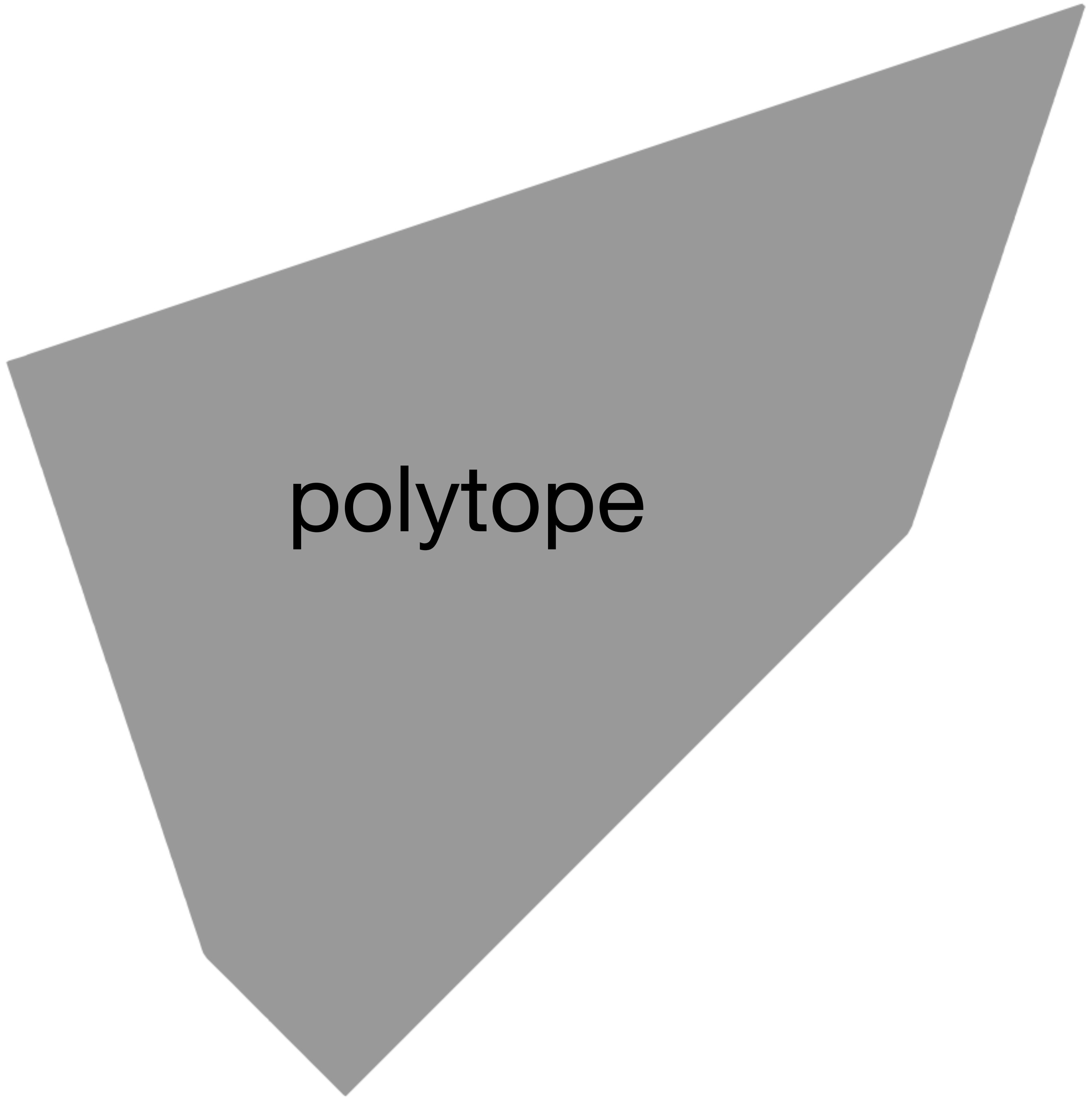
Linear Programming

A really very extremely big hammer

Given: a polytope

Find: the *lowest* point in the polytope



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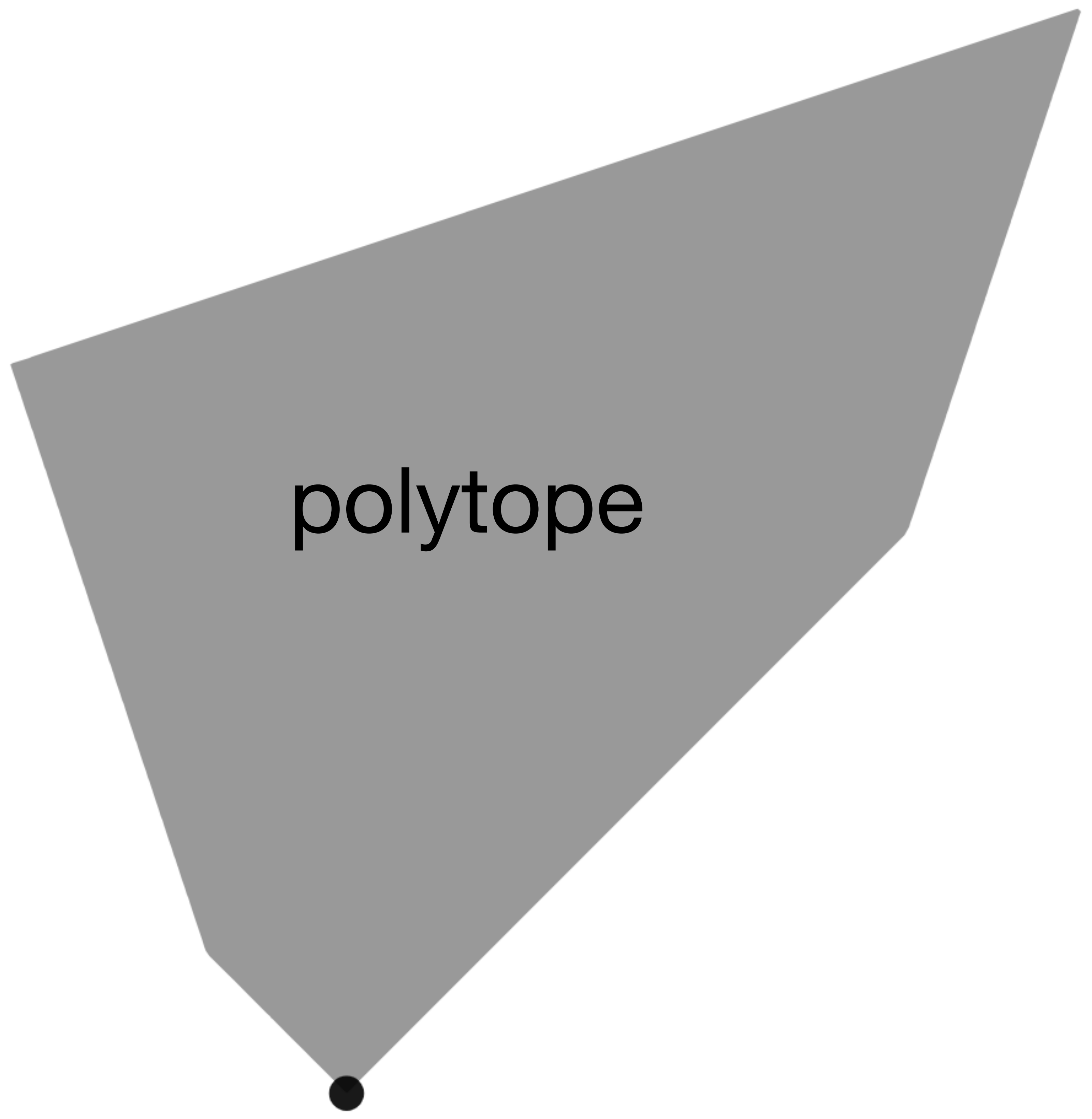

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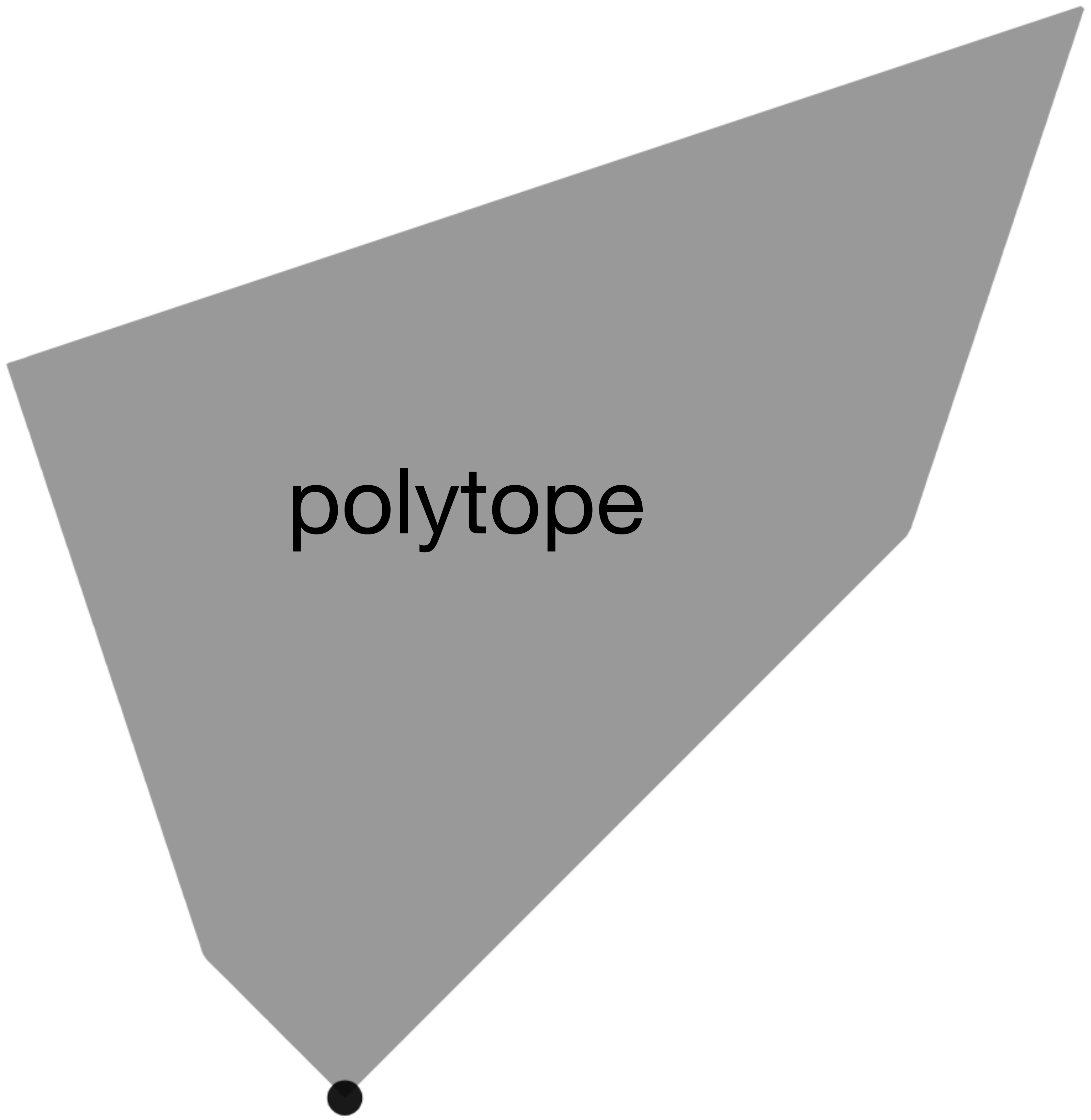

maximize $z_1 + 2z_3$
subject to
$$2z_1 - z_2 + 3z_3 \leq 1$$
$$-z_1 + z_2 - z_3 \leq 5$$

