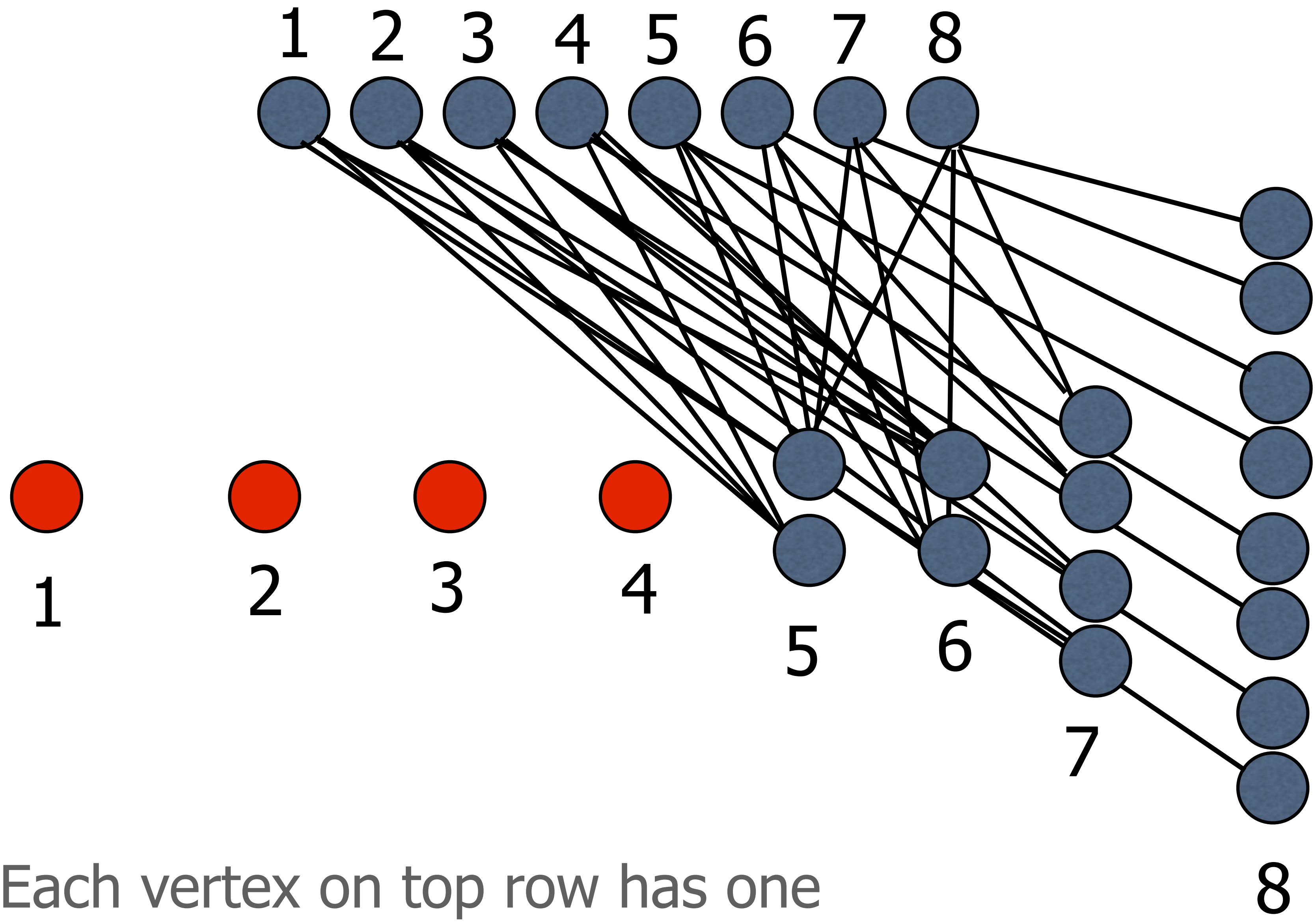

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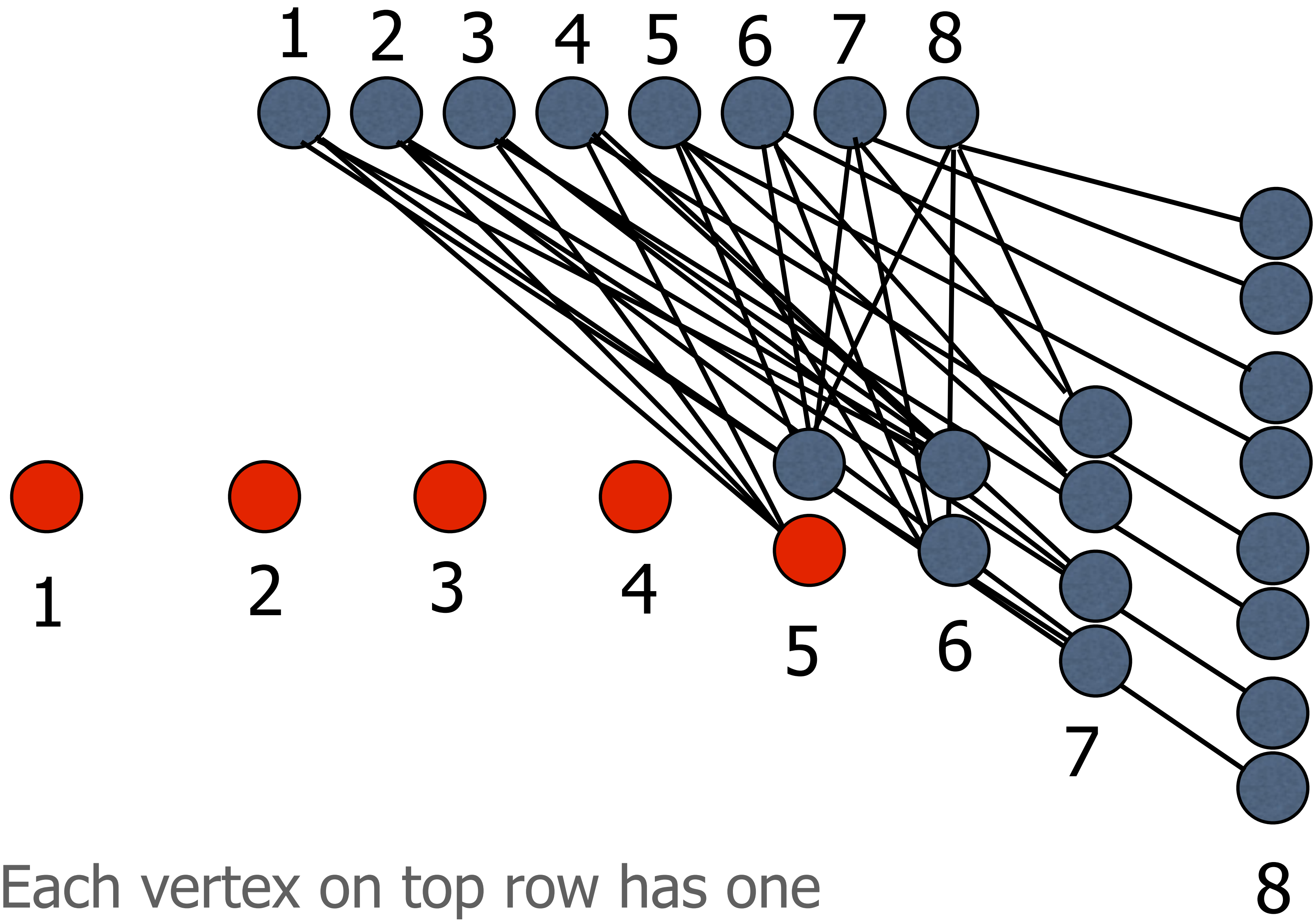

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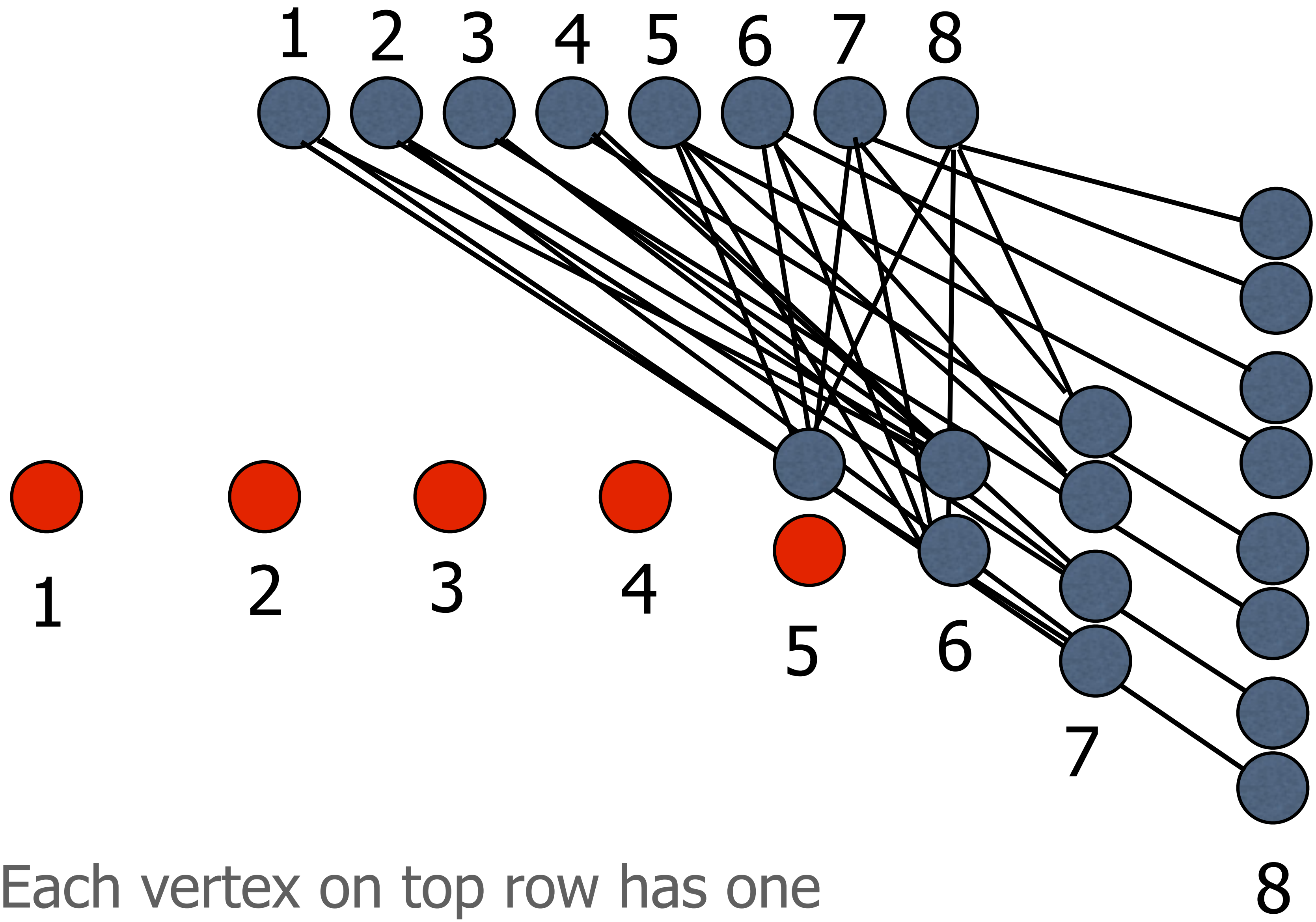

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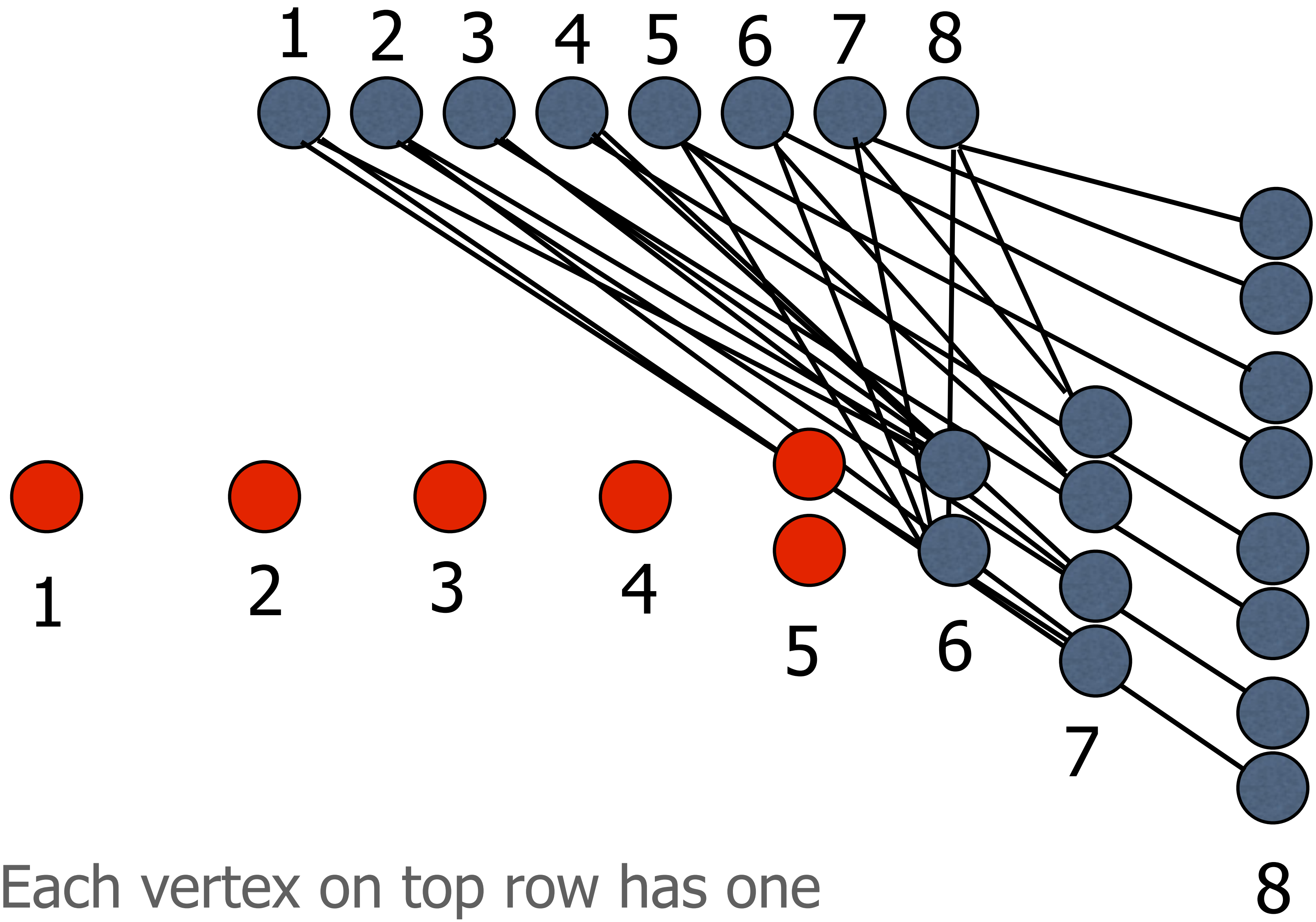

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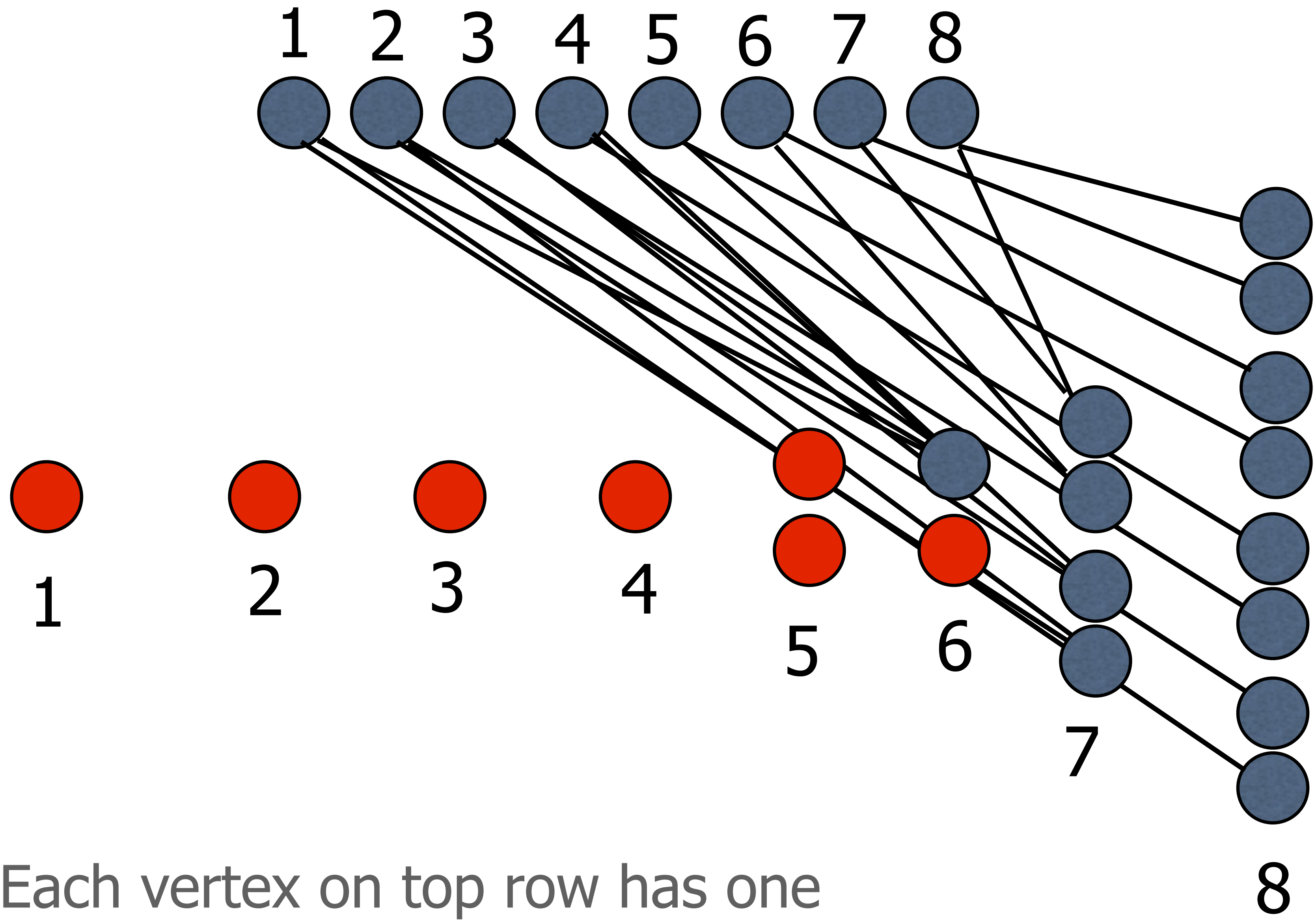