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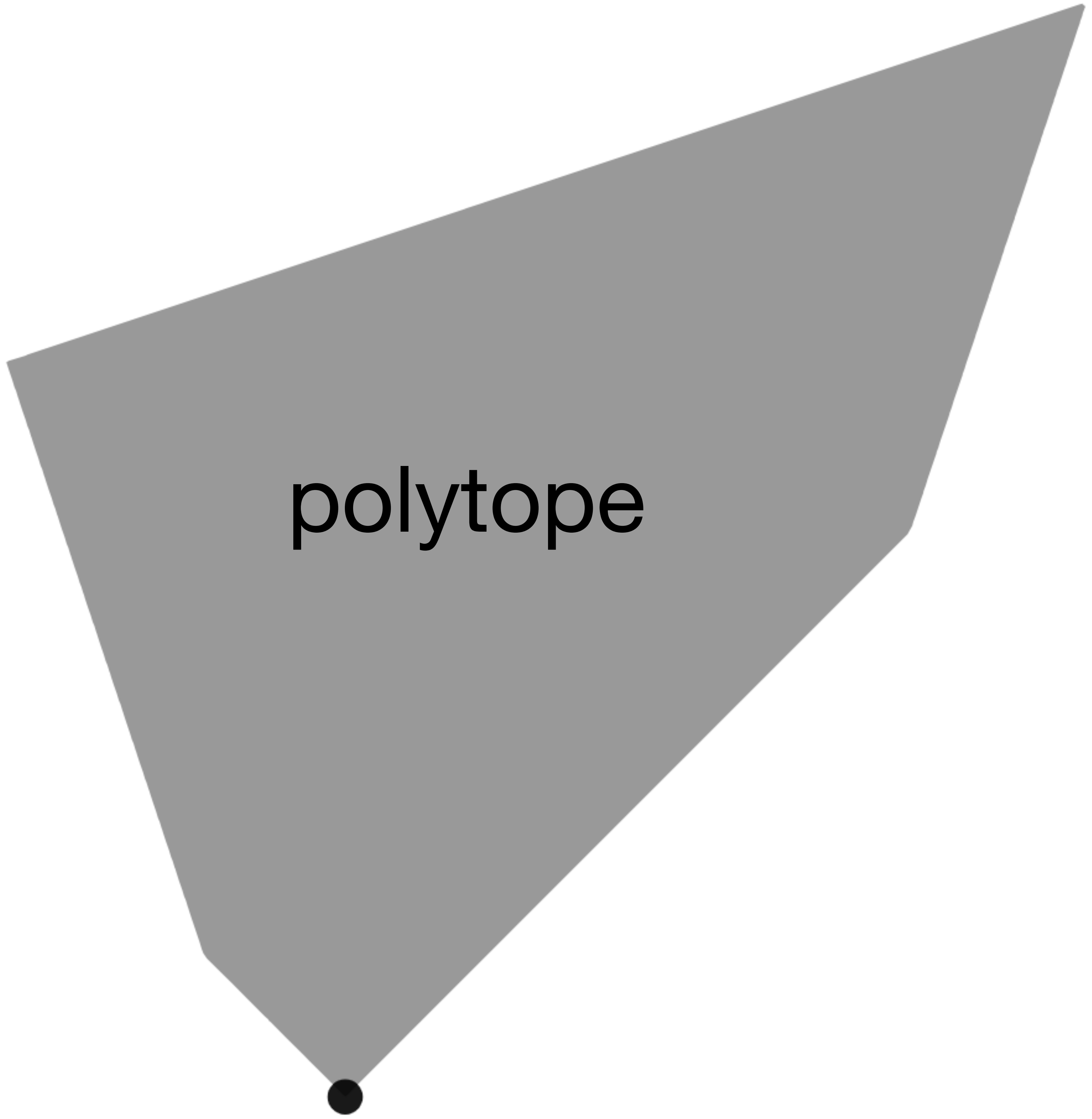


We have fast algorithms for this!

maximize $z_1 + 2z_3$
subject to
$2z_1 - z_2 + 3z_3 \leq 1$
$-z_1 + z_2 - z_3 \leq 5$

Linear Algebra primer

$a, x \in \mathbb{R}^n$, think of them as column vectors.

$a^\intercal x = a_1 x_1 + \ldots + a_n x_n$

The set of x satisfying $a^\intercal x = 0$ is a *hyperplane*.

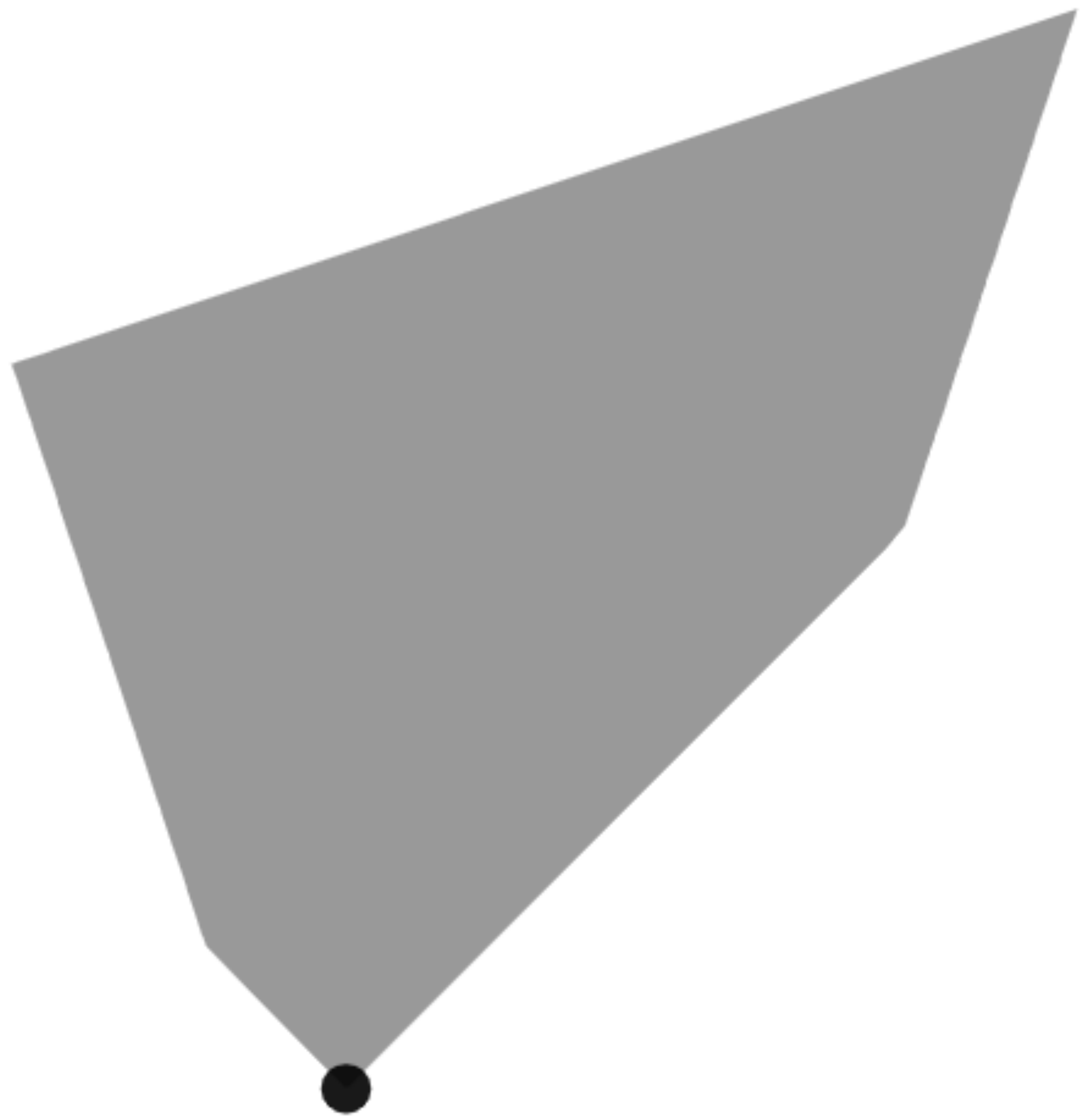

$a^\intercal x = 0$


$a^\intercal x \le 0$


$a^\intercal x \leq b$

Given: a polytope

Find: the *lowest* point in the polytope



Given: a polytope

Find: the *lowest* point in the polytope

Linear Algebra primer

$a, x \in \mathbb{R}^n$, think of them as column vectors.

$a^\intercal x = a_1 x_1 + \ldots + a_n x_n$

$$Ax = \begin{matrix} A_1 x \\ A_2 x \\ A_3 x \\ \\ A_m x \end{matrix}$$

Given: a polytope

Find: the *lowest* point in the polytope

maximize $c^\intercal x$
subject to
$Ax \leq b$

$Ax \leq b$ means
$(Ax)_i \leq b_i$
for all i

Standard form

maximize $c^\intercal x$
subject to
$Ax \leq b$
$x \geq 0$

Standard form

maximize $z_1 + 2z_3$
subject to
$2z_1 - z_2 + 3z_3 \leq 1$
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Standard form

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maximize $z_1 + 2z_3$
subject to
$2z_1 - z_2 + 3z_3 \leq 1$
$-z_1 + z_2 - z_3 \leq 5$

↓

maximize $(x_{1,a} - x_{1,b}) + 2(x_{3,a} - x_{3,b})$
subject to
$2(x_{1,a} - x_{1,b}) - (x_{2,a} - x_{2,b}) + 3(x_{3,a} - x_{3,b}) \leq 1$
$-(x_{1,a} - x_{1,b}) + (x_{2,a} - x_{2,b}) - (x_{3,a} - x_{3,b}) \leq 5$
$x \geq 0$

Max Flow

Given: a flow network

maximize flow out of s

subject to

Respecting capacities and conservation

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maximize $\sum_{e \textbf{ out of } s} x_e$

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for all e,

$0 \leq x_e \leq c(e)$

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subject to

for all e,

$0 \leq x_e \leq c(e)$

for all intermediate v,

$\sum_{e \text{ out of } v} x_e = \sum_{e \text{ into } v} x_e$

maximize $\sum_{e \textbf{ out of } s} x_e$

subject to

for all e,

$0 \leq x_e \leq c(e)$

for all intermediate v,

$$\sum_{e \text{ out of } v} x_e = \sum_{e \text{ into } v} x_e$$

→

maximize $c^\intercal x$

subject to

$Ax \leq b$

$x \geq 0$

maximize $\sum_{e \text{ out of } s} x_e$

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maximize $c^\intercal x$

subject to

$Ax \le b$

$x \ge 0$

1. $c_e = \begin{cases} 1 & \text{if } e \text{ out of } s \\ 0 & \text{otherwise.} \end{cases}$

maximize $\sum_{e \textbf{ out of } s} x_e$

subject to

for all e,

$$0 \leq x_e \leq c(e)$$

for all intermediate v,

$$\sum_{e \text{ out of } v} x_e = \sum_{e \text{ into } v} x_e$$

→

maximize $c^\intercal x$

subject to

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$$x \geq 0$$

1. $c_e = \begin{cases} 1 & \text{if } e \text{ out of } s \\ 0 & \text{otherwise.} \end{cases}$
2. $u^\intercal x \geq r \equiv (-u)^\intercal x \leq -r$

maximize $\sum_{e \textbf{ out of } s} x_e$

subject to

for all e,

$0 \leq x_e \leq c(e)$

for all intermediate v,

$$\sum_{e \text{ out of } v} x_e = \sum_{e \text{ into } v} x_e$$

→

maximize $c^\intercal x$
subject to
$Ax \leq b$
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2. $u^\intercal x \geq r \equiv (-u)^\intercal x \leq -r$
3. $u^\intercal x = r \equiv u^\intercal x \leq r, u^\intercal x \geq r$

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subject to

for all e,

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$$\sum_{e \text{ out of } v} x_e = \sum_{e \text{ into } v} x_e$$

maximize $c^\intercal x$
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1. $c_e = \begin{cases} 1 & \text{if } e \text{ out of } s \\ 0 & \text{otherwise.} \end{cases}$
2. $u^\intercal x \geq r \equiv (-u)^\intercal x \leq -r$
3. $u^\intercal x = r \equiv u^\intercal x \leq r, u^\intercal x \geq r$
4. **maximize** $c^\intercal x \equiv$ **minimize** $(-c)^\intercal x$

Shortest paths

Given: a directed graph

Find: shortest path from s to t

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Given: a directed graph

Find: shortest path from s to t

Claim: Length of the shortest path is solution to program.

minimize $\sum_e x_e$

subject to

for all e,

$x_e \geq 0,$

flow out of s is 1

$$\sum_{e \text{ out of } s} x_e - \sum_{e \text{ in to } s} x_e = 1,$$

flow into t is 1

$$\sum_{e \text{ in to } t} x_e - \sum_{e \text{ out of } t} x_e = 1,$$

for all $v \neq s, t,$

conservation of flow

$$\sum_{e \text{ out of } v} x_e = \sum_{e \text{ into } v} x_e$$

Shortest paths

Given: a directed graph

Find: shortest path from s to t

Claim: Length of the shortest path is solution to program.

Proof sketch: Optimal solution must be a combination of flows on shortest paths. Indeed, if there is a path using edges with $x_e > 0$ that is not a shortest path, delete the flow on this path and reroute it on a shortest path to get a better solution.

minimize $\sum_e x_e$

subject to

for all e,

$x_e \geq 0,$

$$\sum_{e \text{ out of } s} x_e - \sum_{e \text{ in to } s} x_e = 1,$$

$$\sum_{e \text{ in to } t} x_e - \sum_{e \text{ out of } t} x_e = 1,$$

for all $v \neq s, t$,

$$\sum_{e \text{ out of } v} x_e = \sum_{e \text{ into } v} x_e$$

Vertex Cover

Given: an undirected graph

Find: smallest set of vertices touching all edges

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Given: an undirected graph

Find: smallest set of vertices touching all edges

minimize $\sum_v x_v$

subject to

for all v,
$0 \leq x_v \leq 1,$

for all $e = \{u, v\}$
$x_u + x_v \geq 1$

Vertex Cover

Given: an undirected graph

Find: smallest set of vertices touching all edges

Want
$x_v = 0$ or $x_v = 1$

minimize $\sum_v x_v$

subject to

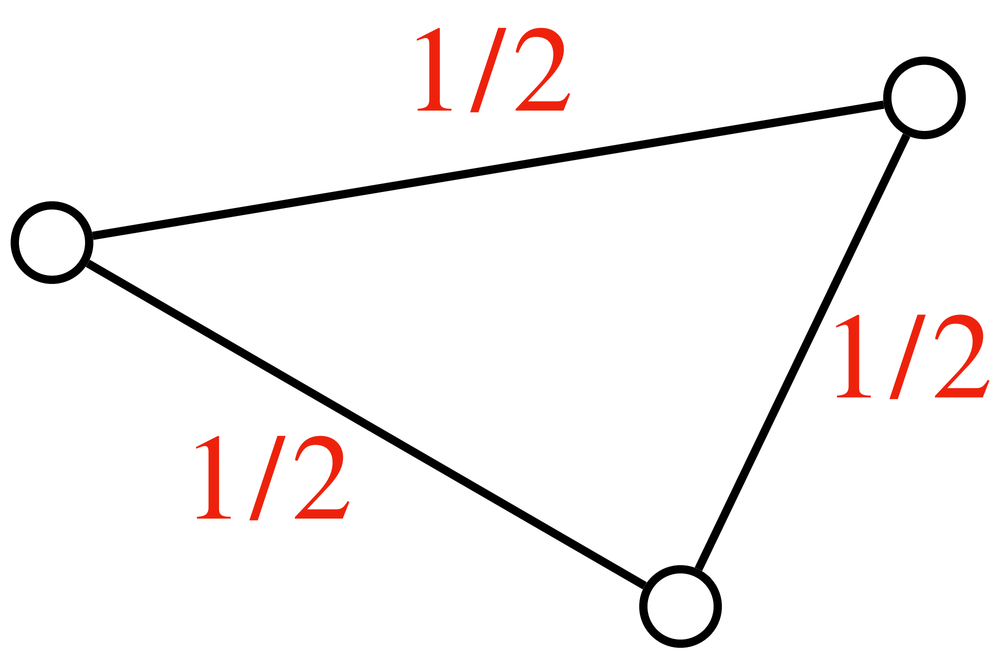
for all v,
$0 \leq x_v \leq 1,$ ❌

for all $e = \{u, v\}$
$x_u + x_v \geq 1$

Vertex Cover

Given: an undirected graph

Find: smallest set of vertices touching all edges



There is a solution of value $3/2$, even though smallest vertex cover has size 2.

Want
$x_v = 0$ or $x_v = 1$

minimize $\sum_v x_v$

subject to

for all v,
$0 \leq x_v \leq 1$, ✗

for all $e = \{u, v\}$
$x_u + x_v \geq 1$

Duality

maximize $x_1 + 2x_3$
subject to
$2x_1 - x_2 + 3x_3 \leq 1$
$-x_1 + x_2 - x_3 \leq 5$
$x \geq 0$

Duality

maximize $x_1 + 2x_3$
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$x \geq 0$

Claim: Optimum ≤ 6

Duality

maximize $x_1 + 2x_3$
subject to
$2x_1 - x_2 + 3x_3 \leq 1$
$-x_1 + x_2 - x_3 \leq 5$
$x \geq 0$

Claim: Optimum ≤ 6

Pf: $x_1 + 2x_3$
$= (2x_1 - x_2 + 3x_3) + (-x_1 + x_2 - x_3)$
≤ 6

Duality

maximize $x_1 + 2x_3$
subject to
a $\quad 2x_1 - x_2 + 3x_3 \leq 1$
b $\quad -x_1 + x_2 - x_3 \leq 5$
$x \geq 0$

Claim: Optimum ≤ 6

Pf: $x_1 + 2x_3$
$= (2x_1 - x_2 + 3x_3) + (-x_1 + x_2 - x_3)$
≤ 6

Claim: For all non-negative a, b, if
$2a - b \geq 1$
$-a + b \geq 0$
$3a - b \geq 2$
then
opt $\leq a + 5b$

Pf:
$x_1 + 2x_3$
$\leq a(2x_1 - x_2 + 3x_3) + b(-x_1 + x_2 - x_3)$
$\leq a + 5b$.

Duality

maximize $x_1 + 2x_3$
subject to
a $\quad 2x_1 - x_2 + 3x_3 \leq 1$
b $\quad -x_1 + x_2 - x_3 \leq 5$
$x \geq 0$

primal

minimize $a + 5b$
subject to
$2a - b \geq 1$
$-a + b \geq 0$
$3a - b \geq 2$
$a, b \geq 0$

dual

Claim: For all non-negative a, b, if
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$x_1 + 2x_3$
$\leq a(2x_1 - x_2 + 3x_3) + b(-x_1 + x_2 - x_3)$
$\leq a + 5b\,.$

Duality

maximize $x_1 + 2x_3$
subject to

a $\quad 2x_1 - x_2 + 3x_3 \leq 1$

b $\quad -x_1 + x_2 - x_3 \leq 5$

$x \geq 0$

primal

maximize $-a - 5b$
subject to

$-2a + b \leq -1$

$a - b \leq 0$

$-3a + b \leq -2$

$a, b \geq 0$

dual

Claim: For all non-negative a, b, if

$2a - b \geq 1$

$-a + b \geq 0$

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then

opt $\leq a + 5b$

Pf:

$x_1 + 2x_3$

$\leq a(2x_1 - x_2 + 3x_3) + b(-x_1 + x_2 - x_3)$

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Duality

maximize $x_1 + 2x_3$
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primal

maximize $-a - 5b$
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dual

What is dual of dual?

Duality

maximize $x_1 + 2x_3$
subject to
a $\quad 2x_1 - x_2 + 3x_3 \leq 1$
b $\quad -x_1 + x_2 - x_3 \leq 5$
$x \geq 0$

primal

maximize $-a - 5b$
subject to
y_1 $\quad -2a + b \leq -1$
y_2 $\quad a - b \leq 0$
y_3 $\quad -3a + b \leq -2$
$a, b \geq 0$

dual

What is dual of dual?

minimize $-y_1 - 2y_3$
subject to
$-2y_1 + y_2 - 3y_3 \geq -1$
$y_1 - y_2 + y_3 \geq -5$
$y \geq 0$

Duality

maximize $x_1 + 2x_3$
subject to
a $\quad 2x_1 - x_2 + 3x_3 \leq 1$
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primal

maximize $-a - 5b$
subject to
y_1 $\quad -2a + b \leq -1$
y_2 $\quad a - b \leq 0$
y_3 $\quad -3a + b \leq -2$
$a, b \geq 0$

dual

What is dual of dual?

minimize $-y_1 - 2y_3$
subject to
$-2y_1 + y_2 - 3y_3 \geq -1$
$y_1 - y_2 + y_3 \geq -5$
$y \geq 0$

equivalent to

maximize $y_1 + 2y_3$
subject to
$2y_1 - y_2 + 3y_3 \leq 1$
$-y_1 + y_2 - y_3 \leq 5$
$y \geq 0$

Duality

primal

maximize $c^\intercal x$
subject to
$Ax \leq b$
$x \geq 0$

dual

minimize $b^\intercal y$
subject to
$A^\intercal y \geq c$
$y \geq 0$

$\equiv$

dual

maximize $(-b)^\intercal y$
subject to
$(-A)^\intercal y \leq -c$
$y \geq 0$

Duality

primal

maximize $c^\intercal x$
subject to
$Ax \leq b$
$x \geq 0$

dual

minimize $b^\intercal y$
subject to
$A^\intercal y \geq c$
$y \geq 0$

$\equiv$

dual

maximize $(-b)^\intercal y$
subject to
$(-A)^\intercal y \leq -c$
$y \geq 0$

Thm: The dual of the dual is the primal.

Duality

primal

maximize $c^\intercal x$
subject to
$Ax \leq b$
$x \geq 0$

dual

minimize $b^\intercal y$
subject to
$A^\intercal y \geq c$
$y \geq 0$

$\equiv$

dual

maximize $(-b)^\intercal y$
subject to
$(-A)^\intercal y \leq -c$
$y \geq 0$

Thm: The dual of the dual is the primal.

dual of dual

minimize $(-c)^\intercal x$
subject to
$((-A)^\intercal)^\intercal x \geq -b$
$x \geq 0$

Duality

primal

maximize $c^\intercal x$
subject to
$Ax \leq b$
$x \geq 0$

dual

minimize $b^\intercal y$
subject to
$A^\intercal y \geq c$
$y \geq 0$

$\equiv$

dual

maximize $(-b)^\intercal y$
subject to
$(-A)^\intercal y \leq -c$
$y \geq 0$

Thm: The dual of the dual is the primal.

dual of dual

minimize $(-c)^\intercal x$
subject to
$((-A)^\intercal)^\intercal x \geq -b$
$x \geq 0$

$\equiv$

maximize $c^\intercal x$
subject to
$Ax \leq b$
$x \geq 0$

Duality

primal

maximize $c^\intercal x$
subject to
$Ax \leq b$
$x \geq 0$

dual

minimize $b^\intercal y$
subject to
$A^\intercal y = c$
$y \geq 0$

Thm: The dual of the dual is the primal.

Thm: (Weak Duality) Every solution to primal is at most every solution to dual.

Duality

primal

maximize $c^\intercal x$
subject to
$Ax \leq b$
$x \geq 0$

dual

minimize $b^\intercal y$
subject to
$A^\intercal y = c$
$y \geq 0$

Thm: The dual of the dual is the primal.

Thm: (Weak Duality) Every solution to primal is at most every solution to dual.

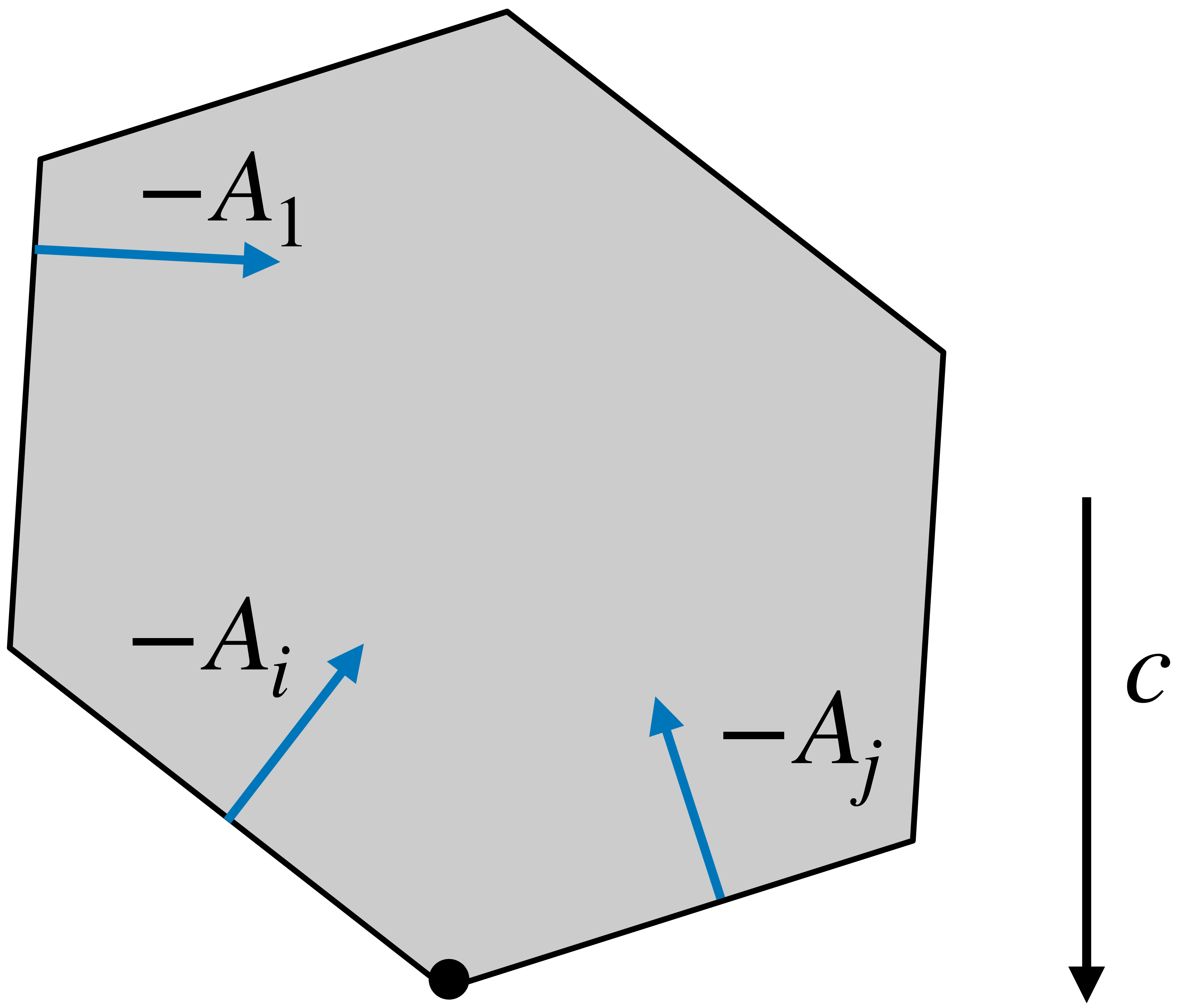
Thm: (Strong Duality) If primal has solution of finite value, then value is equal to optimal solution of dual.