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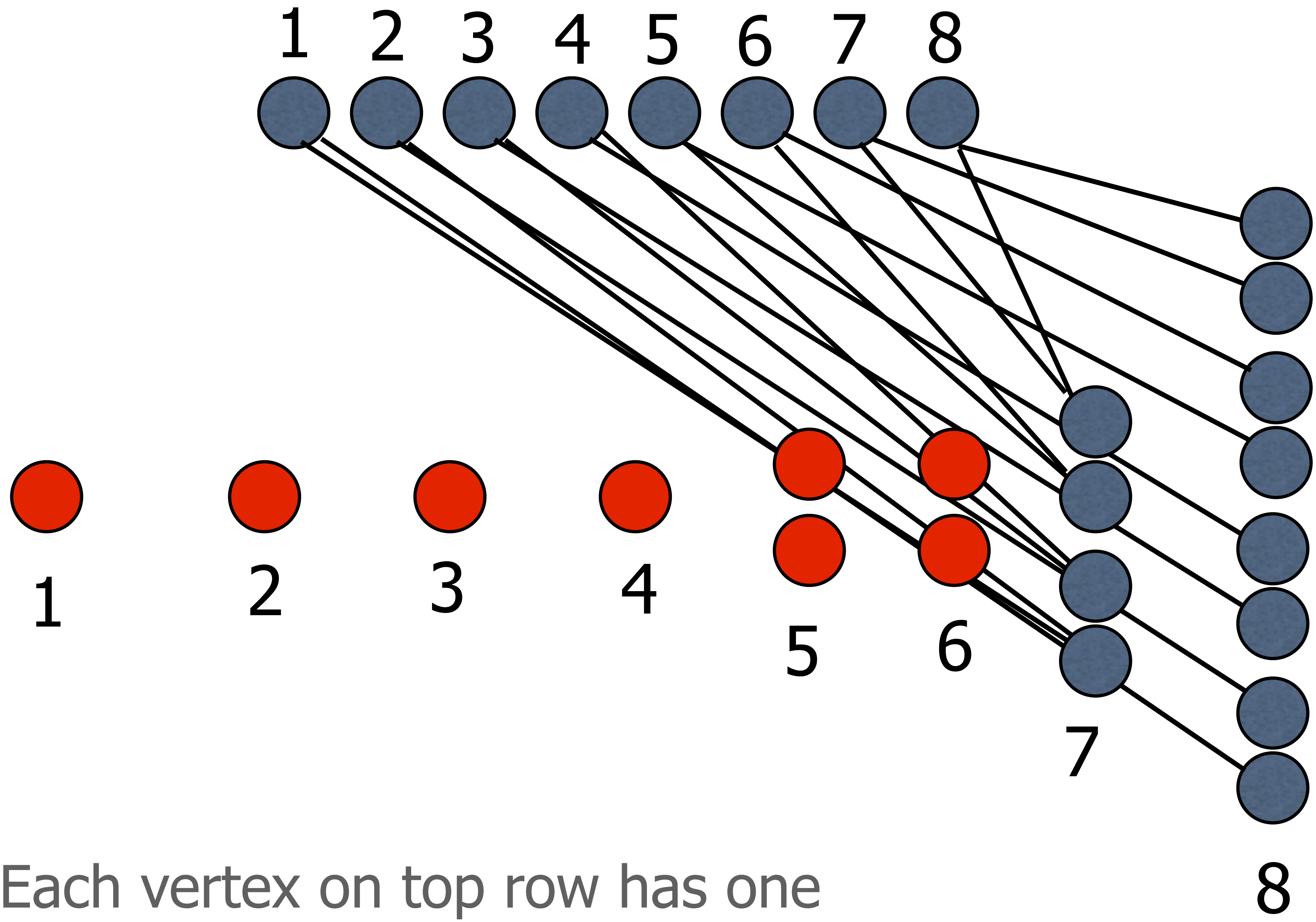

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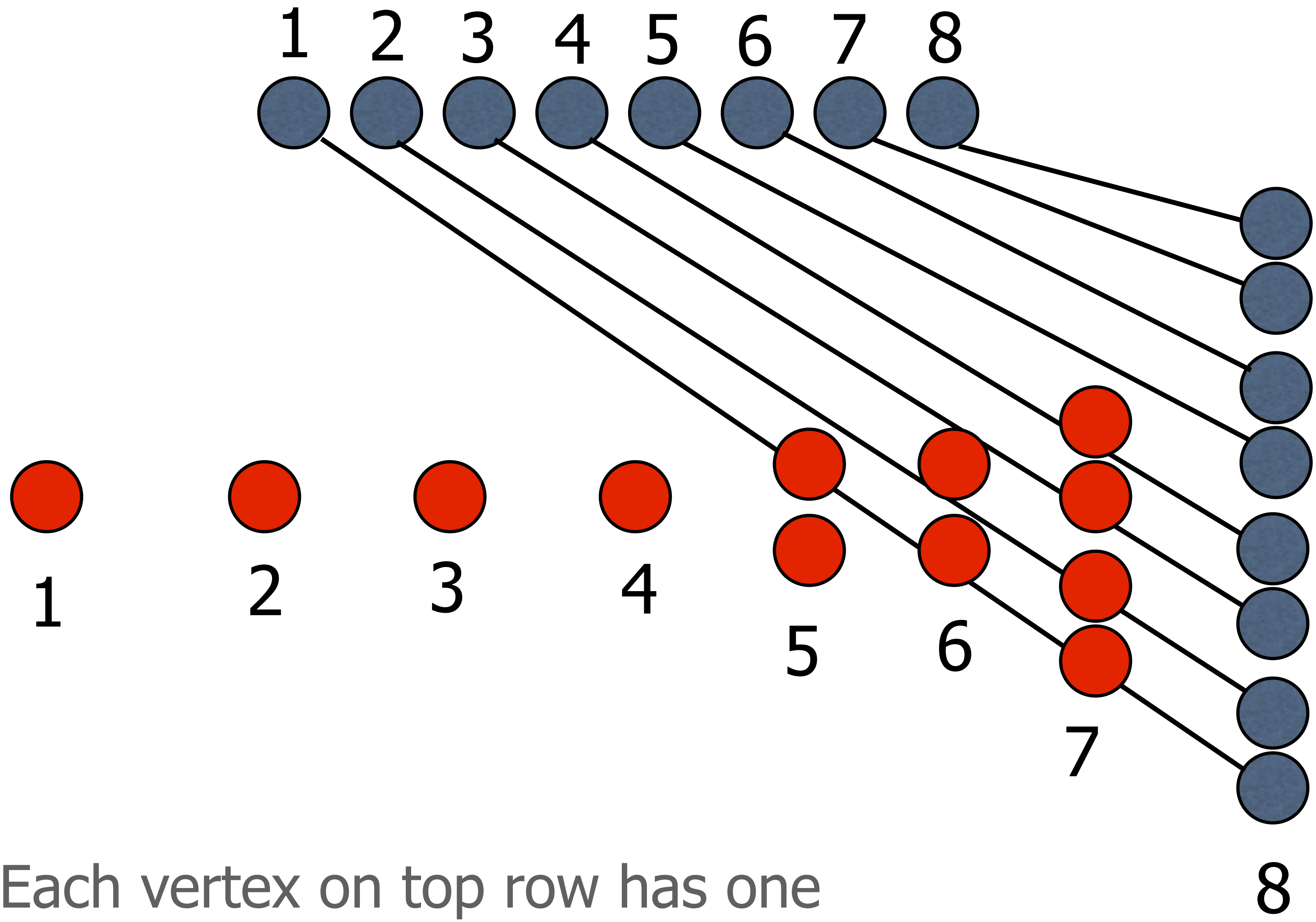

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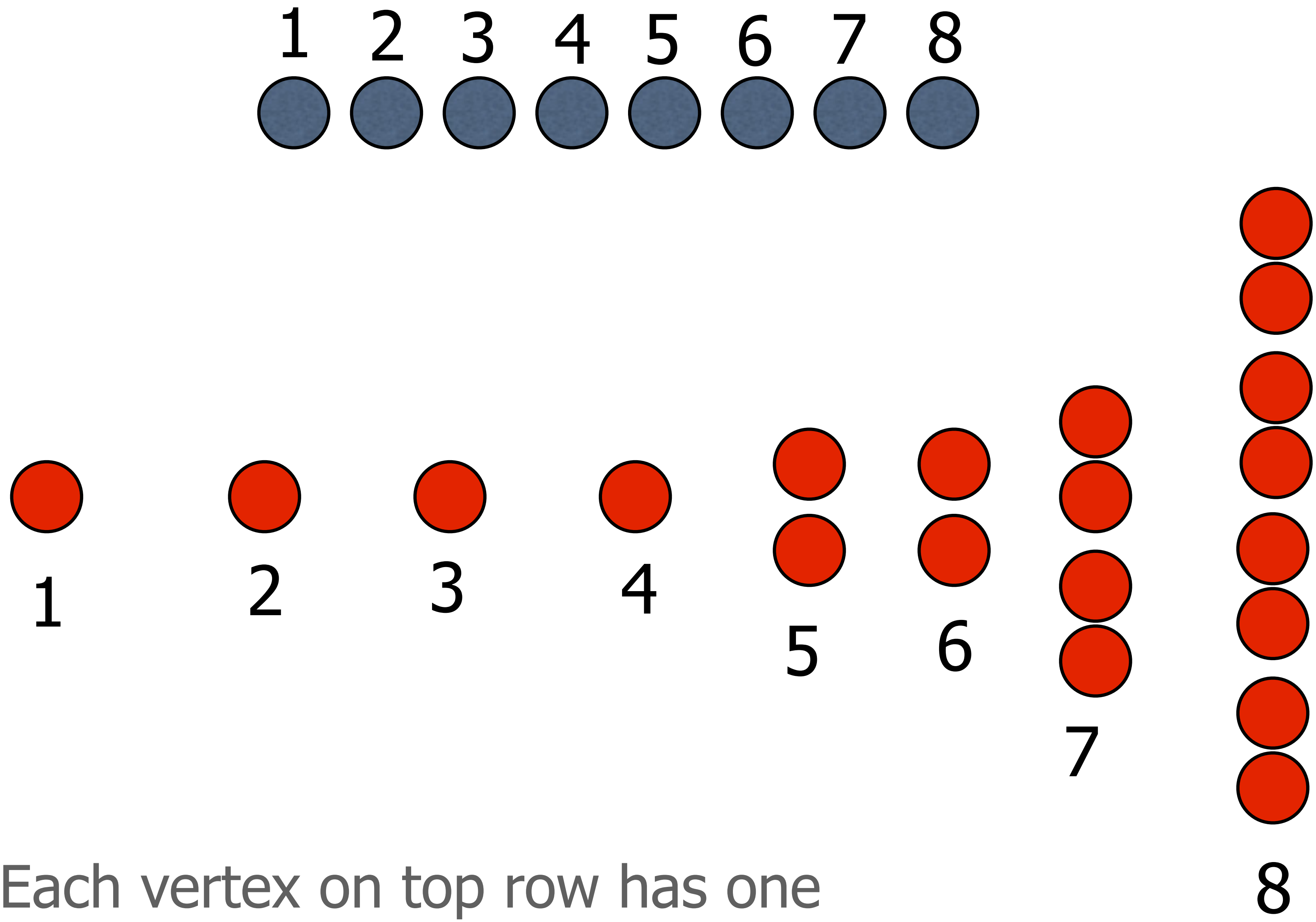

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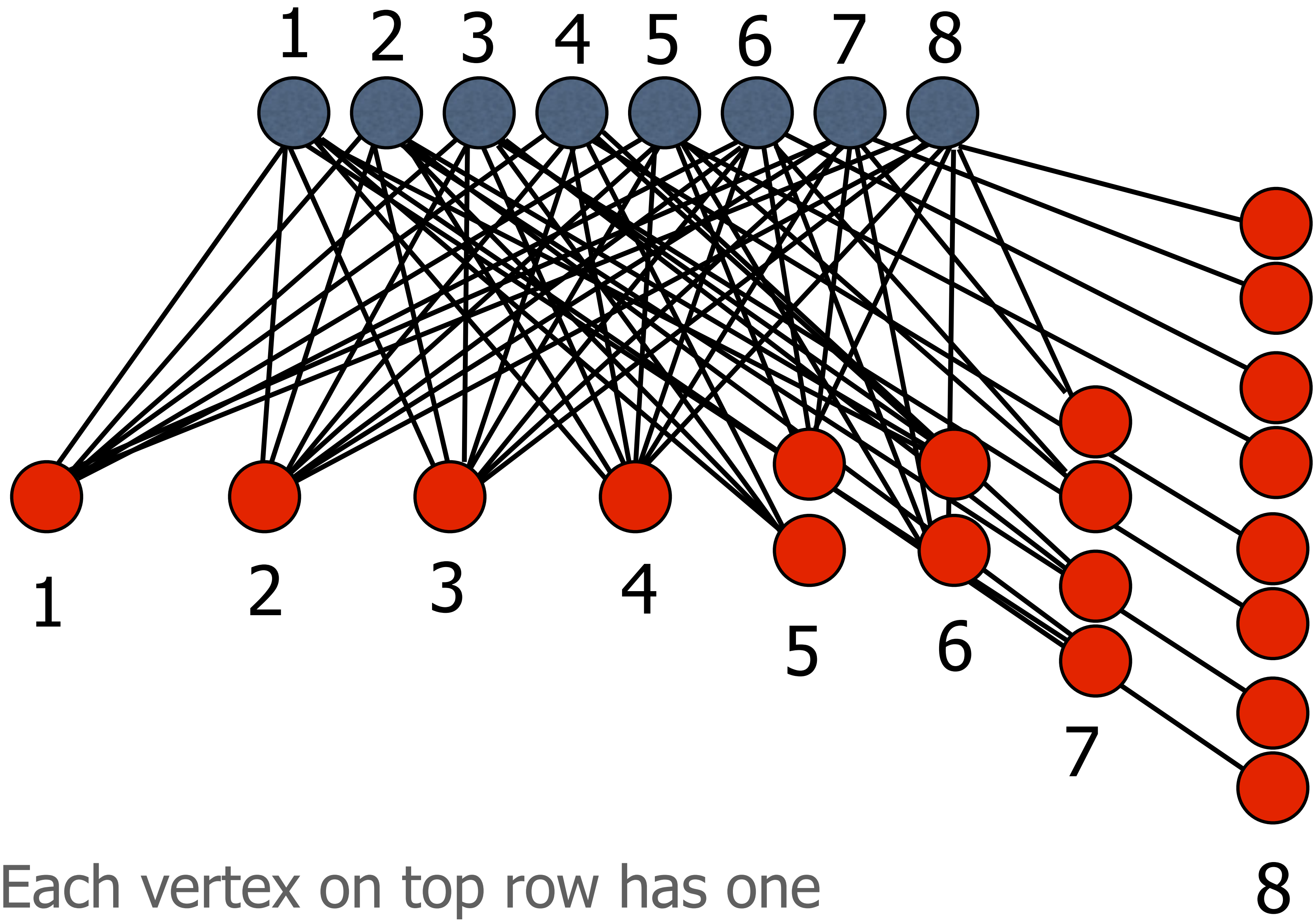