Duality

primal	dual
maximize $c^\intercal x$	**minimize** $b^\intercal y$
subject to	subject to
$Ax \leq b$	$A^\intercal y \leq c$
$x \geq 0$	$y \geq 0$

Thm: (Strong Duality) If primal has solution of finite value, then value is equal to optimal solution of dual.

Fact: A vertex is point for which n of the inequalities become tight.



By physics:

There must be $y_i, y_j \geq 0$

$y_i A_i + y_j A_j = c.$

If $\hat{A}x = \hat{b}$ correspond to sides touching x,

$A^\intercal y = \hat{A}^\intercal \hat{y} = c.$

Then

$b^\intercal y = \hat{b}^\intercal \hat{y} = (\hat{A}x)^\intercal y = x^\intercal \hat{A}^\intercal \hat{y} = x^\intercal c = c^\intercal x$

Duality of Max flow

maximize $\sum_{e \textbf{ out of } s} x_e$

subject to

for all e,

$0 \leq x_e \leq c(e)$

for all intermediate v,

$$\sum_{e \text{ out of } v} x_e = \sum_{e \text{ into } v} x_e$$

minimize $c^{\mathsf{T}} a$

subject to

for all $e = (s, v)$,

$a_e + b_v \geq 1$

for all $e = (u, t)$,

$a_e - b_u \geq 0$

for all other $e = (u, v)$,

$a_e - b_u + b_v \geq 0$

for all e

$a_e \geq 0$

Duality of Max flow

maximize $\sum_{e \textbf{ out of } s} x_e$

subject to

for all e,

$0 \leq x_e \leq c(e)$

for all intermediate v,

$$\sum_{e \text{ out of } v} x_e = \sum_{e \text{ into } v} x_e$$

minimize $c^\intercal a$

subject to

for all $e = (s, v)$,

$a_e + b_v \geq 1$

for all $e = (u, t)$,

$a_e - b_u \geq 0$

for all other $e = (u, v)$,

$a_e - b_u + b_v \geq 0$

for all e

$a_e \geq 0$

$\equiv$

minimize $c^\intercal a$

subject to

$b_s = 1, b_t = 0$

for all $e = (u, v)$,

$a_e \geq b_u - b_v$

for all e

$a_e \geq 0$

minimize $c^\top a$

subject to

for all $e = (s, v)$,
$a_e + b_v \leq 1$
for all $e = (u, t)$,
$a_e - b_u \leq 0$
for all other $e = (u, v)$,
$a_e - b_u + b_v \leq 0$

for all e
$a_e \geq 0$

$\equiv$

minimize $c^\top a$

subject to

$b_s = 1, b_t = 0$

for all $e = (u, v)$,
$a_e \geq b_u - b_v$

for all e
$a_e \geq 0$

$\equiv$

minimize $c^\top a$

subject to

$b_s = 1, b_t = 0$

for all $e = (u, v)$,
$a_e = \max\{0, b_u - b_v\}$

minimize $c^{\intercal}a$

subject to

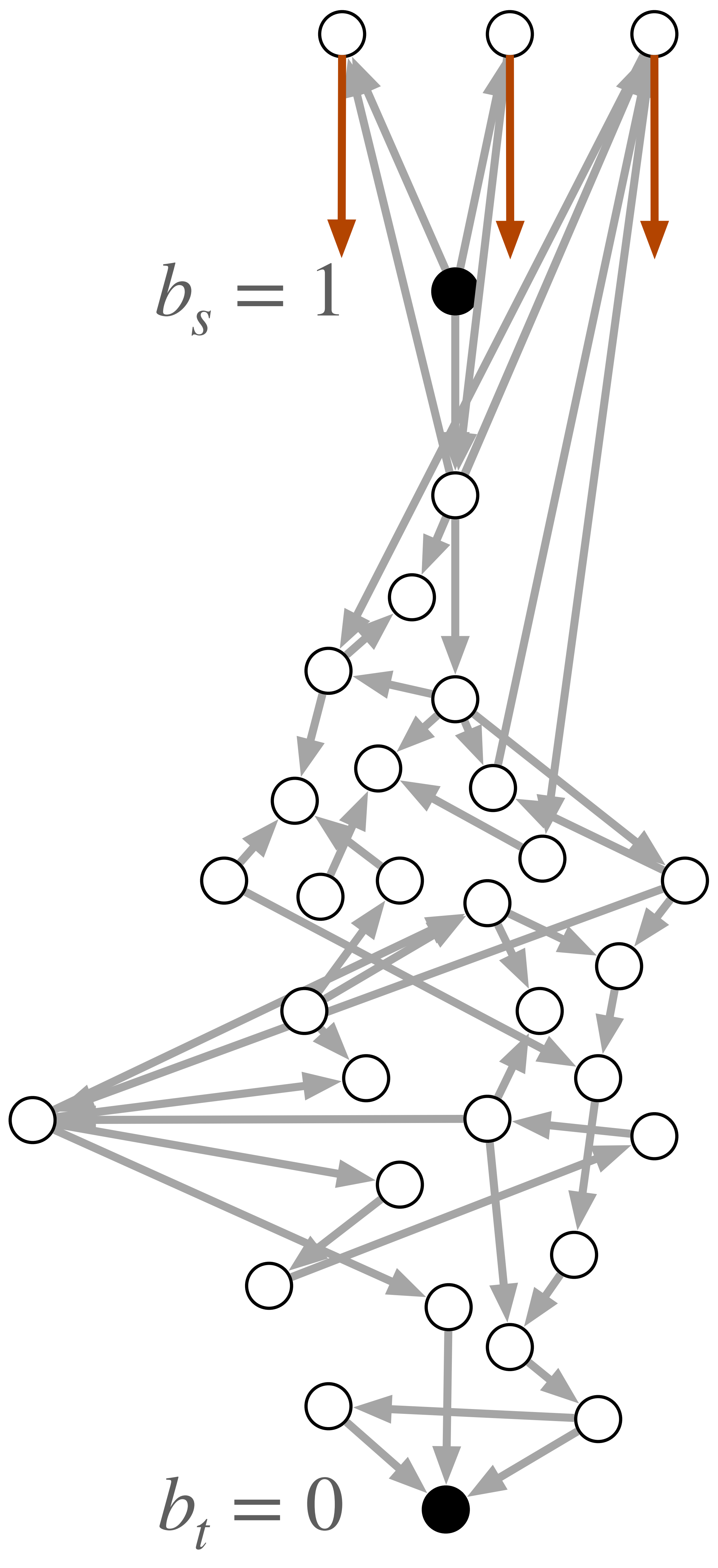
$b_s = 1, b_t = 0$

$0 \leq b_u \leq 1$

for all $e = (u, v)$,

$a_e = \max\{0, b_u - b_v\}$

Claim: Opt is achieved with
$1 \geq b_u \geq 0.$
Pf: Take any solution and move the extreme values up/down. The solution only improves.



minimize $c^{\top}a$

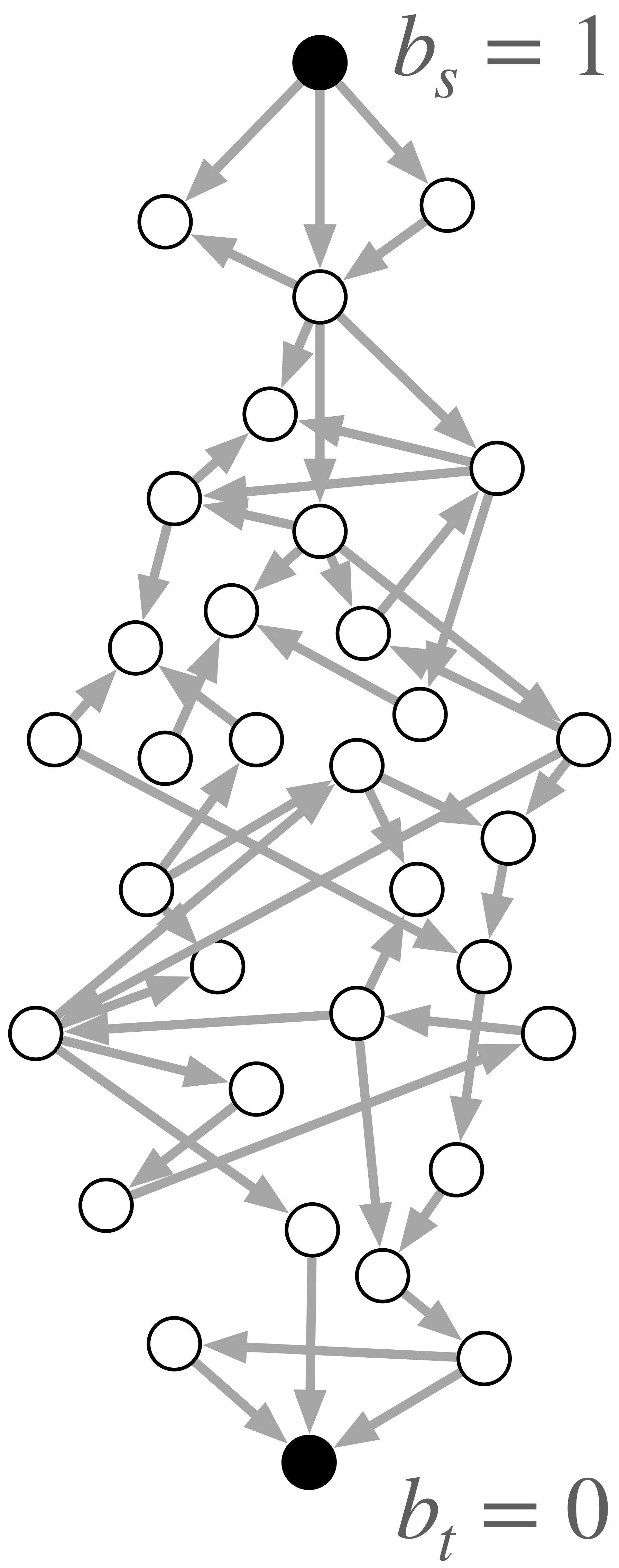
subject to

$b_s = 1, b_t = 0$

$0 \leq b_u \leq 1$

for all $e = (u, v)$,

$a_e = \max\{0, b_u - b_v\}$



minimize $c^{\mathsf{T}}a$

subject to

$b_s = 1, b_t = 0$

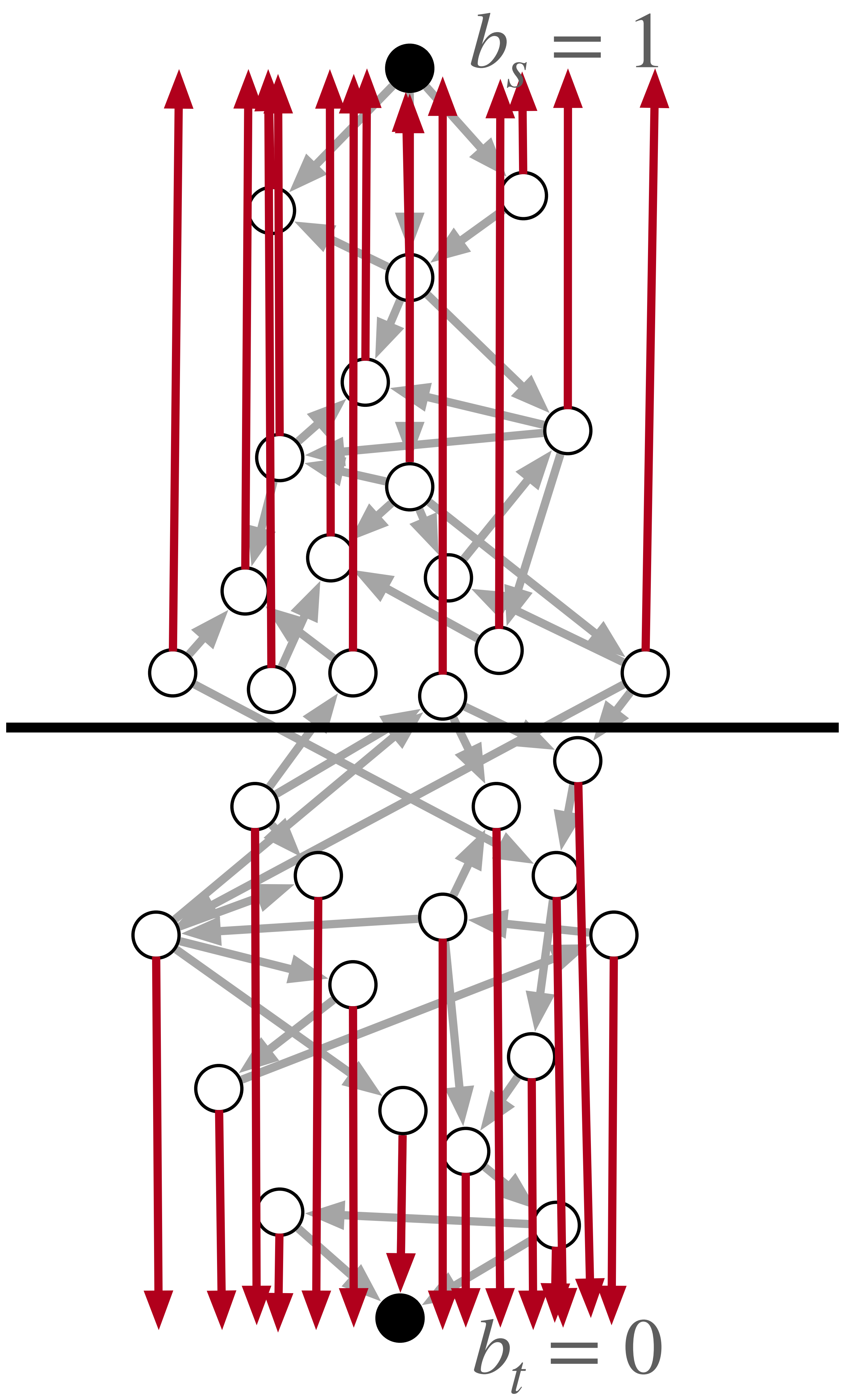
$0 \leq b_u \leq 1$

for all $e = (u, v)$,

$a_e = \max\{0, b_u - b_v\}$

Claim: Opt is achieved with $b_u = 0/1$.

Pf: Pick $0 \leq t \leq 1$ uniformly at random. If $b_u \geq t$, set $b_u = 1$, otherwise set it to 0. The expected value of resulting solution is the same as original!



minimize $c^\top a$

subject to

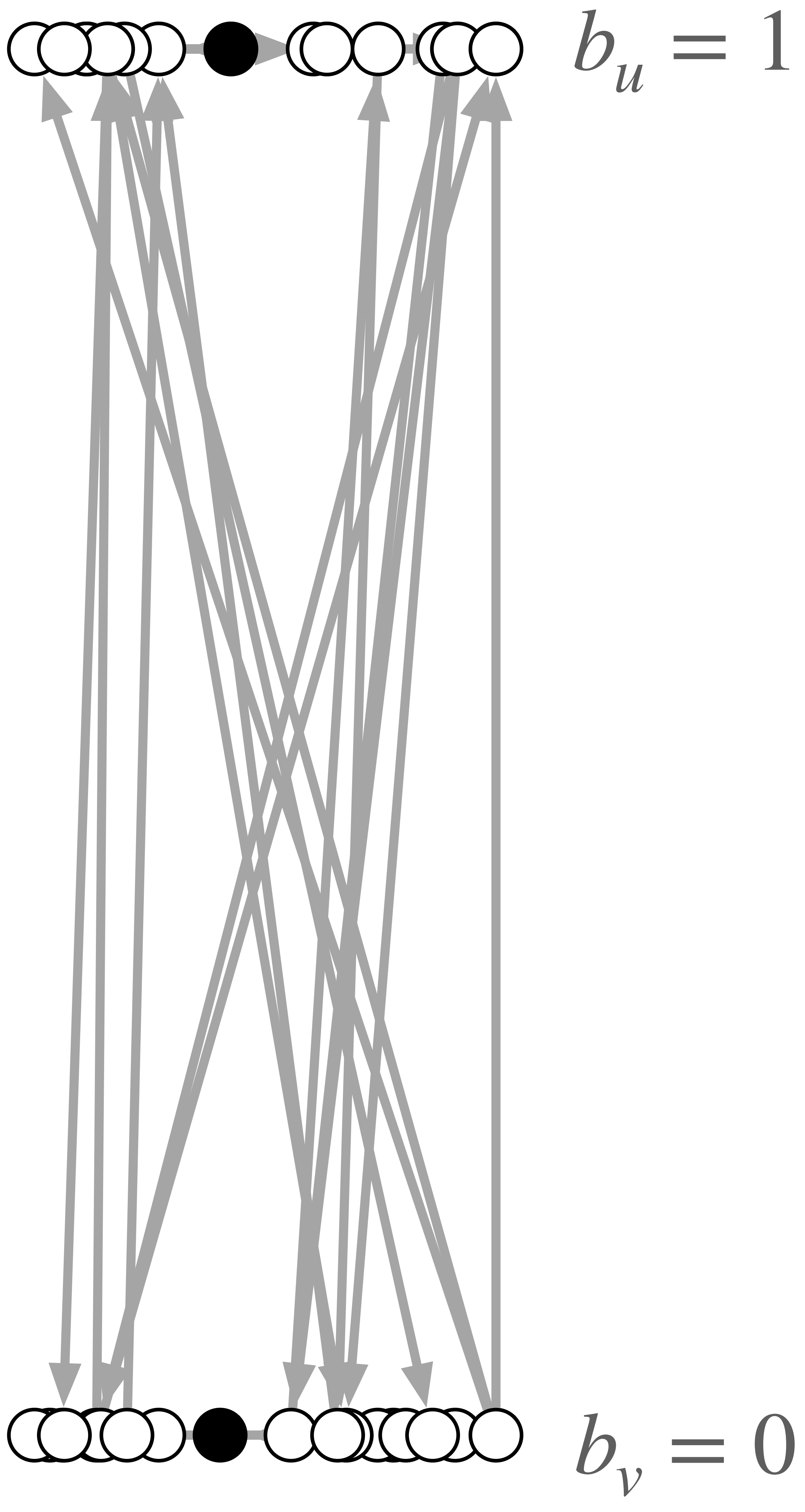
$b_s = 1, b_t = 0$

$b_u \in \{0,1\}$

for all $e = (u, v)$,

$a_e = \max\{0, b_u - b_v\}$

Min-Cut!



Duality of Shortest Path

minimize $\sum_e x_e$

subject to

for all e,

$x_e \geq 0$,

$$\sum_{e \text{ out of } s} x_e - \sum_{e \text{ in to } s} x_e = 1,$$

$$\sum_{e \text{ out of } t} x_e - \sum_{e \text{ in to } t} x_e = -1,$$

for all $v \neq s, t$,

$$\sum_{e \text{ out of } v} x_e - \sum_{e \text{ into } v} x_e = 0$$

Duality of Shortest Path

minimize $\sum_{e} x_e$

subject to

for all e,

$x_e \geq 0,$

$$\sum_{e \text{ out of } s} x_e - \sum_{e \text{ in to } s} x_e = 1,$$

$$\sum_{e \text{ out of } t} x_e - \sum_{e \text{ in to } t} x_e = -1,$$

for all $v \neq s, t,$

$$\sum_{e \text{ out of } v} x_e - \sum_{e \text{ into } v} x_e = 0$$

dual

maximize $a_s - a_t$

subject to

for all edges $e = (u, v),$

$a_u - a_v \leq 1$

Duality of Shortest Path

minimize $\sum_{e} x_e$

subject to

for all e,

$x_e \geq 0$,

$$\sum_{e \text{ out of } s} x_e - \sum_{e \text{ in to } s} x_e = 1,$$

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for all $v \neq s, t$,

$$\sum_{e \text{ out of } v} x_e - \sum_{e \text{ into } v} x_e = 0$$

dual

maximize $a_s - a_t$

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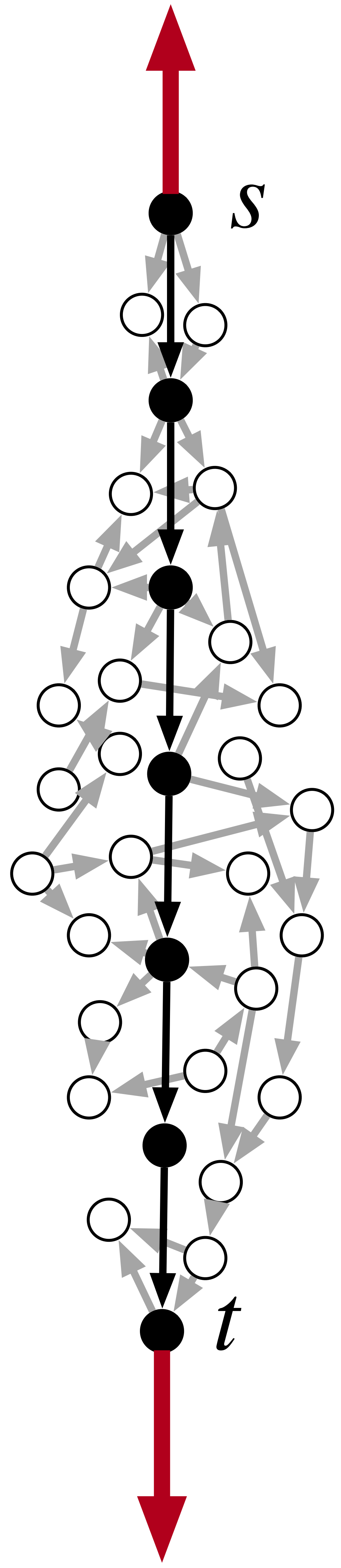


Duality and zero-sum games

Two player zero-sum game:

an $m \times n$ matrix G

$G_{i,j}$: payoff to row player, assuming row player uses strategy i, and column player uses strategy j.

$-G_{i,j}$: payoff to column player.

Example: Chess

i: specifies how white would move in every possible board configuration.

j: specifies how black would move.

$$G_{i,j} = \begin{cases} 1 & \text{if white wins} \\ -1 & \text{if black wins} \\ 0 & \text{stalemate} \end{cases}$$

Randomized strategy:

probability distribution on row strategies

A column vector x with

$$x_i \geq 0, \ \sum_i x_i = 1$$

probability distribution on column strategies

$$y_i \geq 0, \ \sum_j y_j = 1$$

expected payoff to row player

$x^\intercal G y$

Who decides on their strategy first?

If row player commits to x

Row player will get payoff

$$\min_y x^\intercal G y = \min_j (x^\intercal G)_j$$

So, if row player has to play first:

$$\max_x \min_y x^\intercal G y$$

If column player commits to y

Row player will get payoff

$$\max_x x^\intercal G y = \max_i (G y)_i$$

So, if column player has to play first

$$\min_y \max_x x^\intercal G y$$

Randomized strategy:

probability distribution on row strategies

A column vector x with

$$x_i \geq 0, \ \sum_i x_i = 1$$

probability distribution on column strategies

$$y_i \geq 0, \ \sum_j y_j = 1$$

expected payoff to row player

$$x^\intercal G y$$

von-Neumann's min-max Theorem

If row player commits to x

Row player will get payoff

$$\min_y x^\intercal G y = \min_j (x^\intercal G)_j$$

So, if row player has to play first:

$$\max_x \min_y x^\intercal G y$$

If column player commits to y

Row player will get payoff

$$\max_x x^\intercal G y = \max_i (Gy)_i$$

So, if column player has to play first

$$\min_y \max_x x^\intercal G y$$

Doesn't matter who plays first:

Thm:

$$\max_x \min_y x^\intercal G y = \min_y \max_x x^\intercal G y.$$

Using strong duality

Thm: $\max_x \min_y x^\intercal G y = \min_y \max_x x^\intercal G y$.