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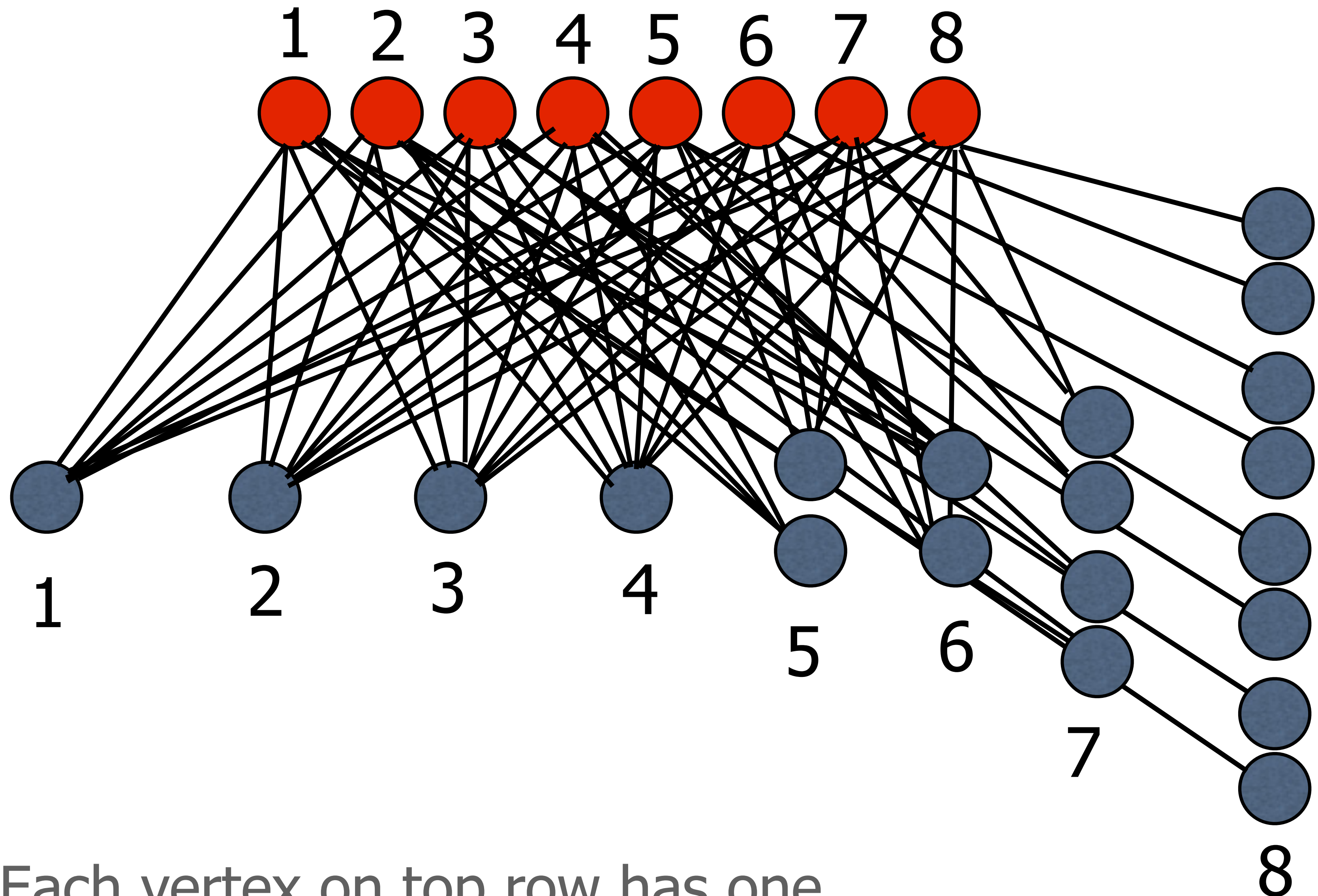
Vertex Cover size 20

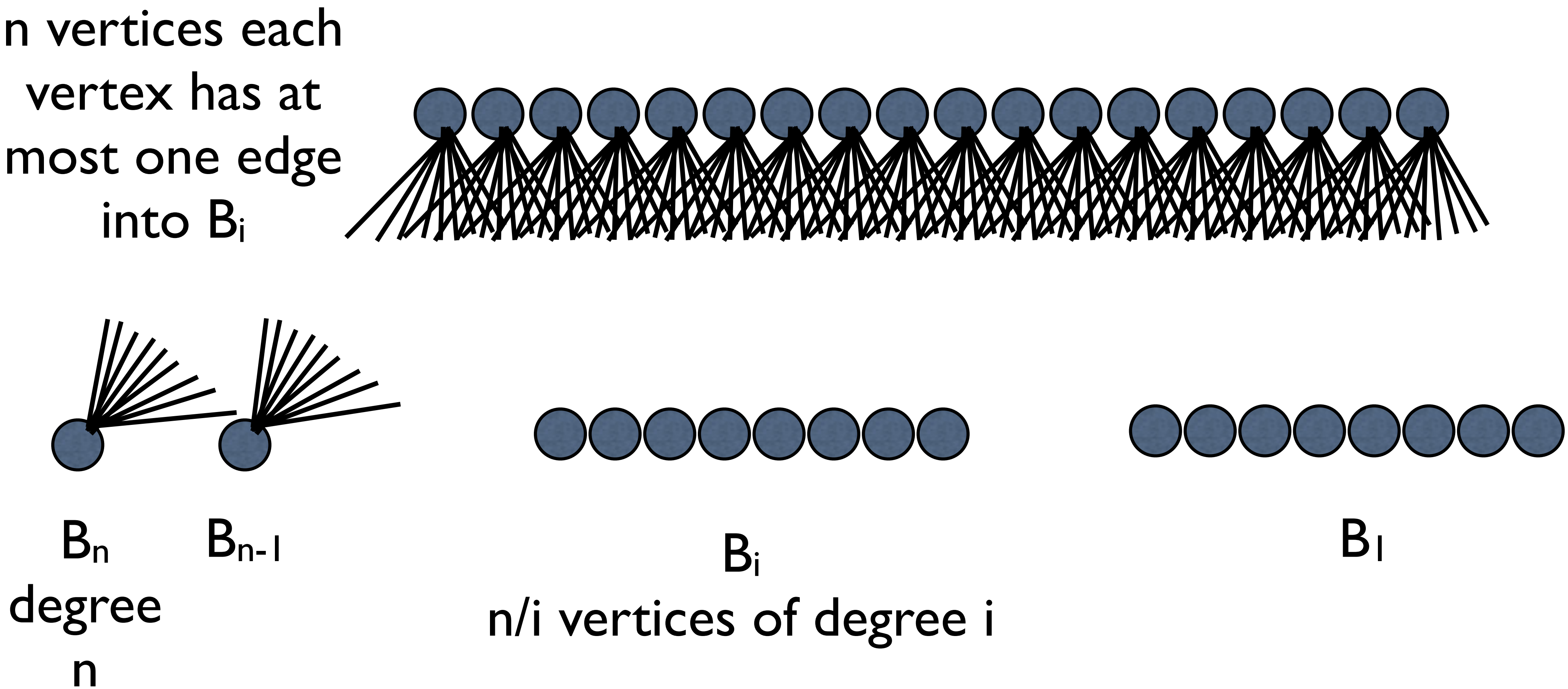
Algorithm: Pick vertex that covers most new edges



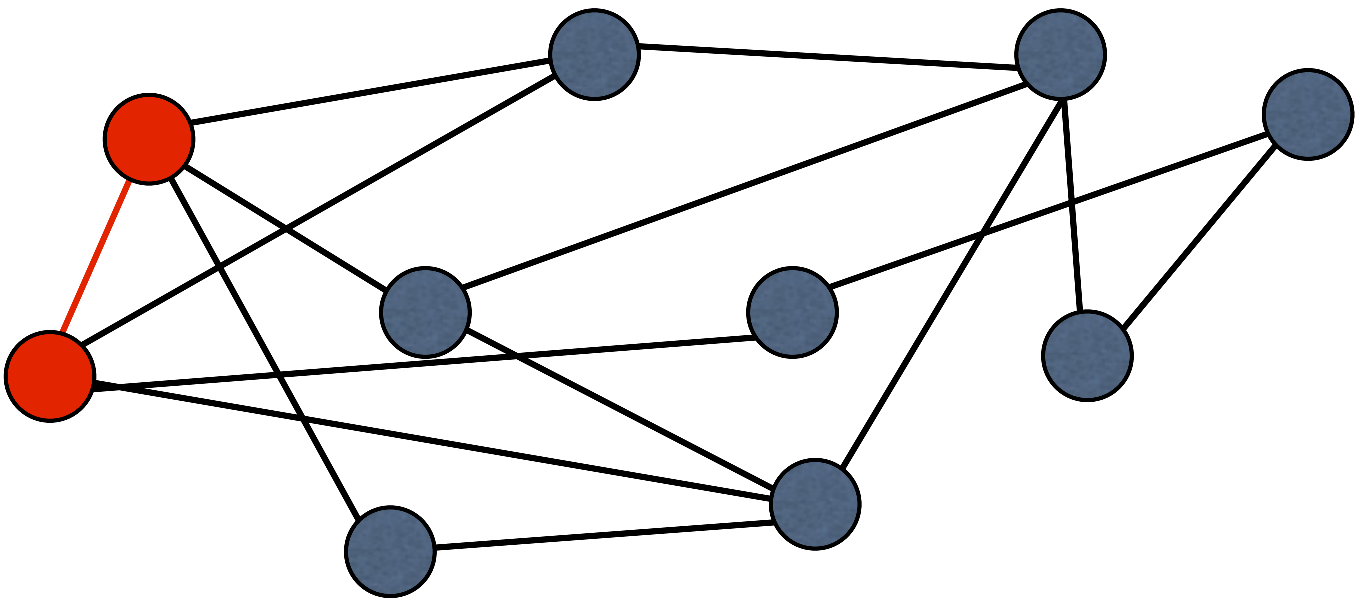
Each vertex on top row has one edge into each of the groups below.

Optimal Vertex Cover size 8

Greedy Rule: Pick vertex that covers the most edges
Could pick $B_1,\ldots,B_n$: $n\log(n)$ vertices



Greedy Rule:
Pick uncovered edge, add its end points



Find smallest set of
vertices touching
every edge

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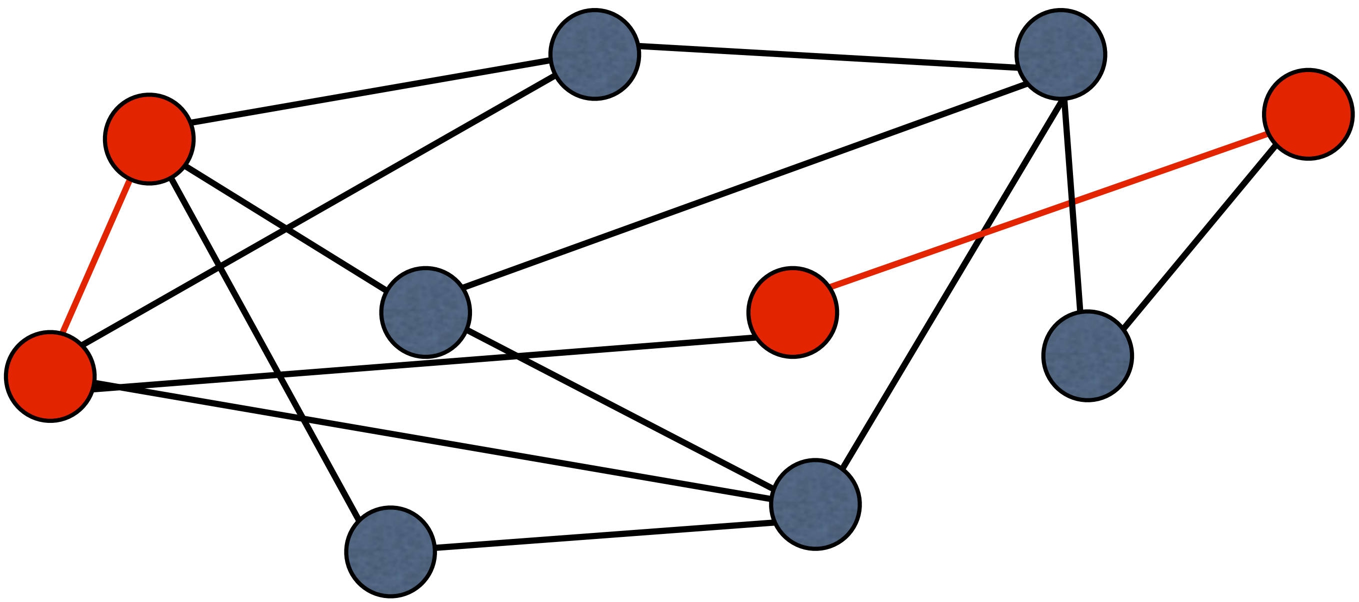

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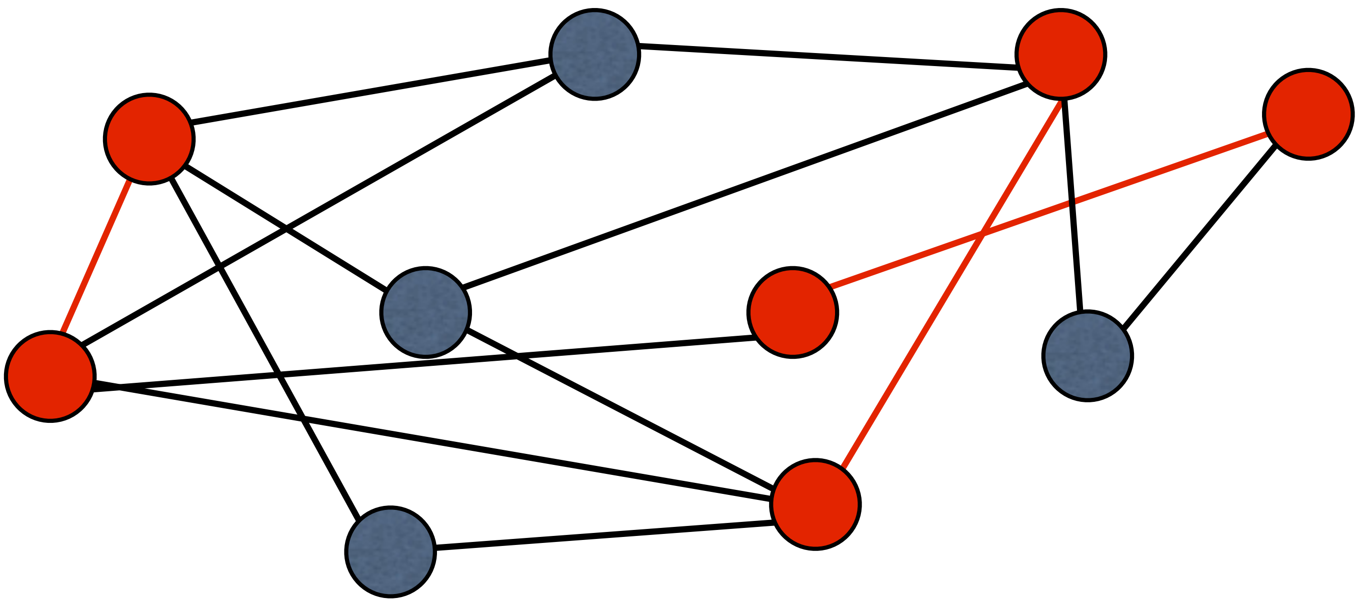
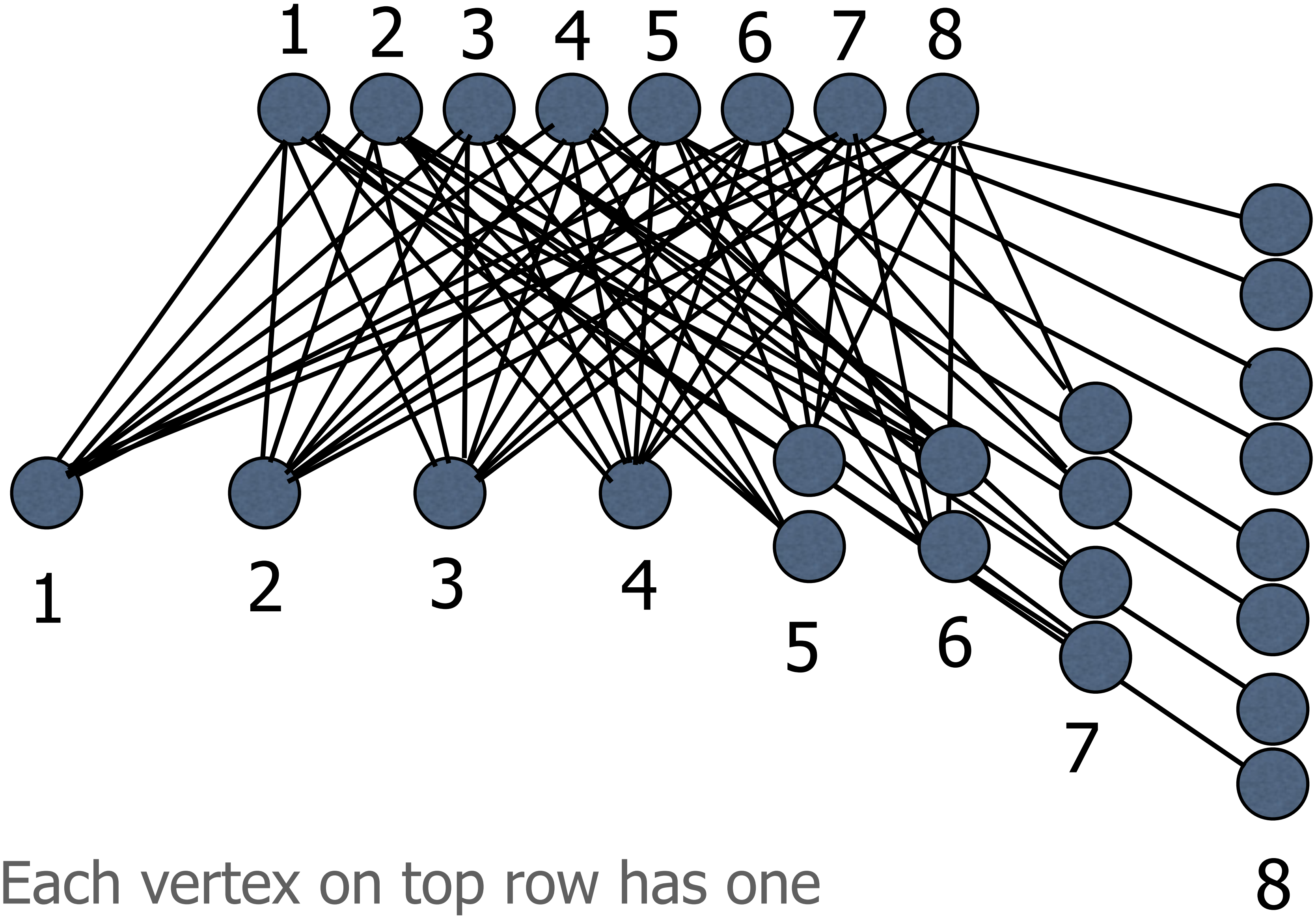