$$\max_x \min_j (x^\intercal G)_j = \min_y \max_i (Gy)_i$$

primal

maximize z
subject to

w $x_1 + \ldots + x_m = 1$

for all j,

y_j $z \leq (x^\intercal G)_j$

$x \geq 0$

coefficient of z must be 1

coefficient of x_i must be ≥ 0

dual

minimize w
subject to

$y_1 + \ldots + y_m = 1$

for all i,

$w \geq (Gy)_i$

$y \geq 0$

Algorithms for Linear programs

Simplex Algorithm

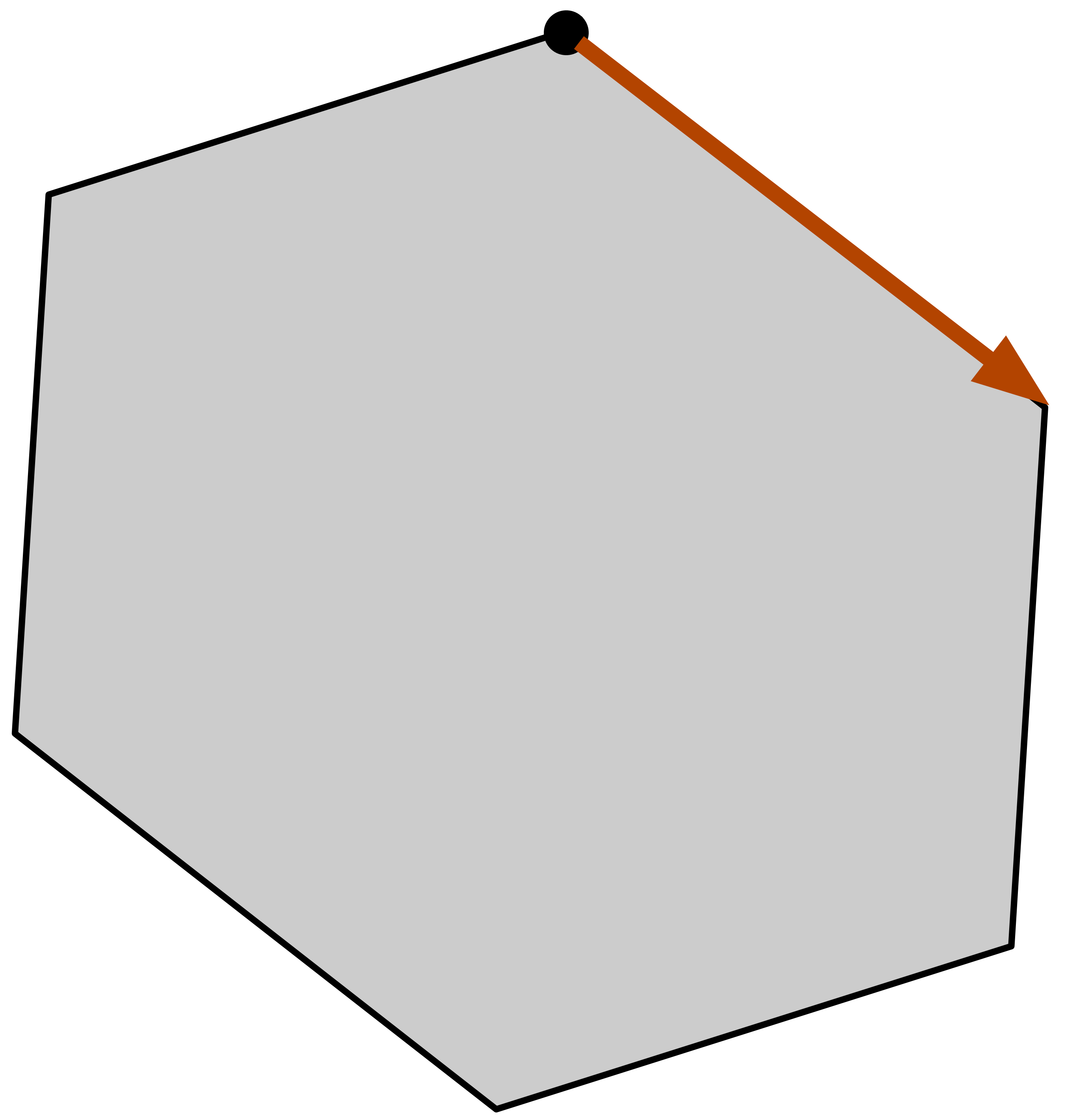
Simple
Often fast in practice
Not polynomial time (on pathological counterexamples)

Ellipsoid Algorithm

More complicated
Polynomial time, but not always fast

Simplex

Start with a vertex
In each step,
move to a lower vertex



Problem: Number of vertices
on this path can be
exponential!

Simplex: how to find initial vertex?

maximize $c^\intercal x$
subject to
$Ax \leq b$
$x \geq 0$

→

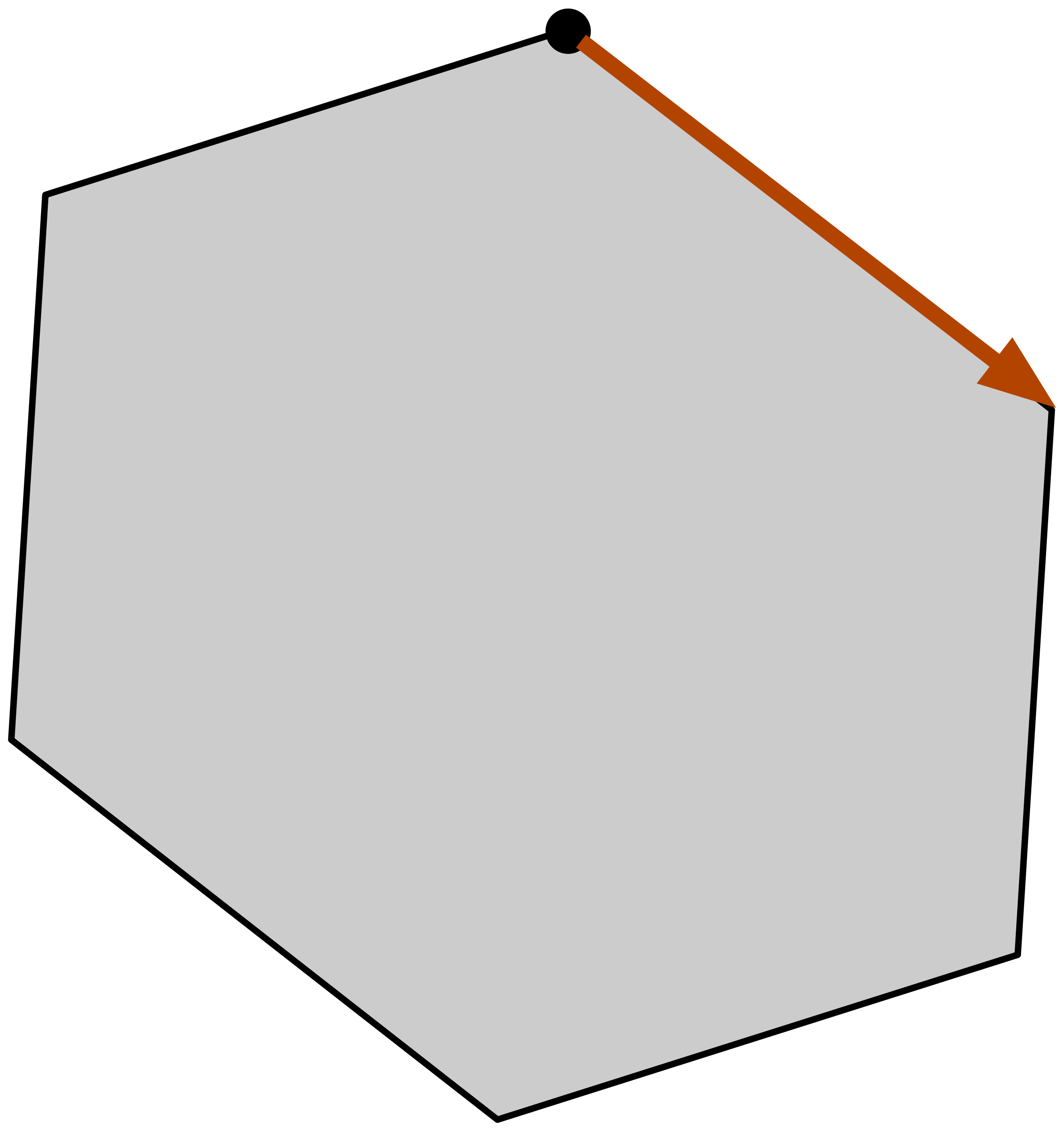
minimize $z_1 + z_2 + \ldots$
subject to
$Ax \leq b + z$
$x, z \geq 0$

For this program, $z_i = \max\{0, -b_i\}, x = 0$ is a vertex. Run simplex to find a solution with $z = 0$. The x value of solution will be a a vertex of original program!

Simplex: how to go to better vertex?

maximize $c^\intercal x$
subject to
$Ax \leq b$
$x \geq 0$

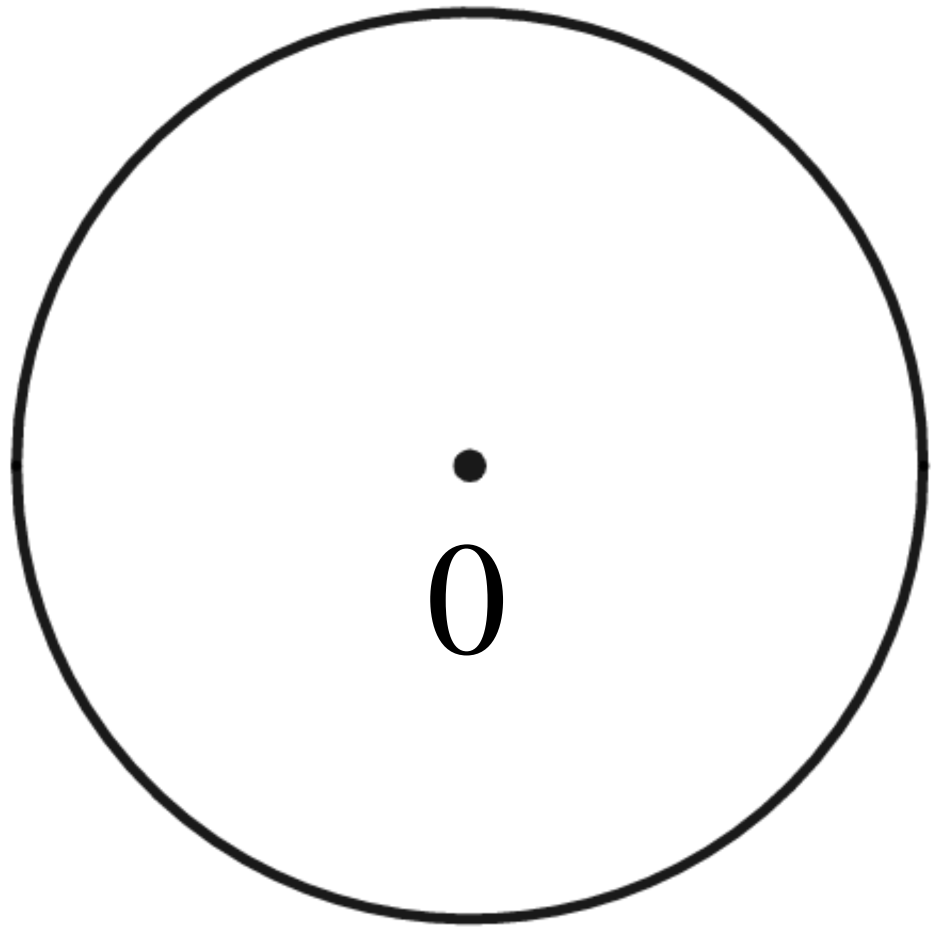
1. There must be $\hat{A}x = \hat{b}$.
2. Find y satisfying $n-1$ of the equations, $c^\intercal y > 0$.
3. Change $x = x + \epsilon y$, until some new equation becomes tight.