Find smallest set of
vertices touching
every edge

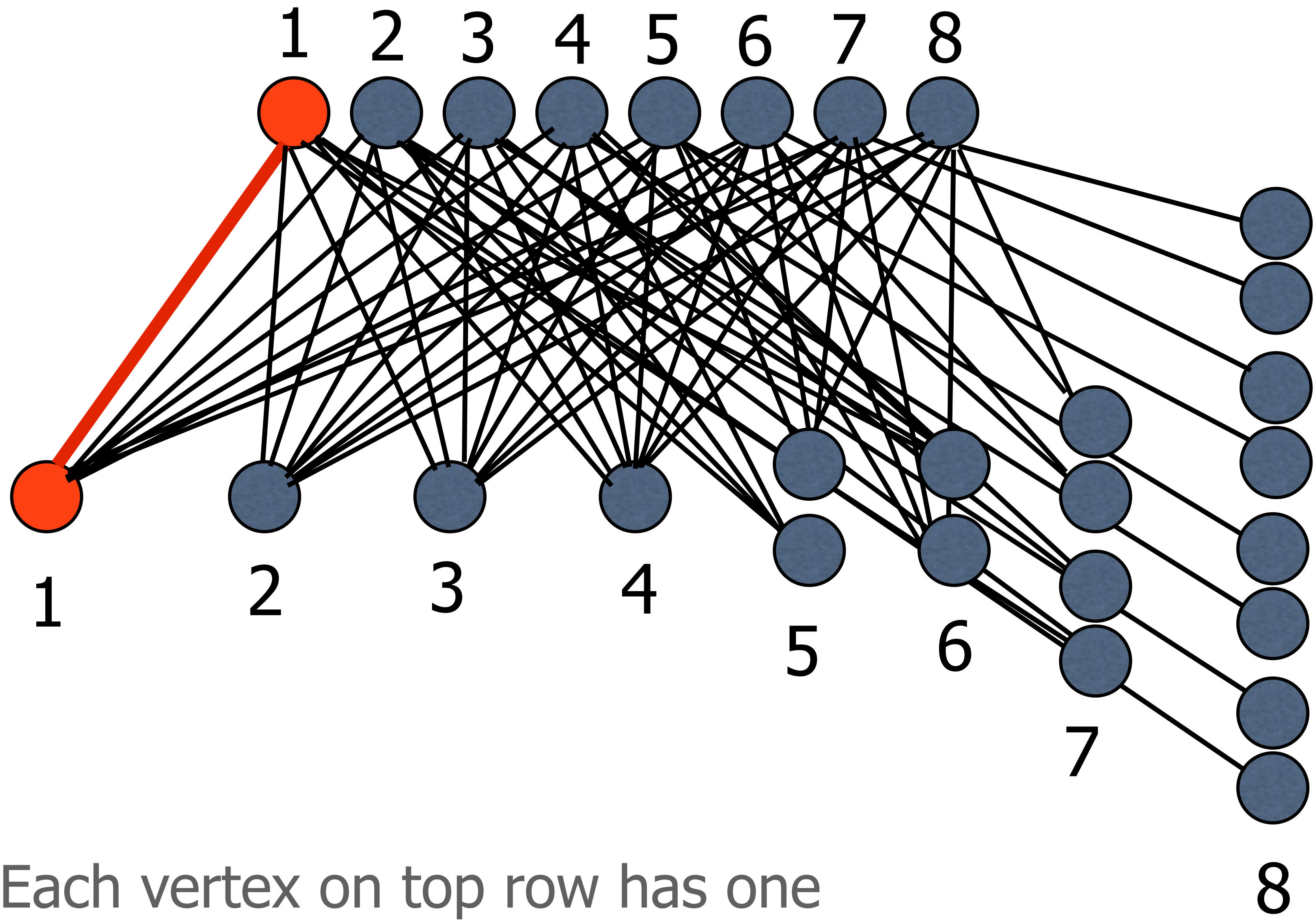
Vertex Cover size 6

Greedy Rule:
Pick uncovered edge, add its end points



Each vertex on top row has one edge into each of the groups below.

Greedy Rule:
Pick uncovered edge, add its end points



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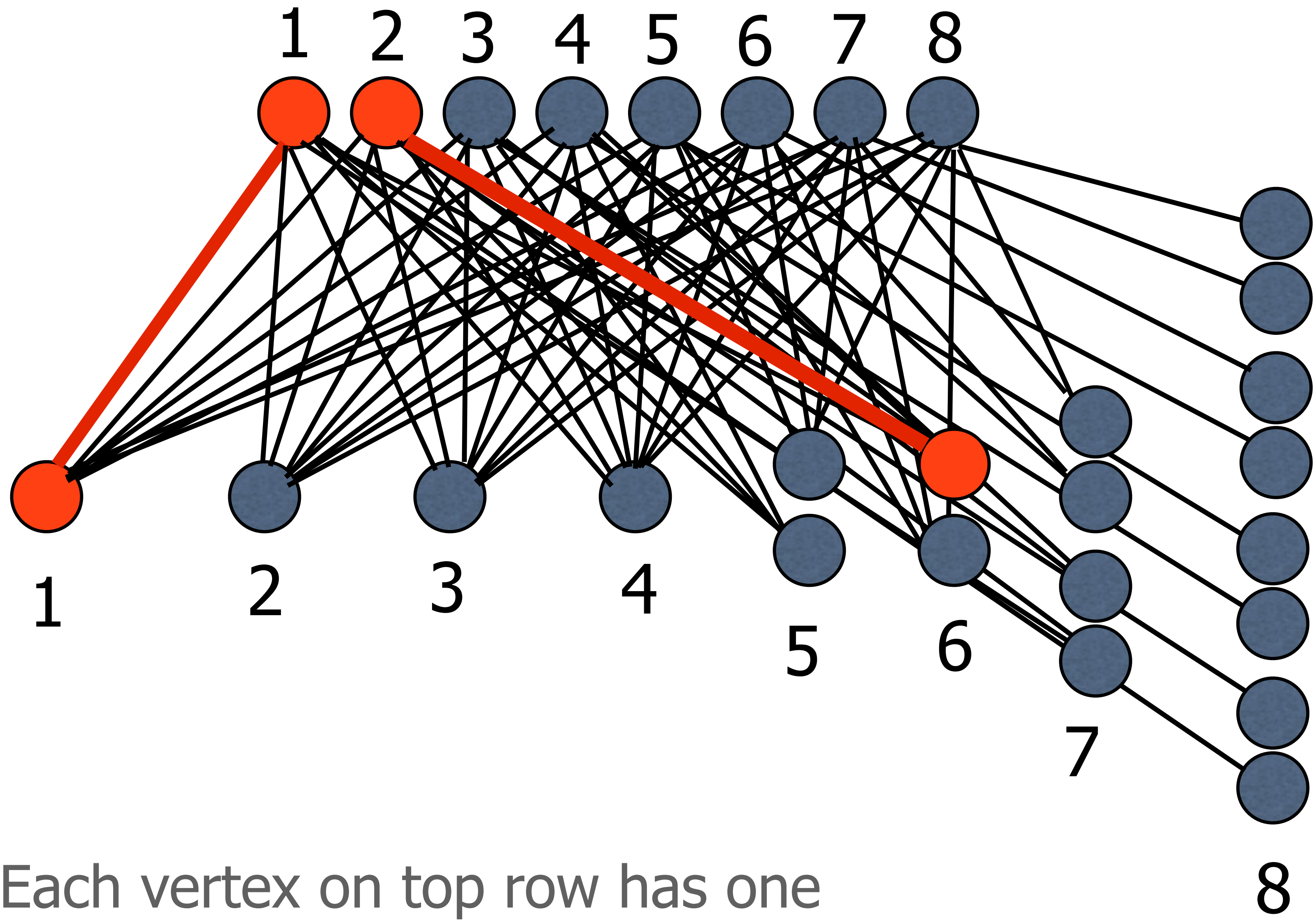

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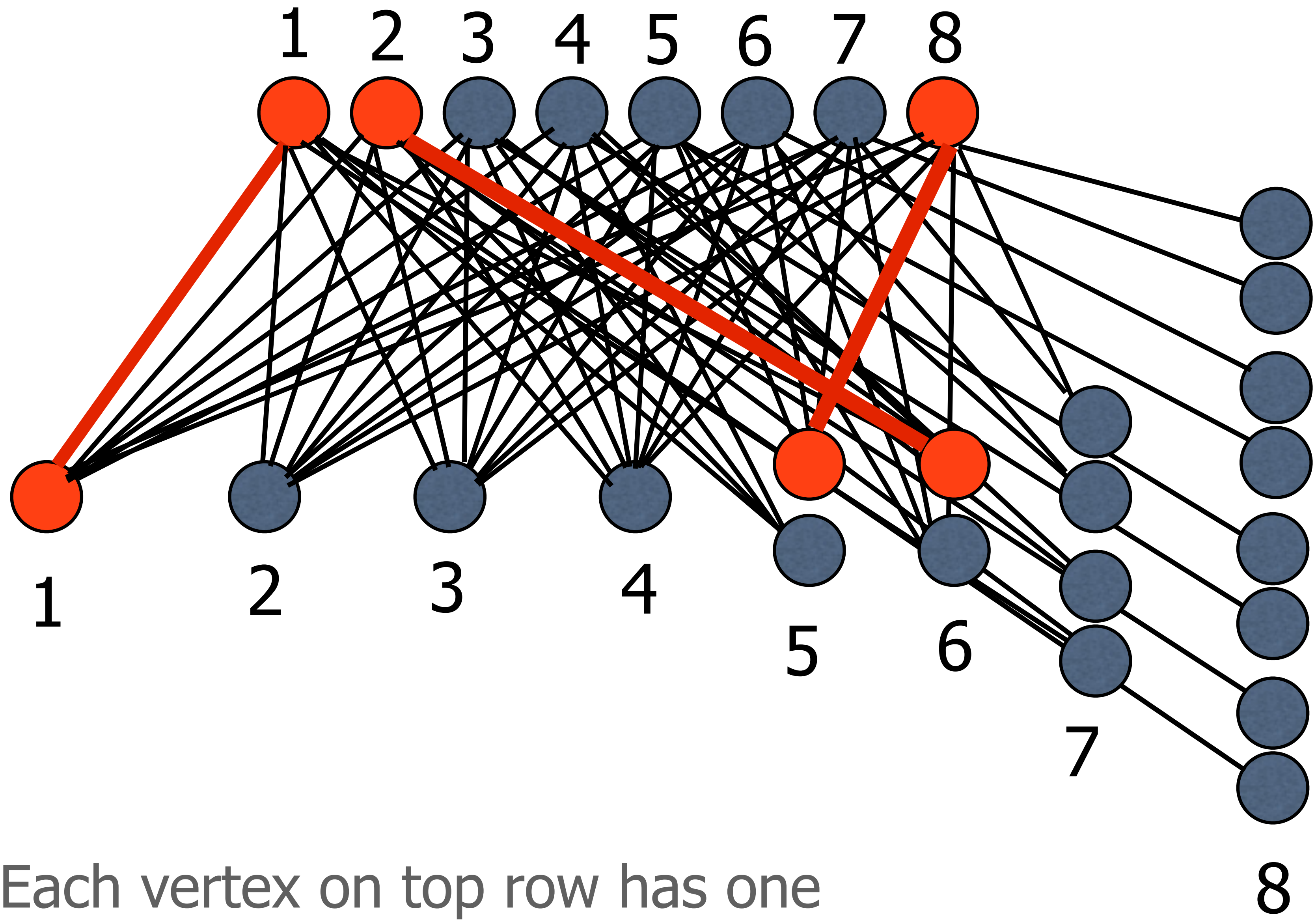

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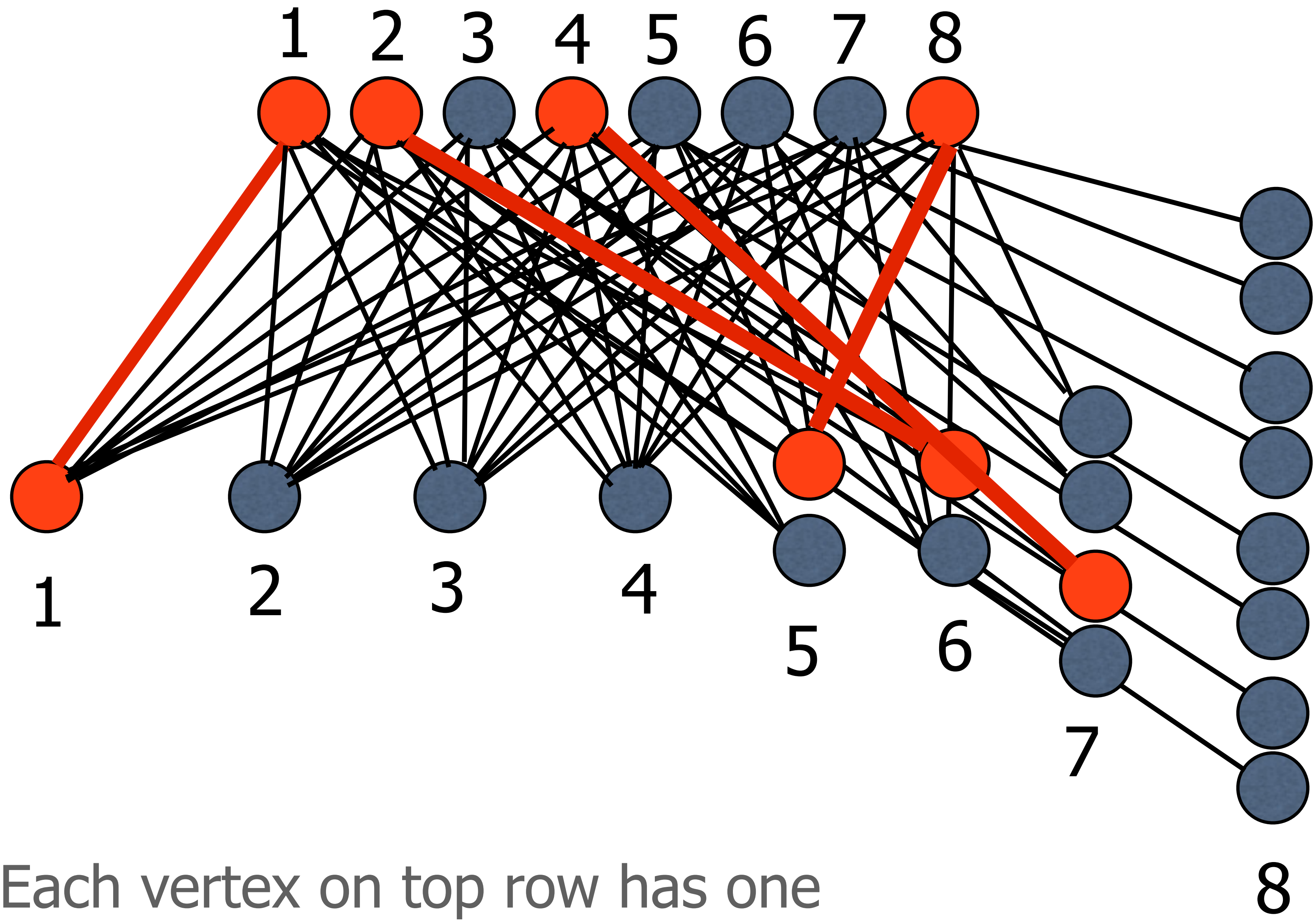

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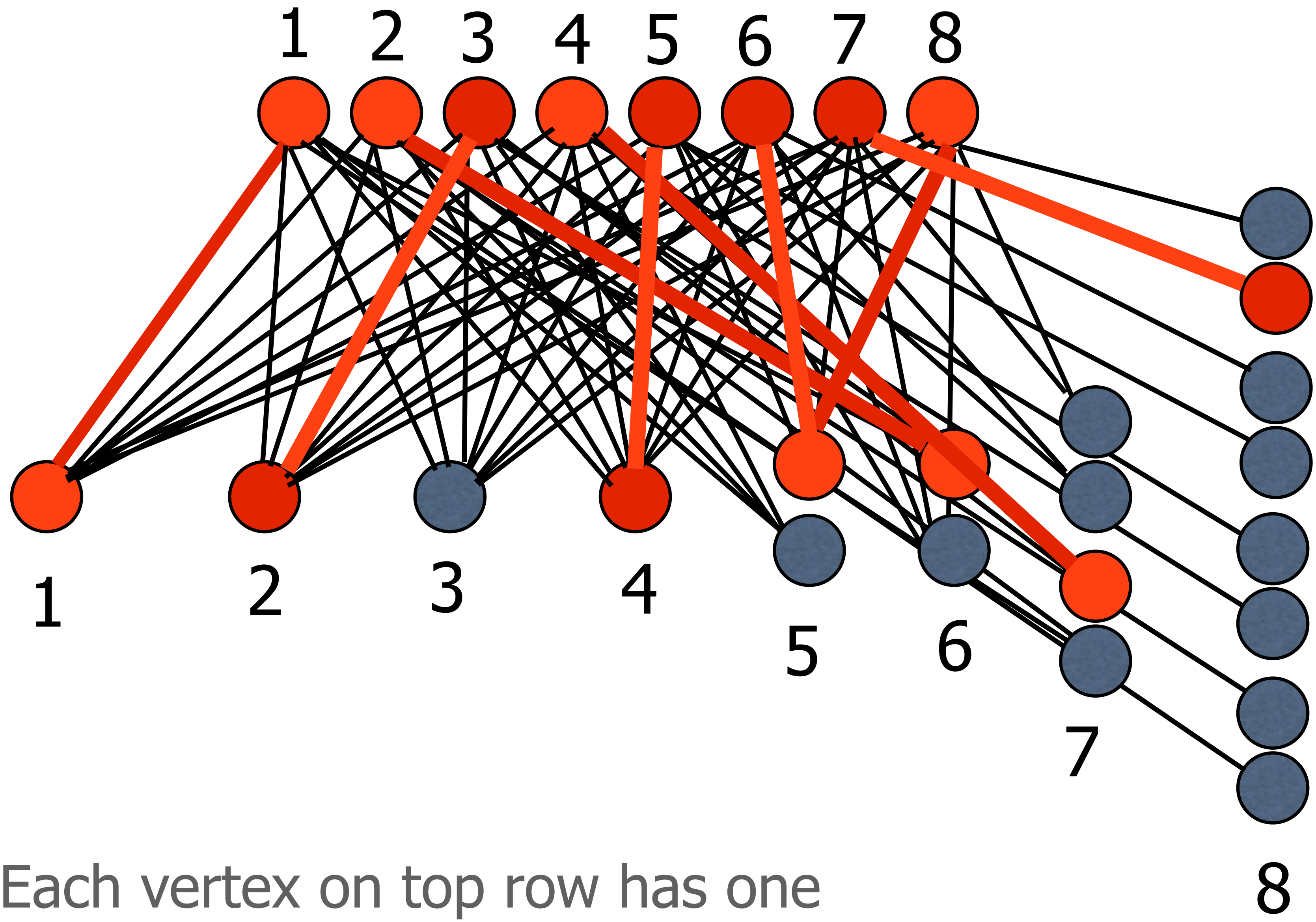
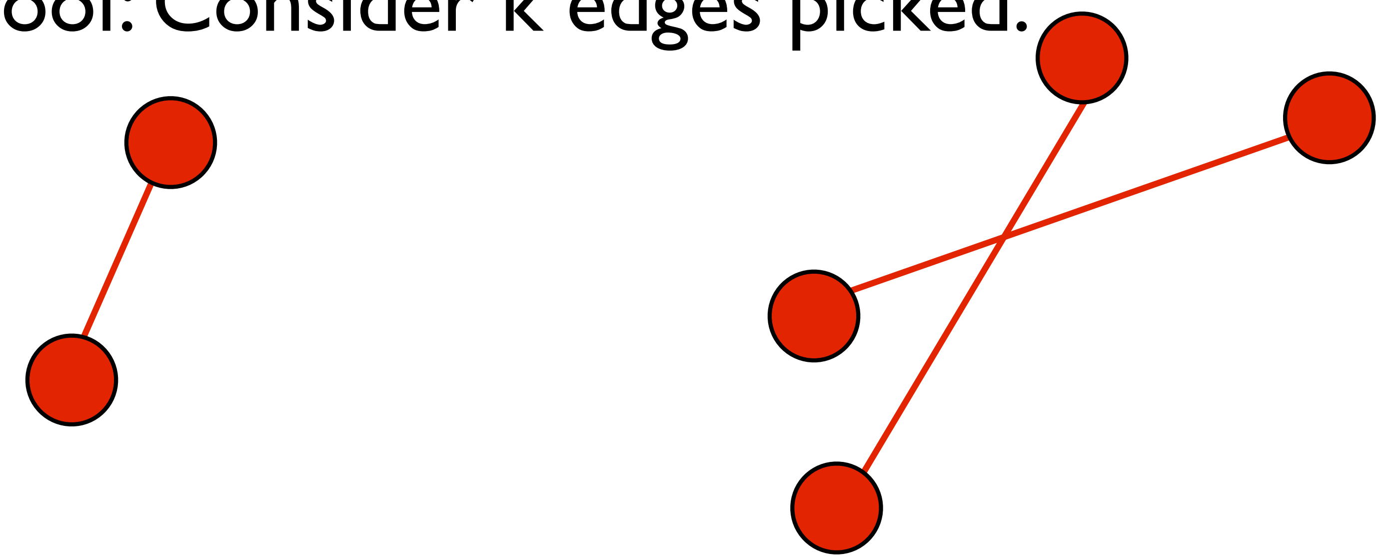


Each vertex on top row has one edge into each of the groups below.

Vertex Cover size 16

Theorem: Size of greedy vertex cover is at most twice as big as size of optimal cover

Proof: Consider k edges picked.



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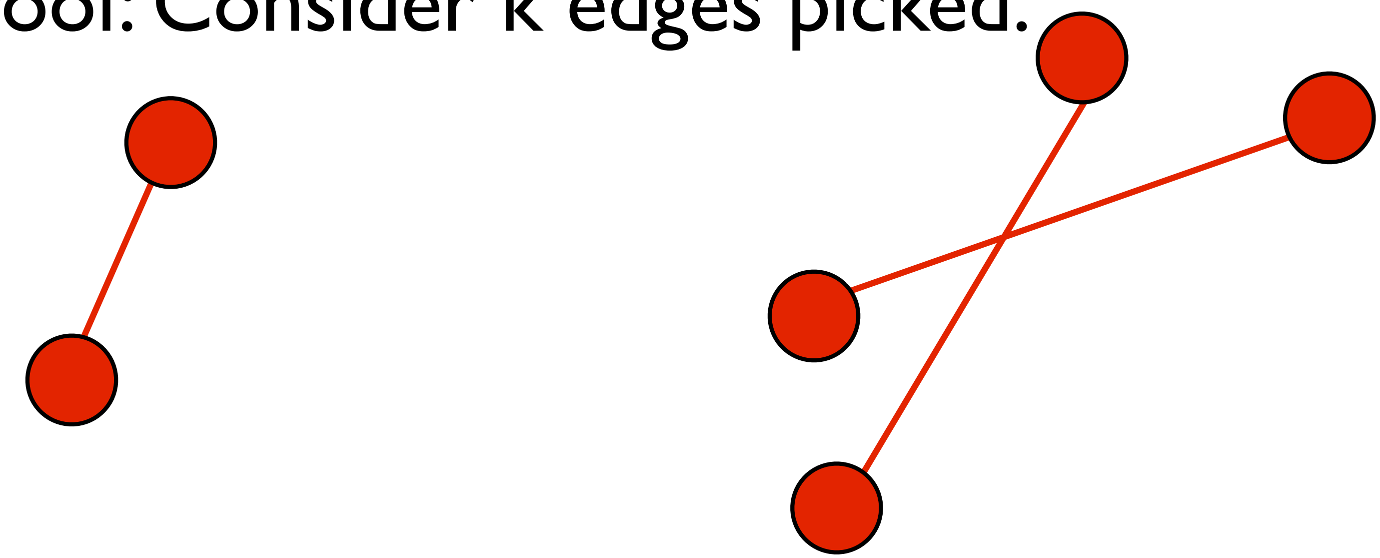
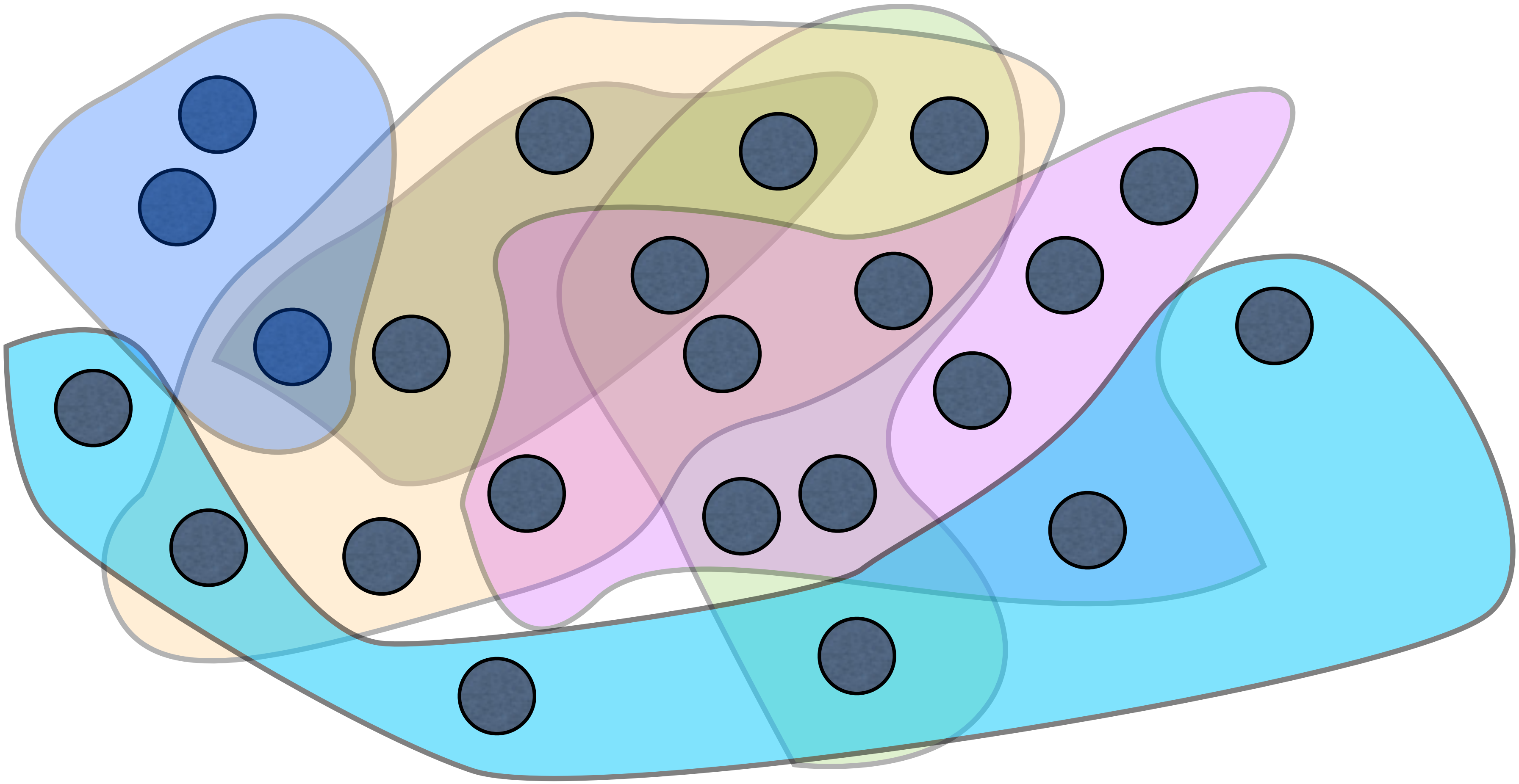


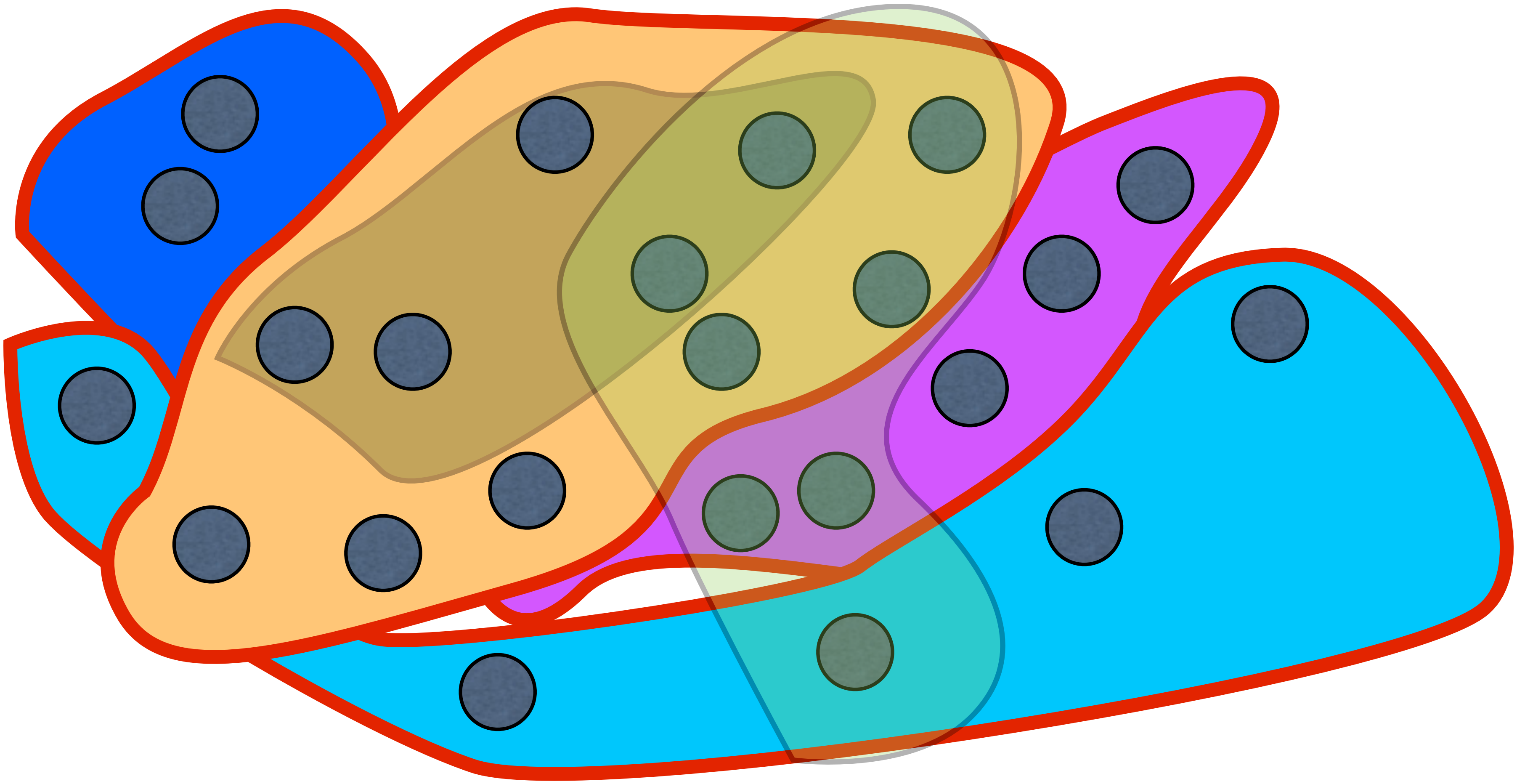
Edges do not touch: every cover must pick one vertex per edge! Thus every vertex cover has k vertices.

Set Cover



Find smallest
collection of sets
containing every point

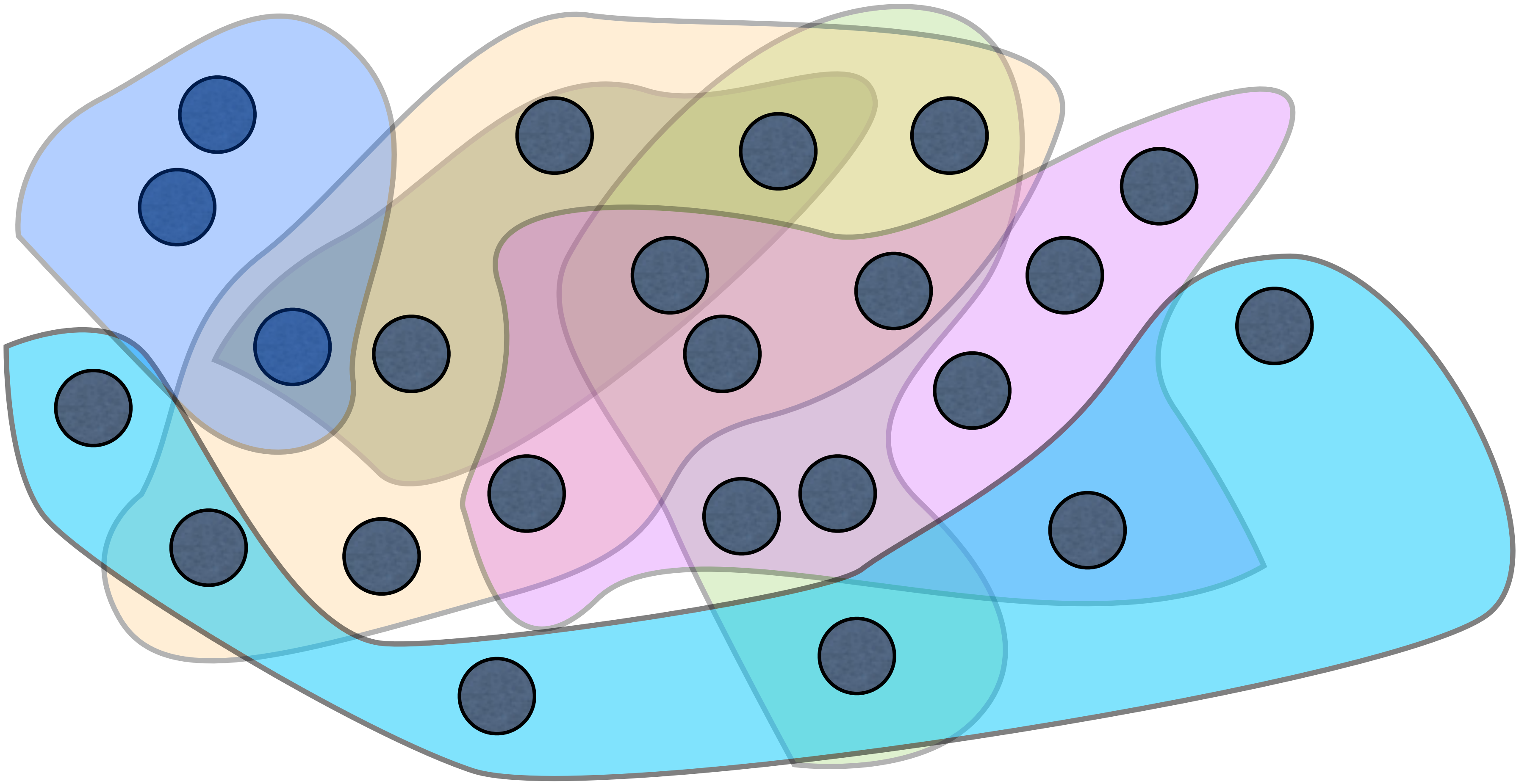
Set Cover



Find smallest collection of sets containing every point

Set Cover size 4

Greedy Set Cover: Pick the set that maximizes # new elements covered



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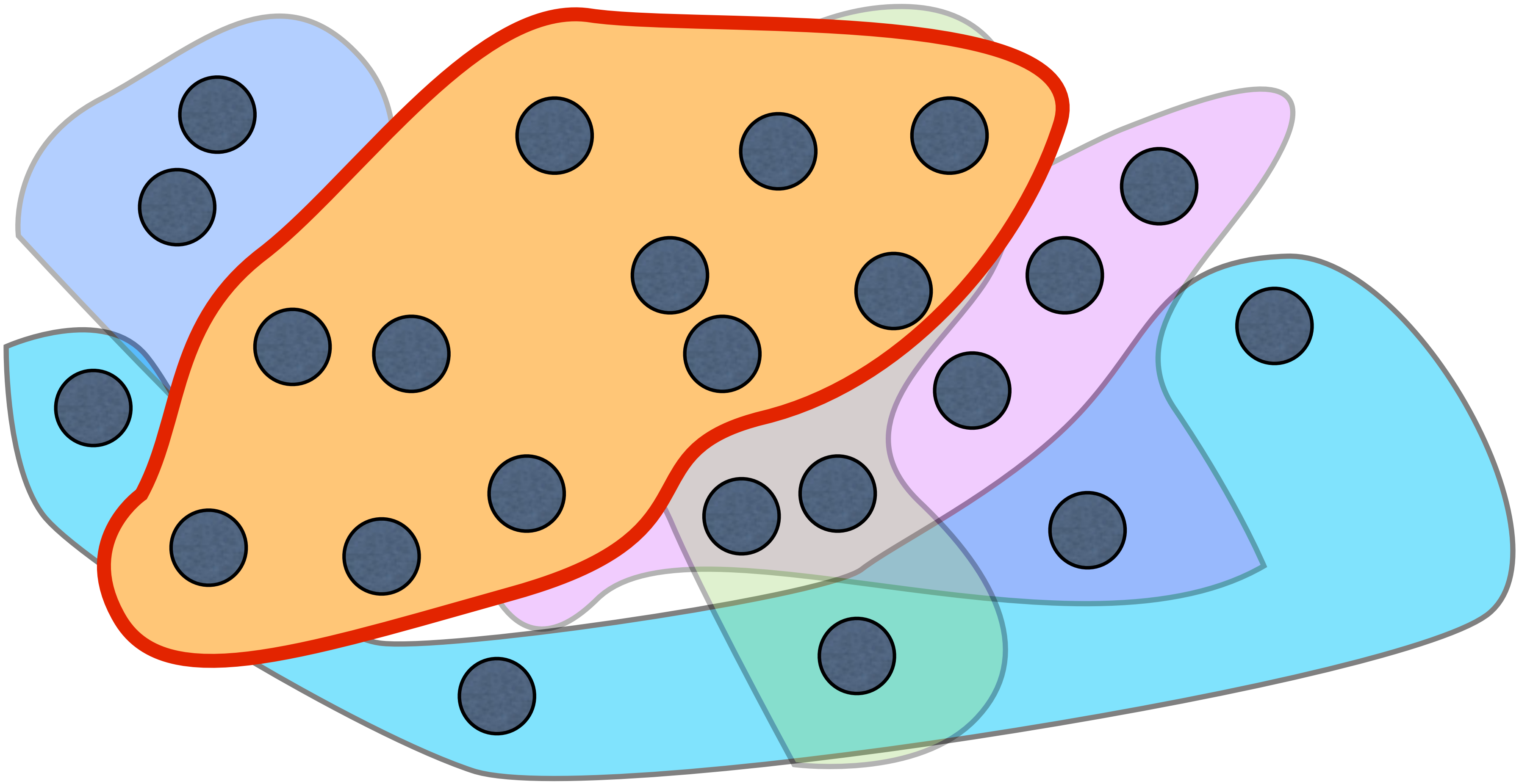

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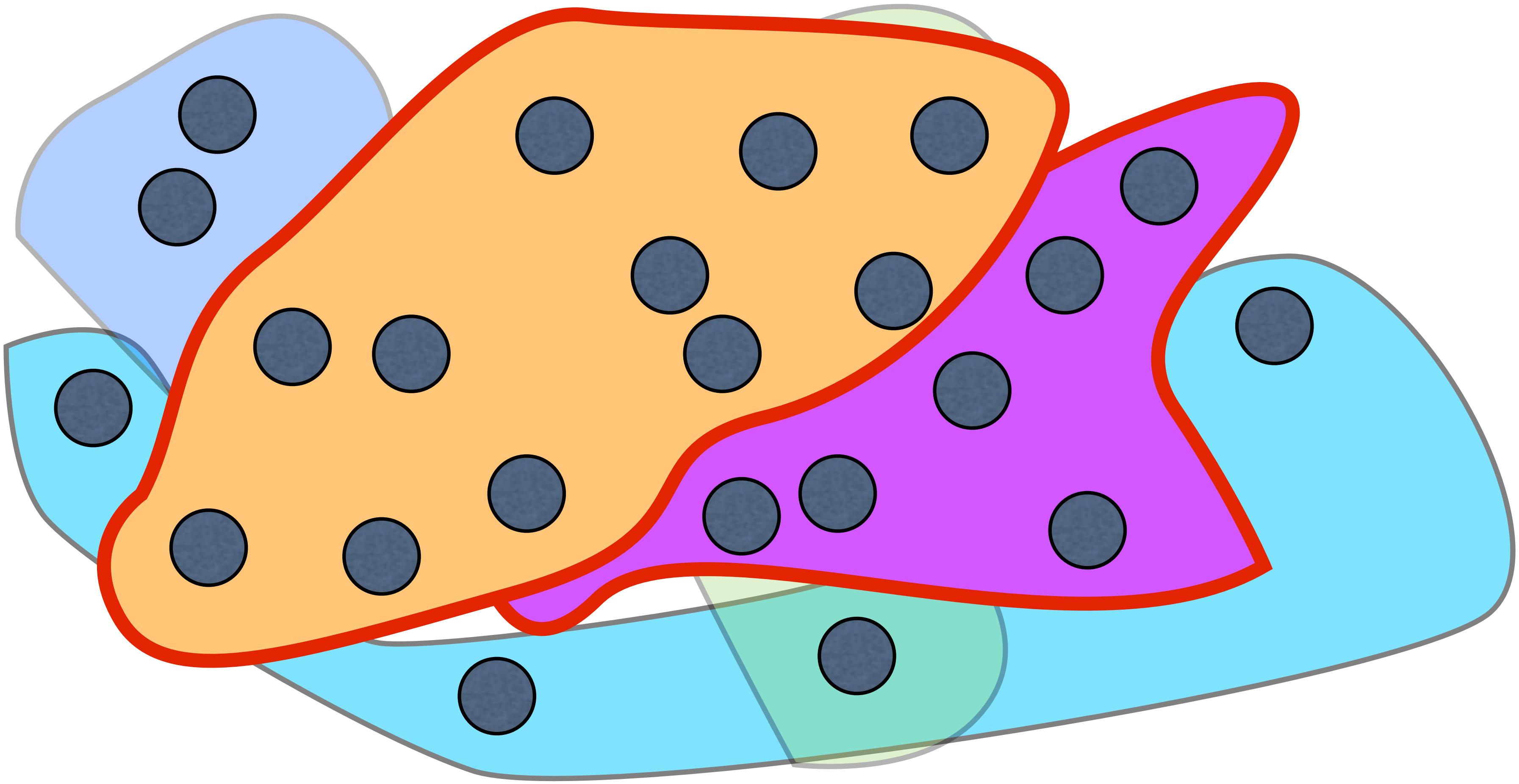

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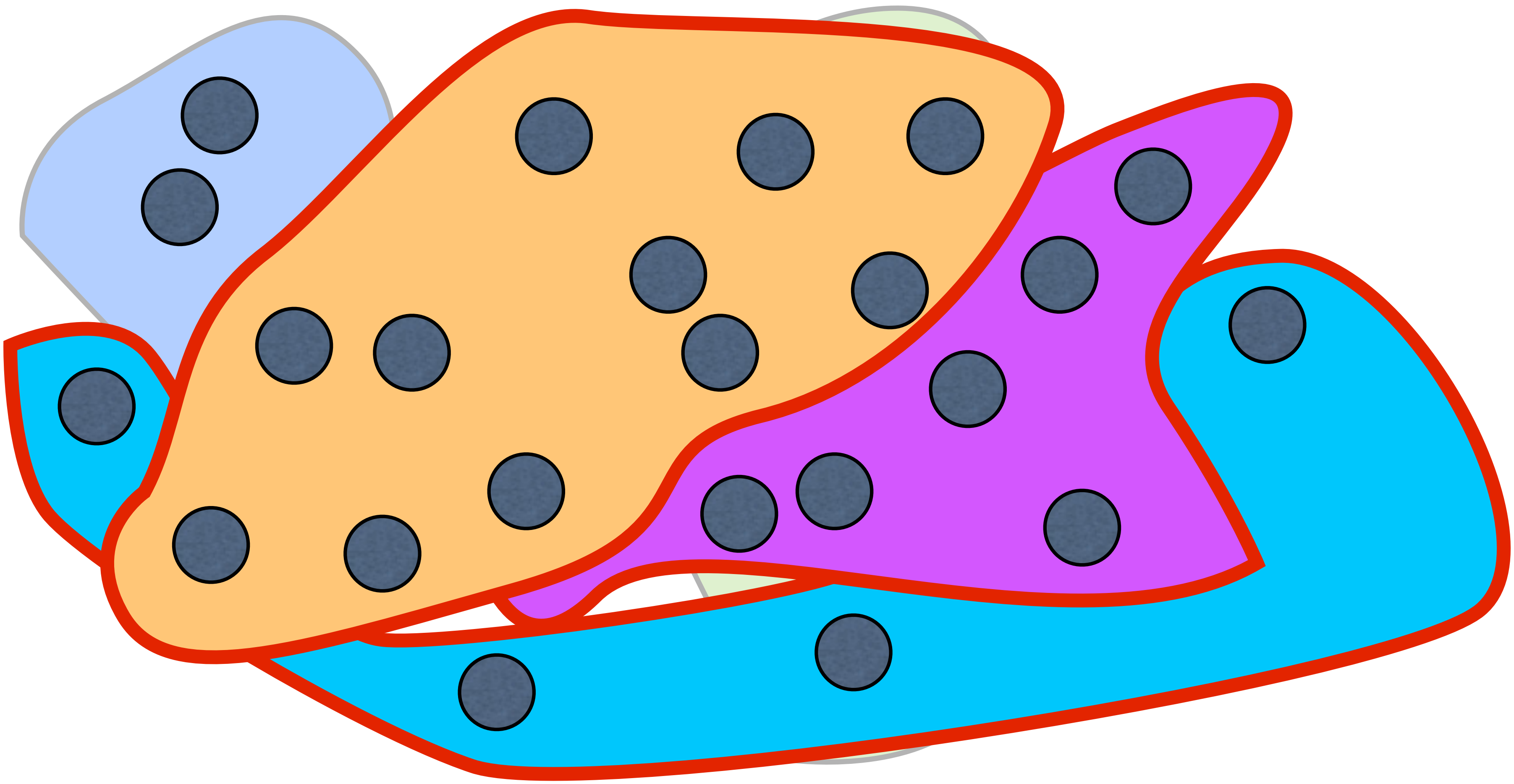
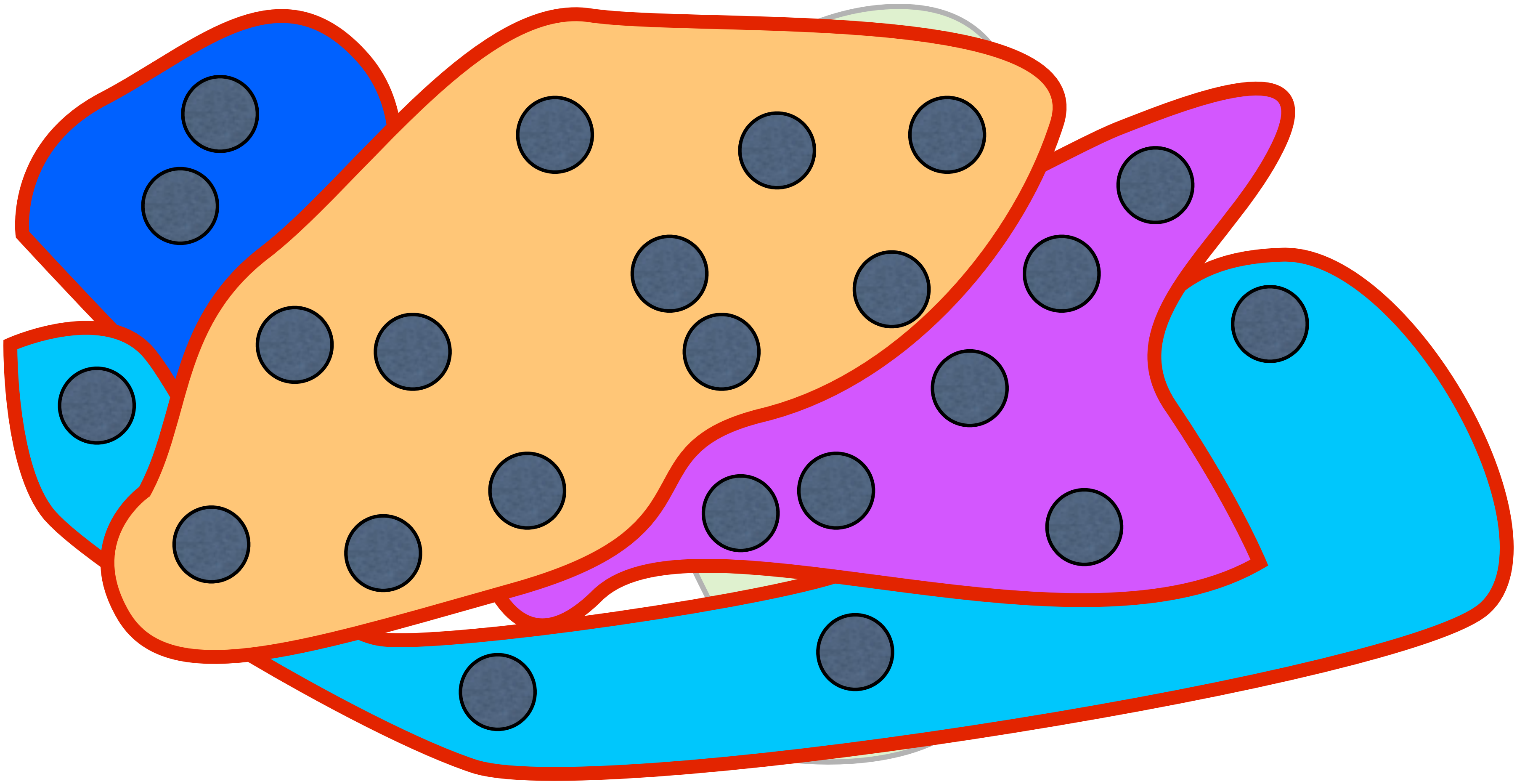
Greedy Set Cover: Pick the set that maximizes # new elements covered



Find smallest collection of sets containing every point

Greedy Set Cover: Pick the set that maximizes # new elements covered



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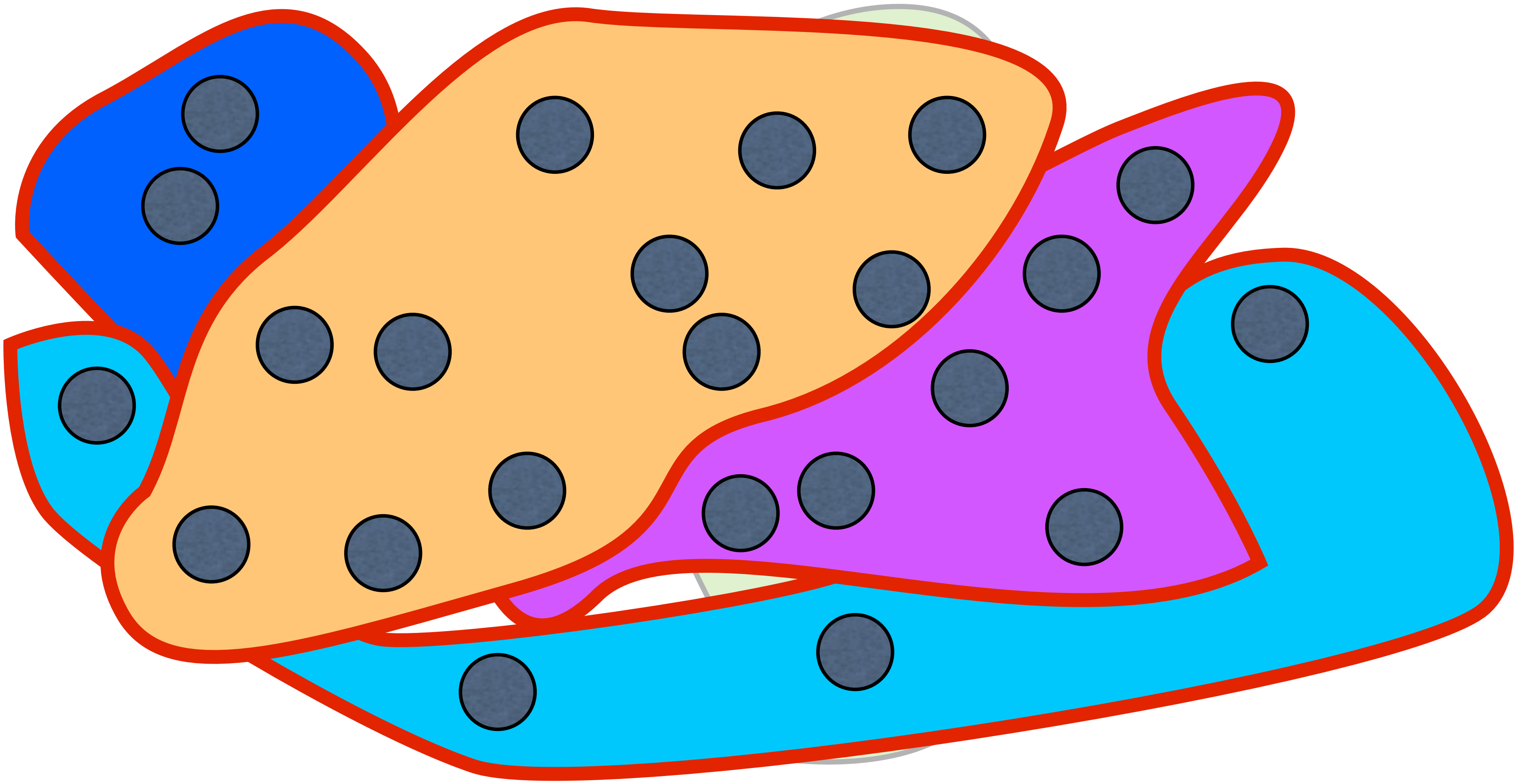
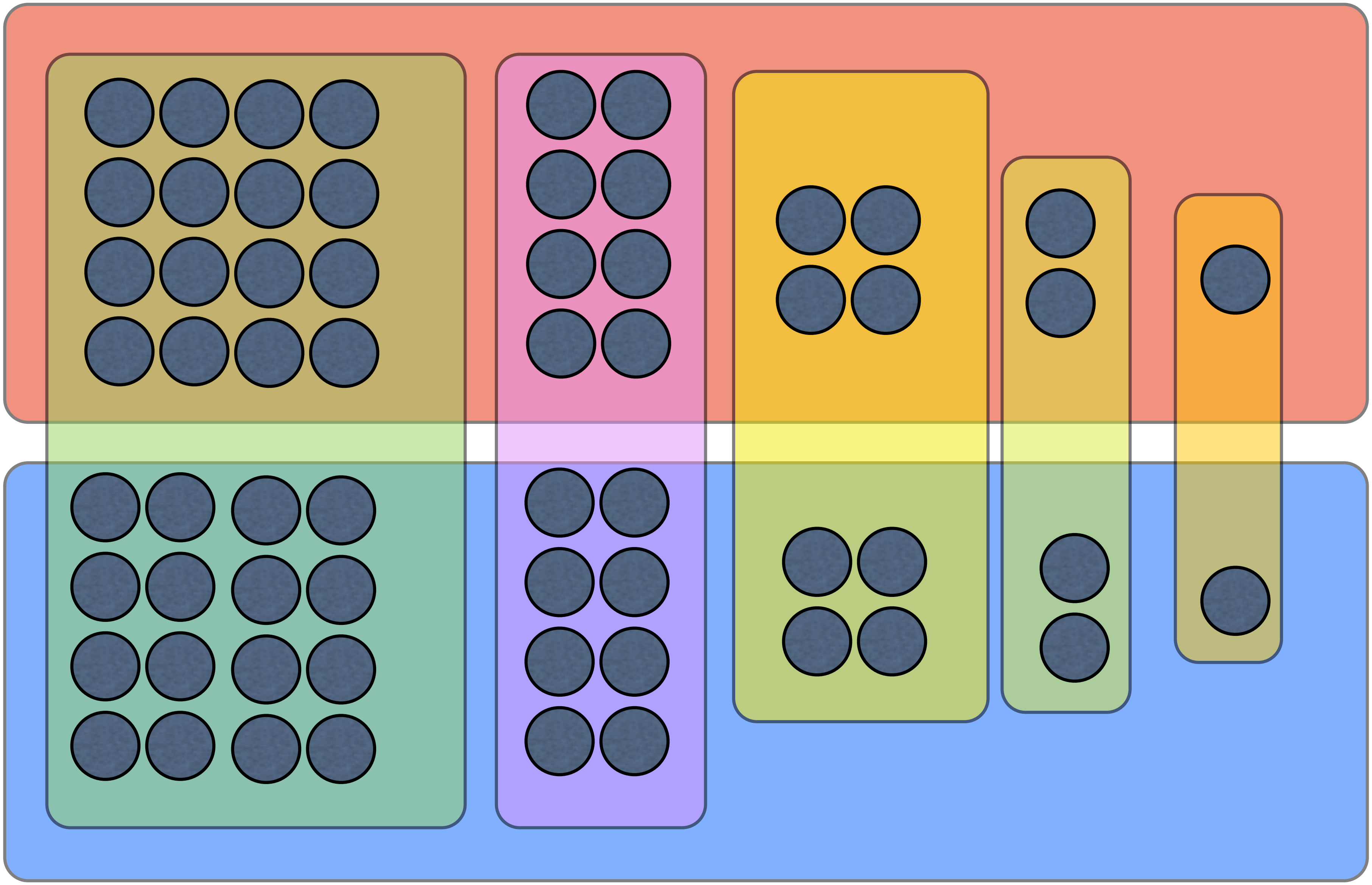


Theorem: Greedy finds best cover upto a factor of ln(n).

Greedy Set Cover: Pick the set that maximizes # new elements covered



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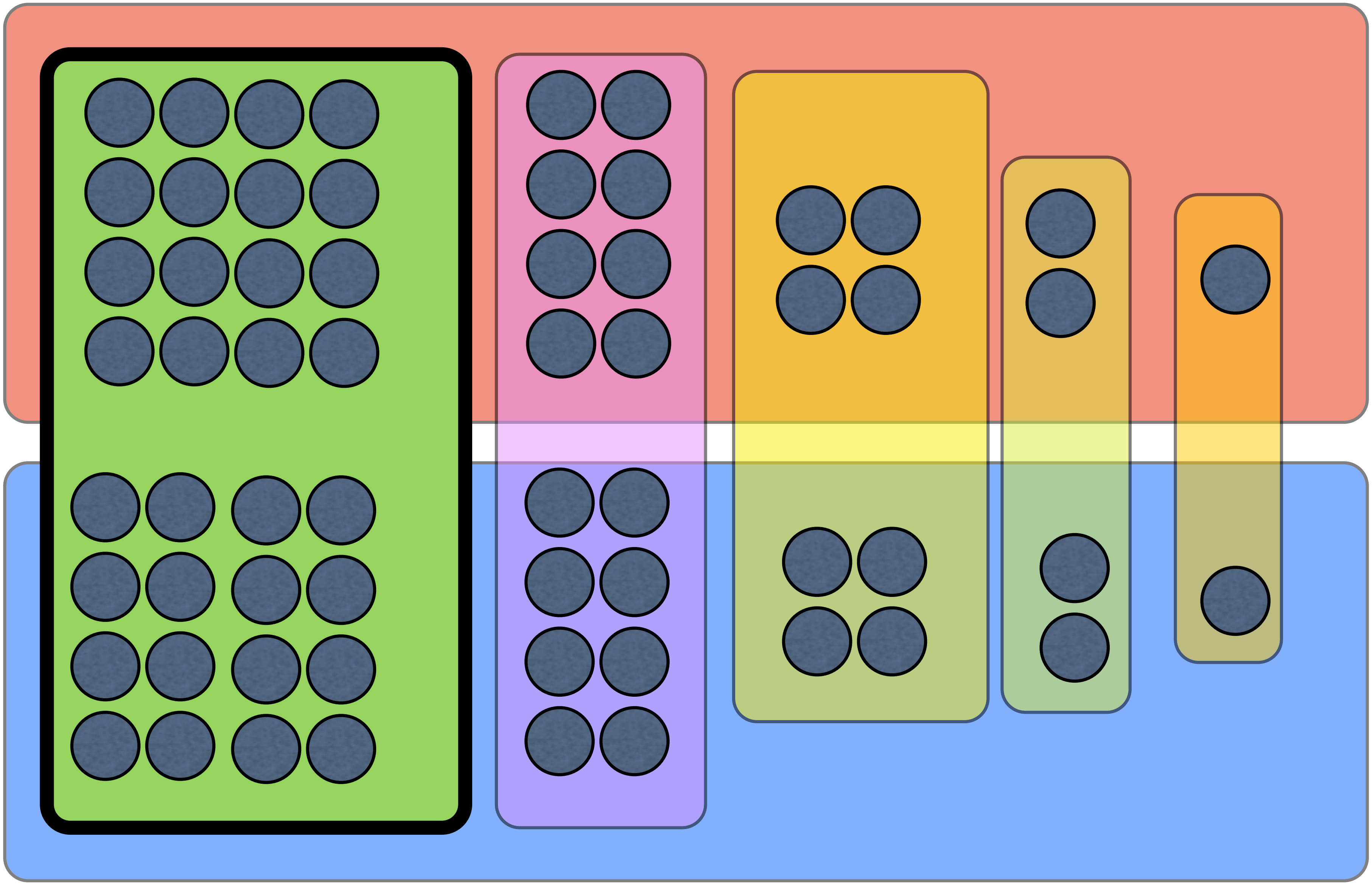

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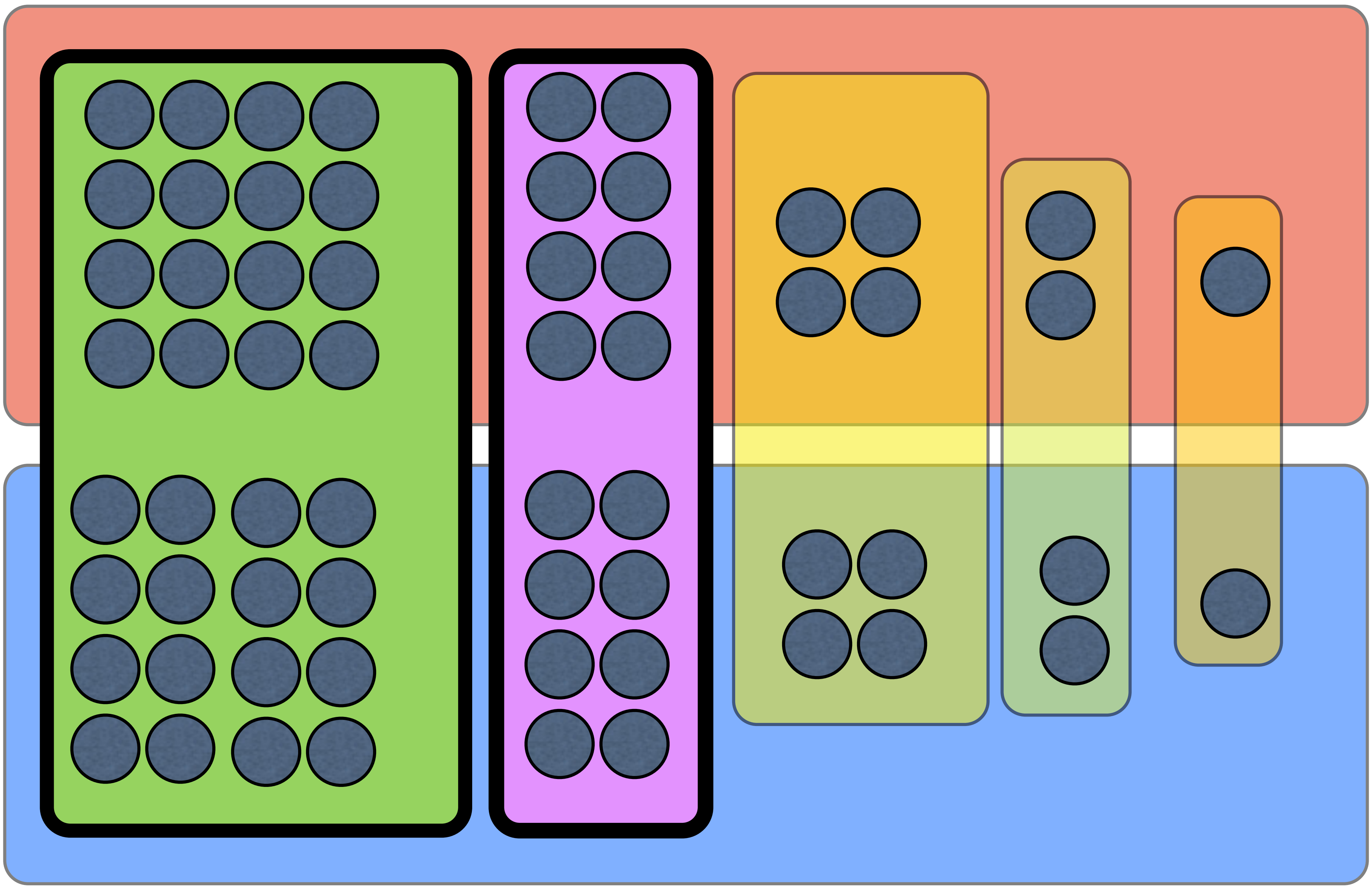

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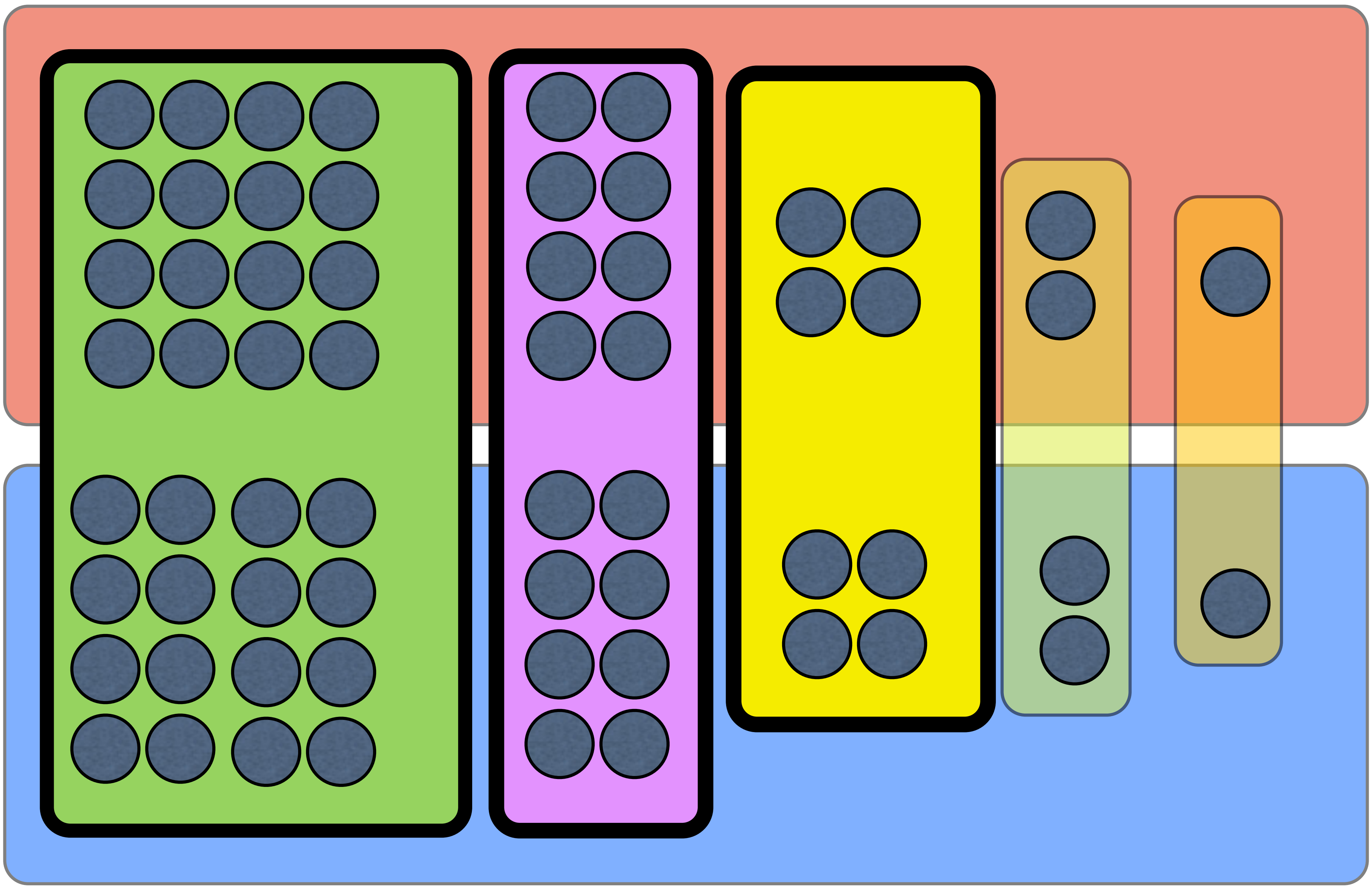

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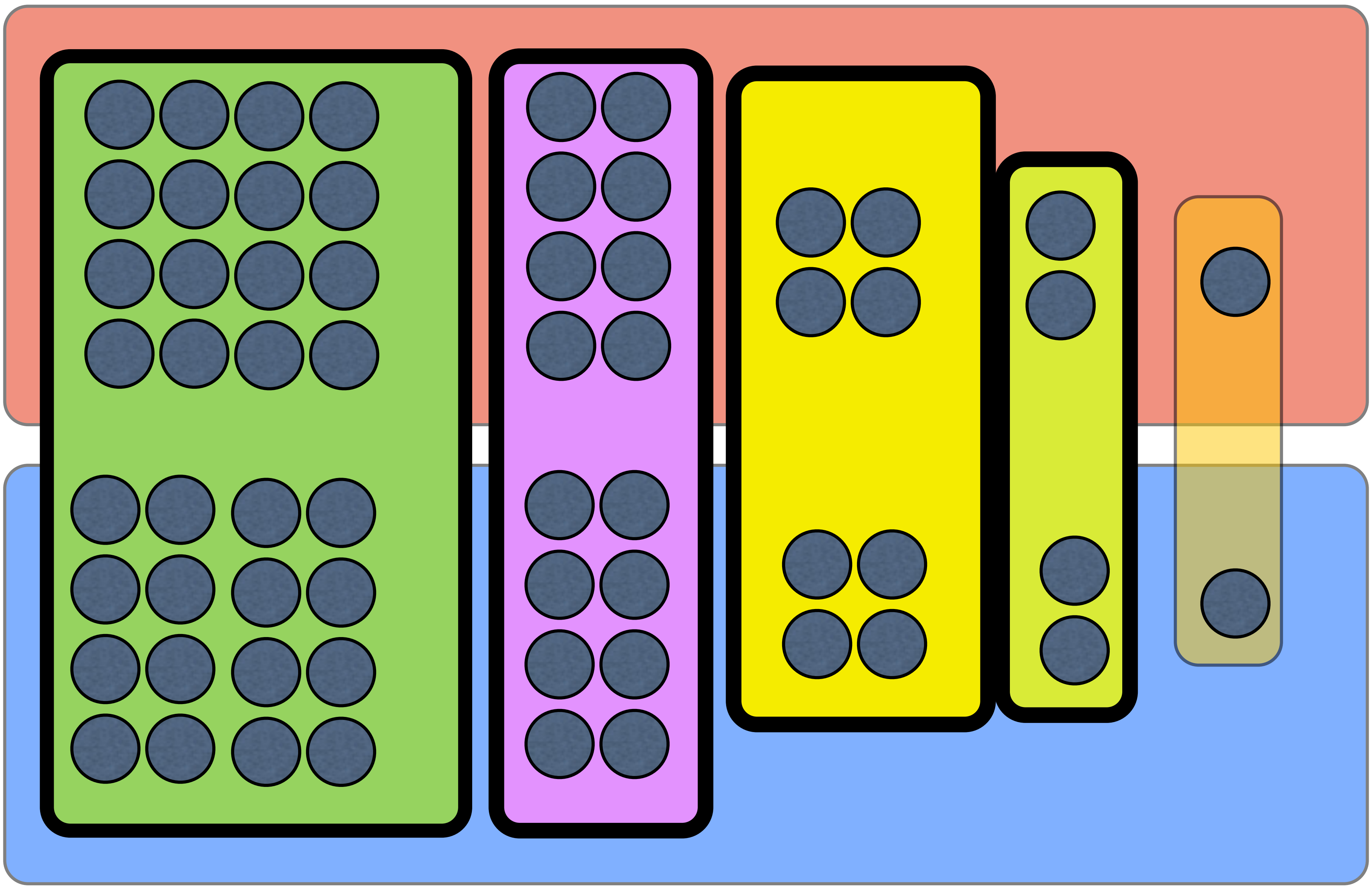
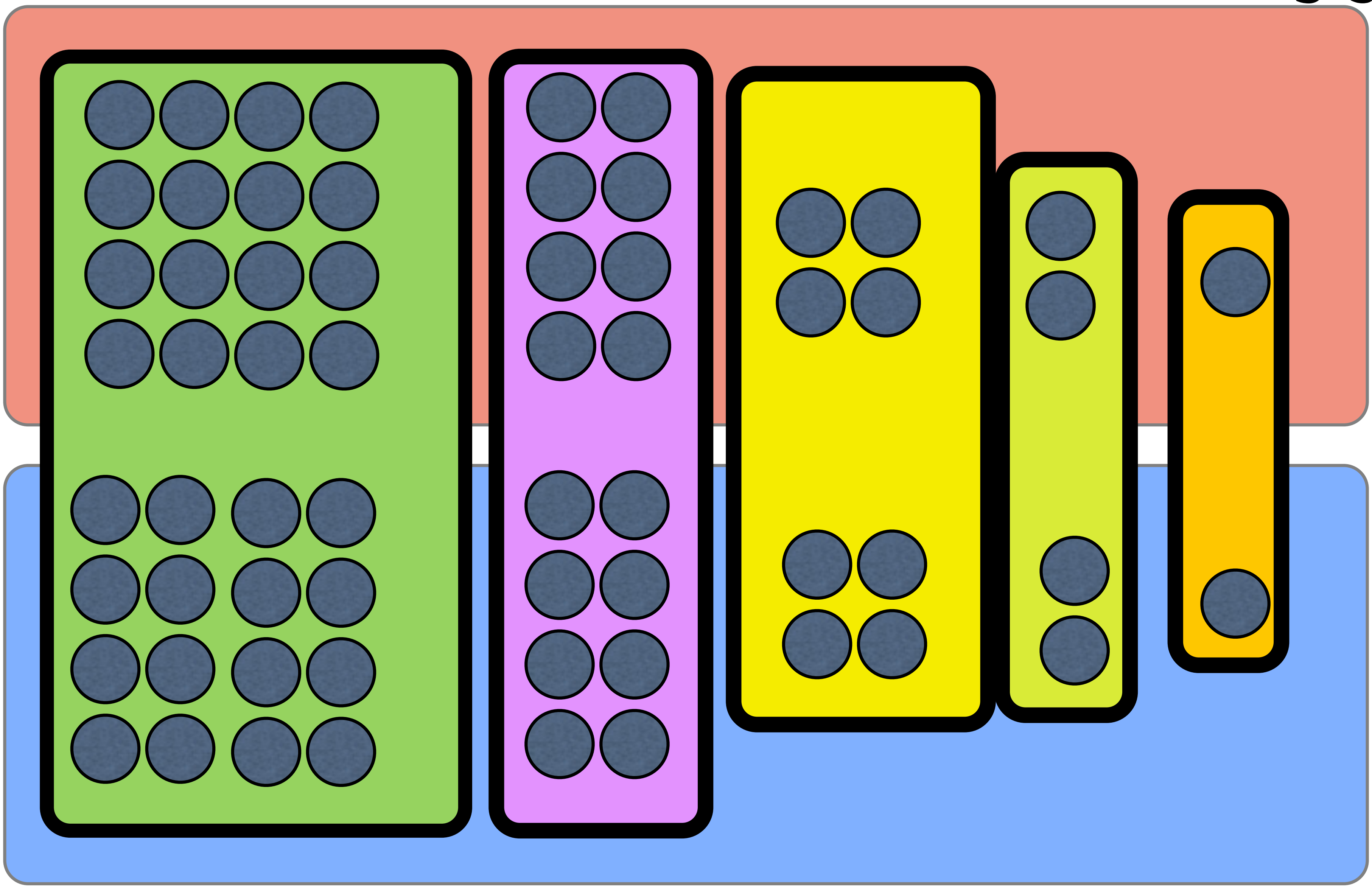


Greedy Set Cover: Pick the set that maximizes # new elements covered

greedy solution: 5 sets



Greedy Set Cover: Pick the set that maximizes # new elements covered

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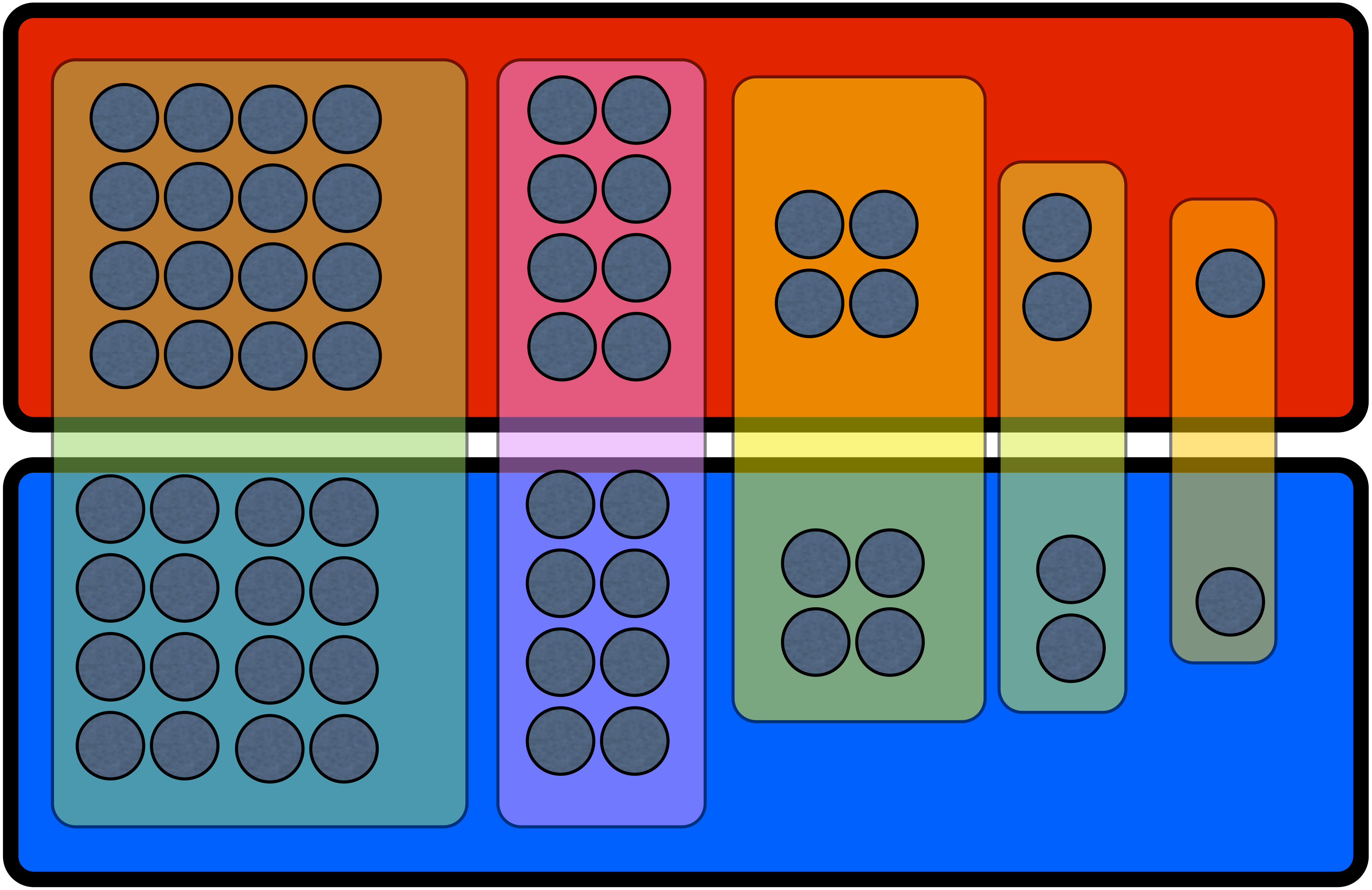
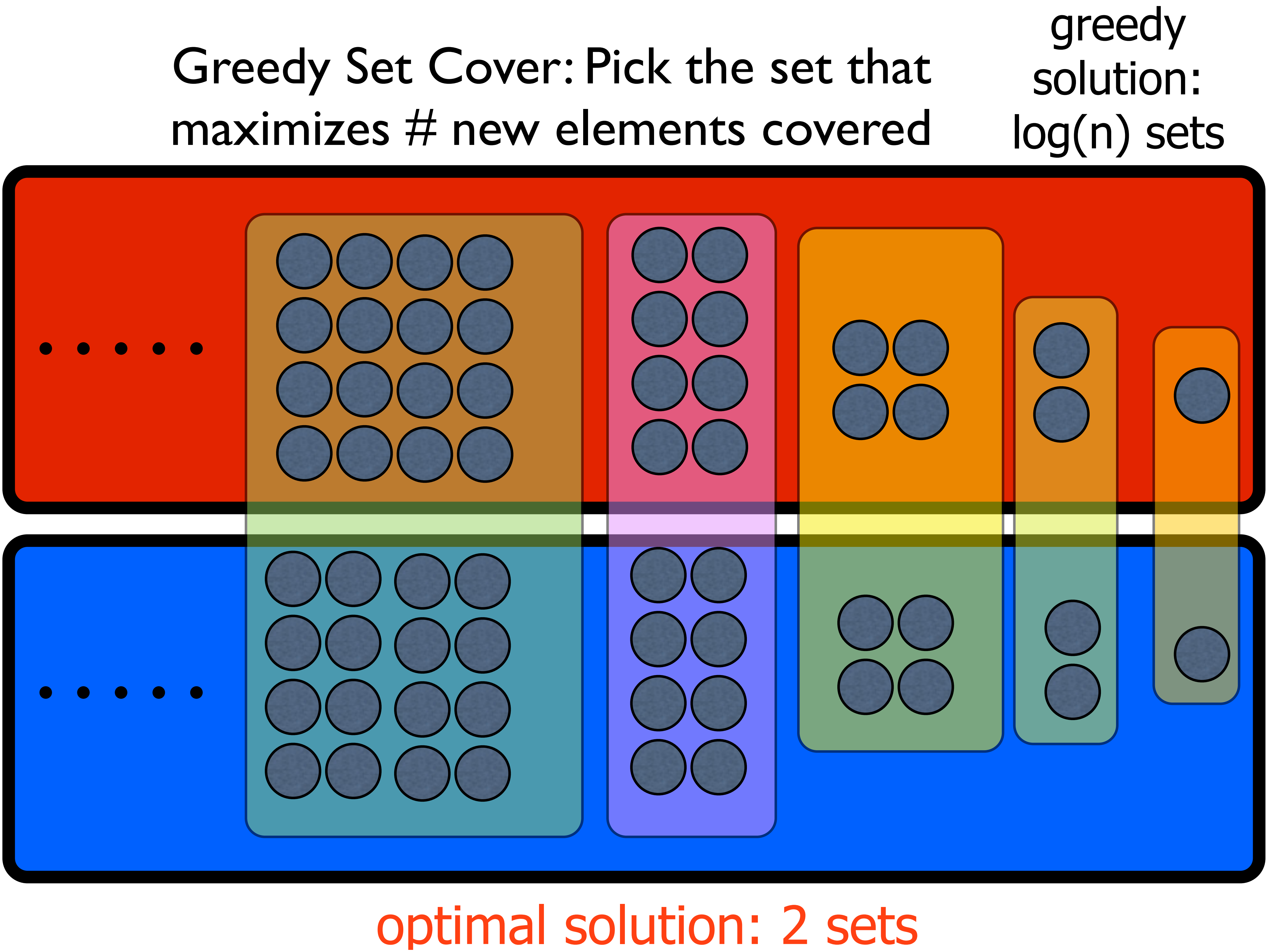


optimal solution: 2 sets

Greedy Set Cover: Pick the set that maximizes # new elements covered



Greedy Set Cover: Pick the set that maximizes # new elements covered

Theorem: If the best solution has k sets, greedy finds at most k ln(n) sets.

Pf: Suppose there is a set cover of size k.

Greedy Set Cover: Pick the set that maximizes # new elements covered

Theorem: If the best solution has k sets, greedy finds at most k ln(n) sets.

Pf:
Suppose there is a set cover of size k.

There is set that covers 1/k fraction of remaining elements, since there are k sets that cover all remaining elements. So in each step, algorithm will cover 1/k fraction of remaining elements.

Greedy Set Cover: Pick the set that maximizes # new elements covered

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#elements uncovered after t steps $\leq n(1-1/k)^t < ne^{-t/k}$. So after $t = k \ln(n)$ steps, number of uncovered elements < 1.