Ellipsoid method

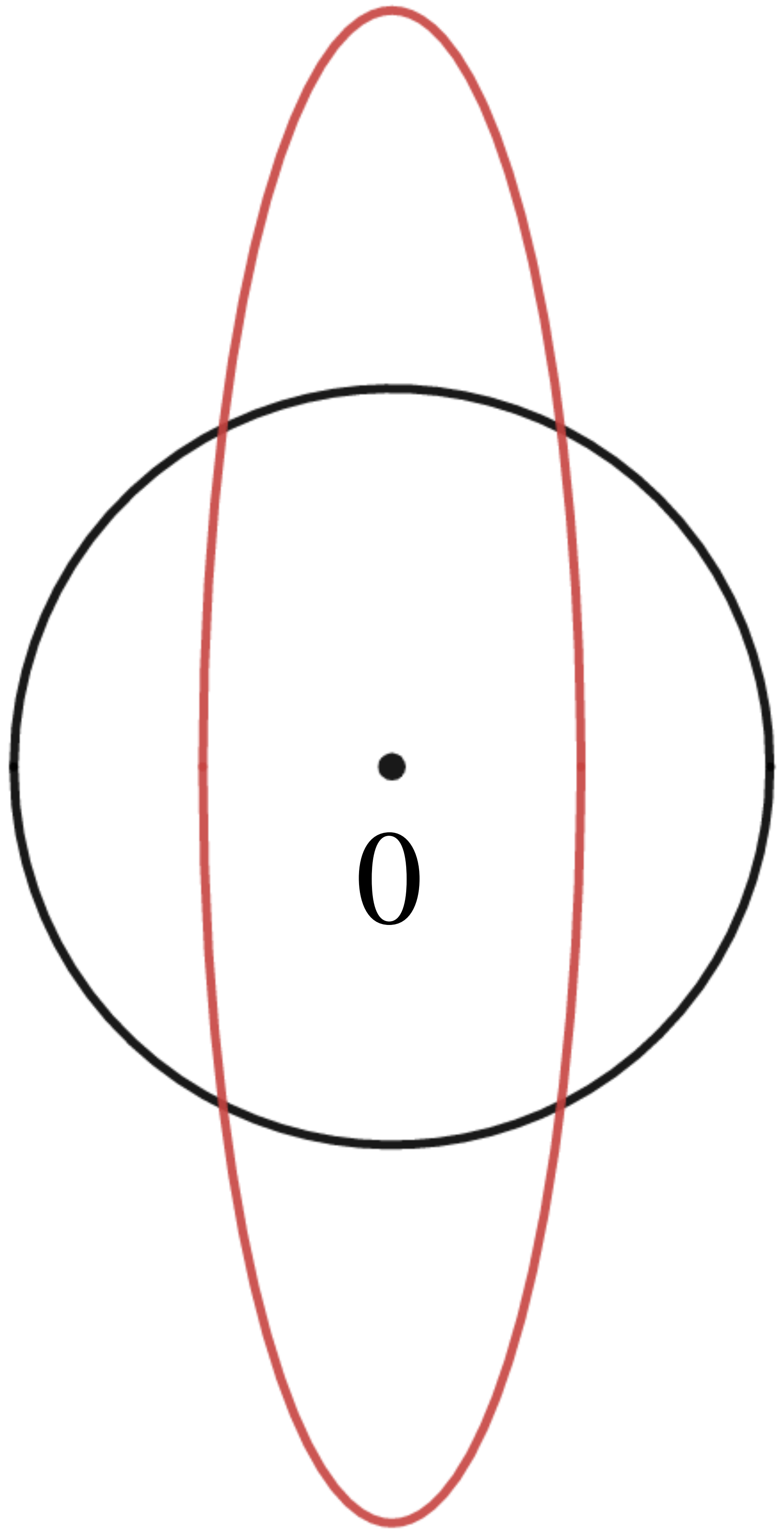
Ellipsoid: a squished ball

$$x^2 + y^2 \leq 1$$



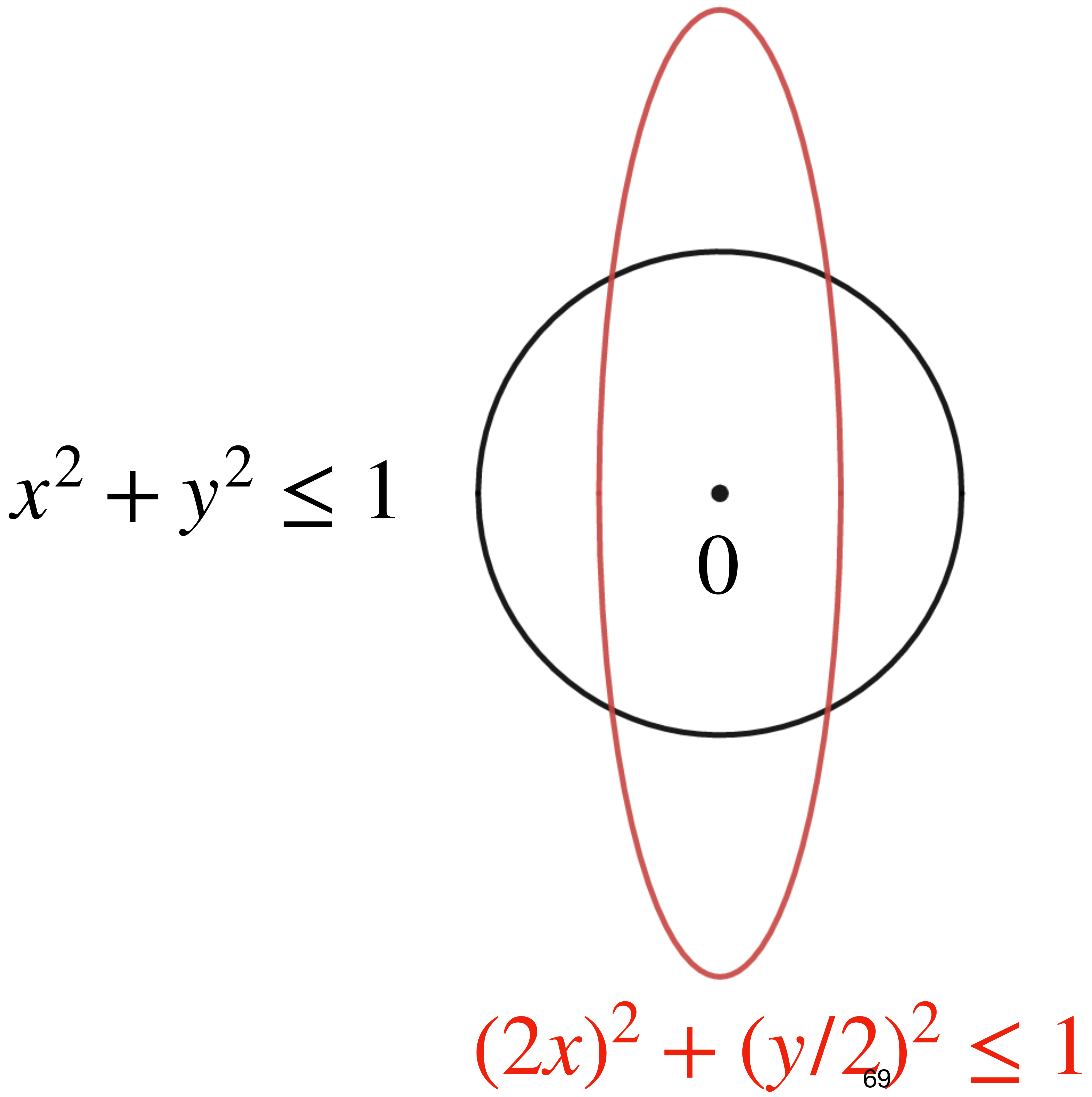
Ellipsoid method

Ellipsoid: a squished ball



Ellipsoid method

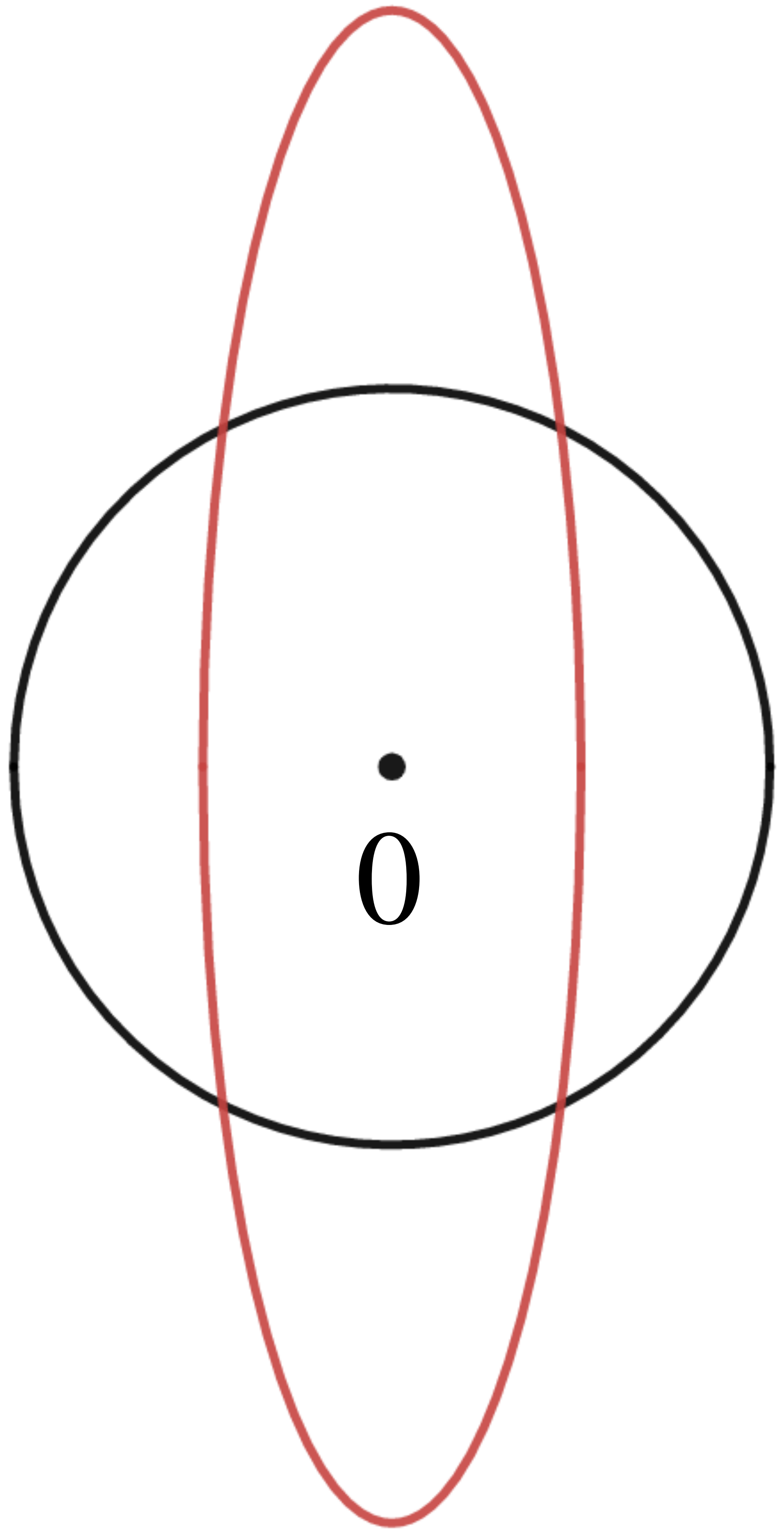
Ellipsoid: a squished ball



Ellipsoid method

Ellipsoid: a squished ball

$$x^2 + y^2 \leq 1$$



$$(2x)^2 + (y/2)^2 \leq 1$$

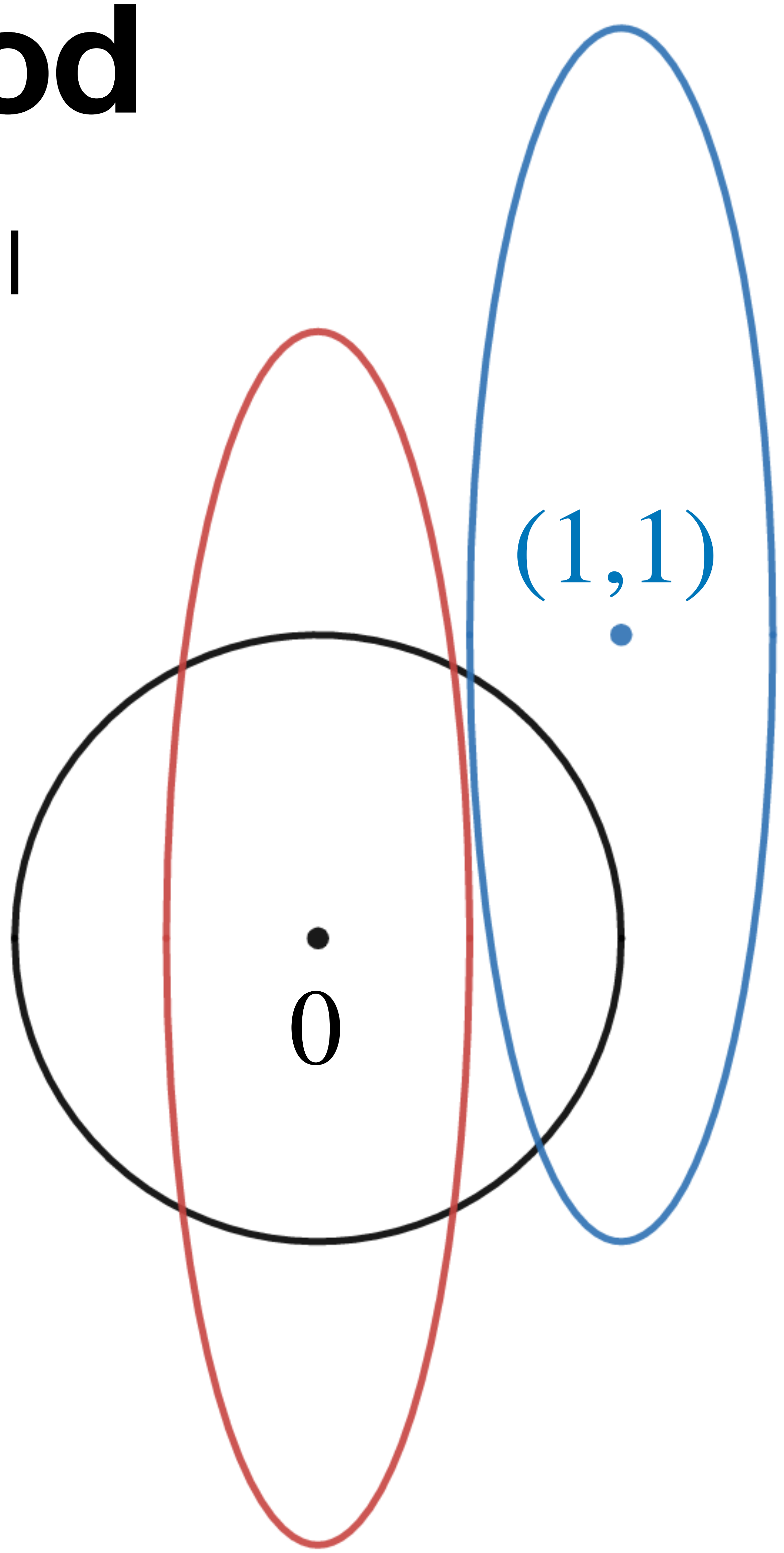
Ratio of area of ellipsoid to sphere:

$$\frac{1}{2} \cdot \frac{2}{1} = 1$$

Ellipsoid method

Ellipsoid: a squished ball

$x^2 + y^2 \leq 1$



$(2(x-1))^2 + ((y-1)/2)^2 \leq 1$

$(2x)^2 + (y/2)^2 \leq 1$

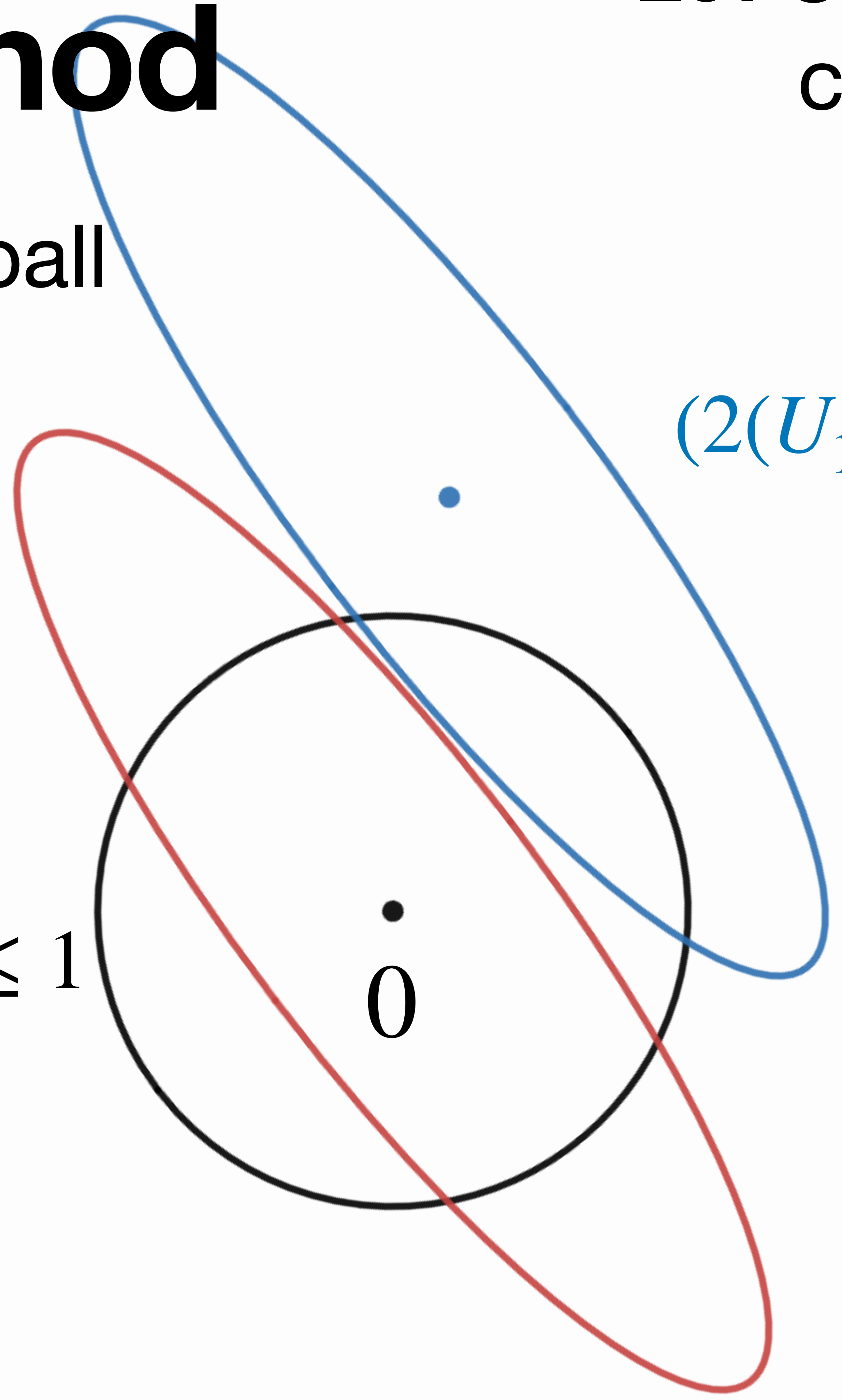
Ratio of area of ellipsoid to sphere:

$$\frac{1}{2} \cdot \frac{2}{1} = 1$$

Ellipsoid method

Let U^{-1} be the linear transformation corresponding to a rotation.

Ellipsoid: a squished ball



$(2(U_1(x,y)-1))^2 + ((U_2(x,y)-1)/2)^2 \leq 1$

$(U_1(x,y))^2 + (U_2(x,y))^2 \leq 1$

Ratio of area of ellipsoid to sphere:

$$\frac{1}{2} \cdot \frac{2}{1} = 1$$

$(2U_1(x,y))^2 + (U_2(x,y)/2)^2 \leq 1$

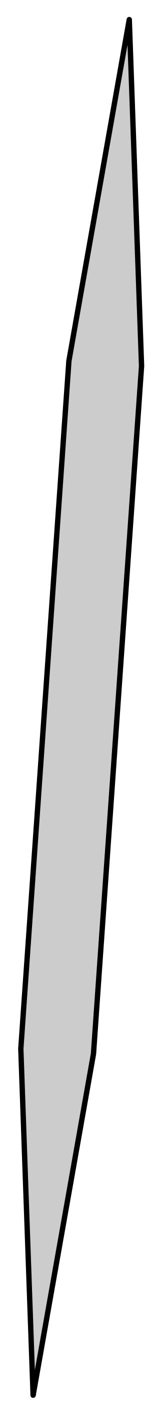
The desired solution is bounded

Fact: If the solution is finite, then its magnitude is at most $2^{O(\text{poly}(\text{input length}))}$.

Pf: If finite, the solution occurs at a vertex. Since every vertex satisfies $Bx = d$, for some B, d, we have $x = B^{-1}d$, and the size of coefficients of B^{-1} are polynomially related to the size of coefficients of A.

Fact: If there is finite solution, then volume of feasible region (i.e. polytope) is at least $2^{-O(\text{poly}(\text{input length}))}$.

Pf sketch: The smallest angle that can be generated is $2^{-O(\text{poly}(\text{input length}))}$.



Ellipsoid method

maximize $c^{\intercal}x$
subject to
$Ax \leq b$
$x \geq 0$

Is there x
with
$c^{\intercal}x \geq d$
$Ax \leq b$
$x \geq 0$

Claim: If we can find x inside polytope in poly time, we can use binary search to find the best value of d in poly time!

Fact: If the solution is finite, then its magnitude is at most $2^{O(\text{poly}(\text{input length}))}$.

Fact: If there is finite solution, then volume of feasible region (i.e. polytope) is at least $2^{-O(\text{poly}(\text{input length}))}$.

Consequence: We know $-T \leq c^{\intercal}x \leq T$, where $T \leq 2^{O(\text{poly}(\text{input length}))}$.

Using binary search

Check polytope is non-empty

Add new constraint

Find point

Add new constraint

Find point: polytope is empty!

Add new constraint

Add new constraint

Find point


$y \leq -T/4$
$y \leq -T/2$

Add new constraint

Find point: polytope is empty!

$y \leq -T/4$

$y \leq -3T/8$

$y \leq -T/2$

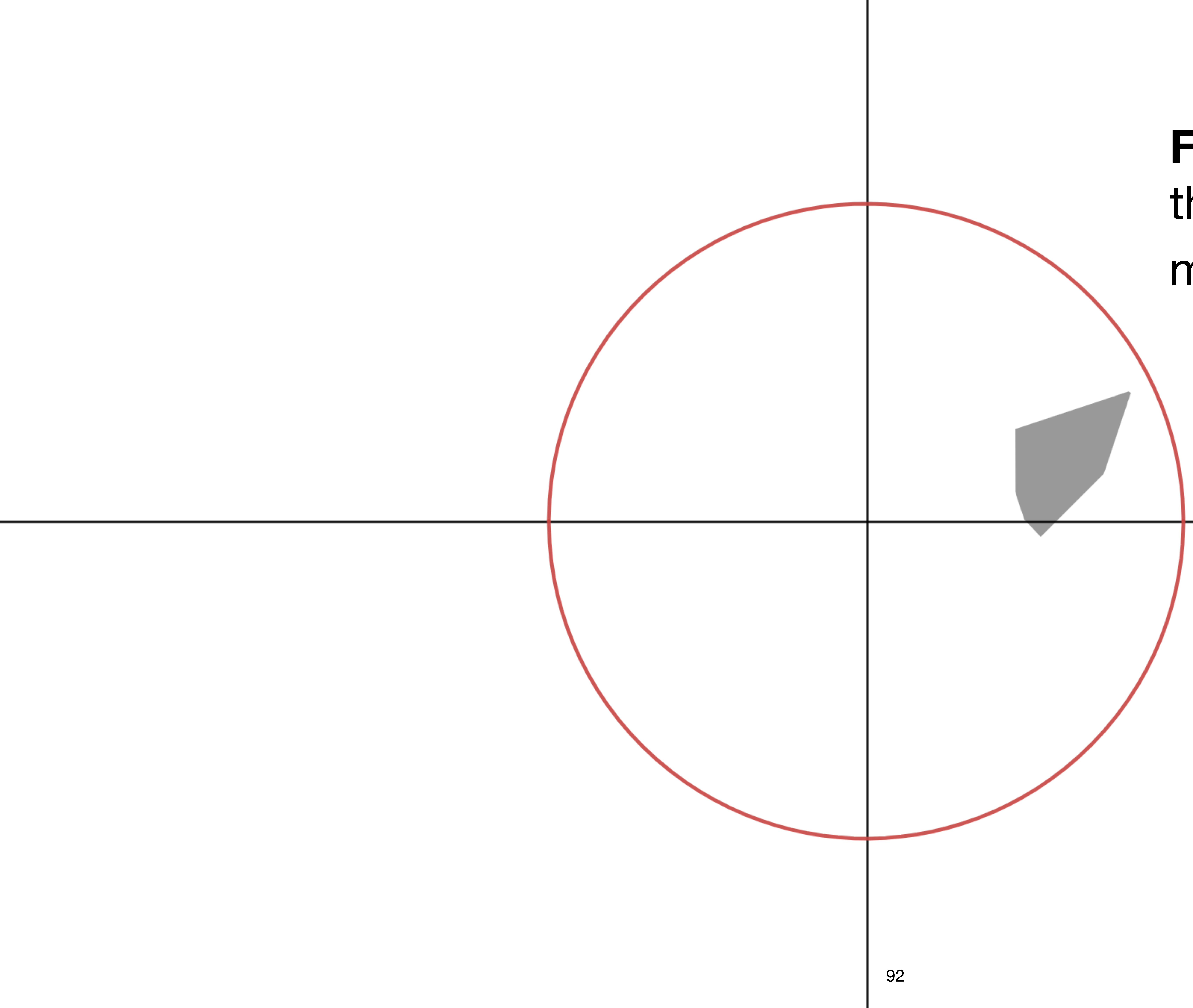

$y \leq -T/4$
$y \leq -3T/8$

Find point

Conclusion: It is enough to give an algorithm to find a point in a polytope.

Ellipsoid algorithm for finding points in polytopes

Idea: Iteratively find ellipsoids where the density of the polytope is larger and larger, until a point is found



Fact: If the solution is finite, then its magnitude is at most $2^{O(\text{poly}(\text{input length}))}$.

Check 0

Find violated inequality

Shift inequality to origin

Find ellipsoid containing half-sphere

Find ellipsoid containing half-sphere

Shift to center

Stretch to get sphere

Check 0

Find violated inequality

Shift inequality to origin

Find ellipsoid containing half-sphere

Find ellipsoid containing half-sphere

Shift to center

Stretch to get sphere

Check 0

Ellipsoid method

Is there x
with
$c^\intercal x \geq d$
$Ax \leq b$
$x \geq 0$

Algorithm to find element of non-empty P:

1. Let E be circle of radius R containing polytope P.
2. If $0 \in P$, output 0.
3. Otherwise half-circle containing P, and ellipsoid E' containing half-circle.
4. Scale and shift E' to get E, and find element of P using new E.

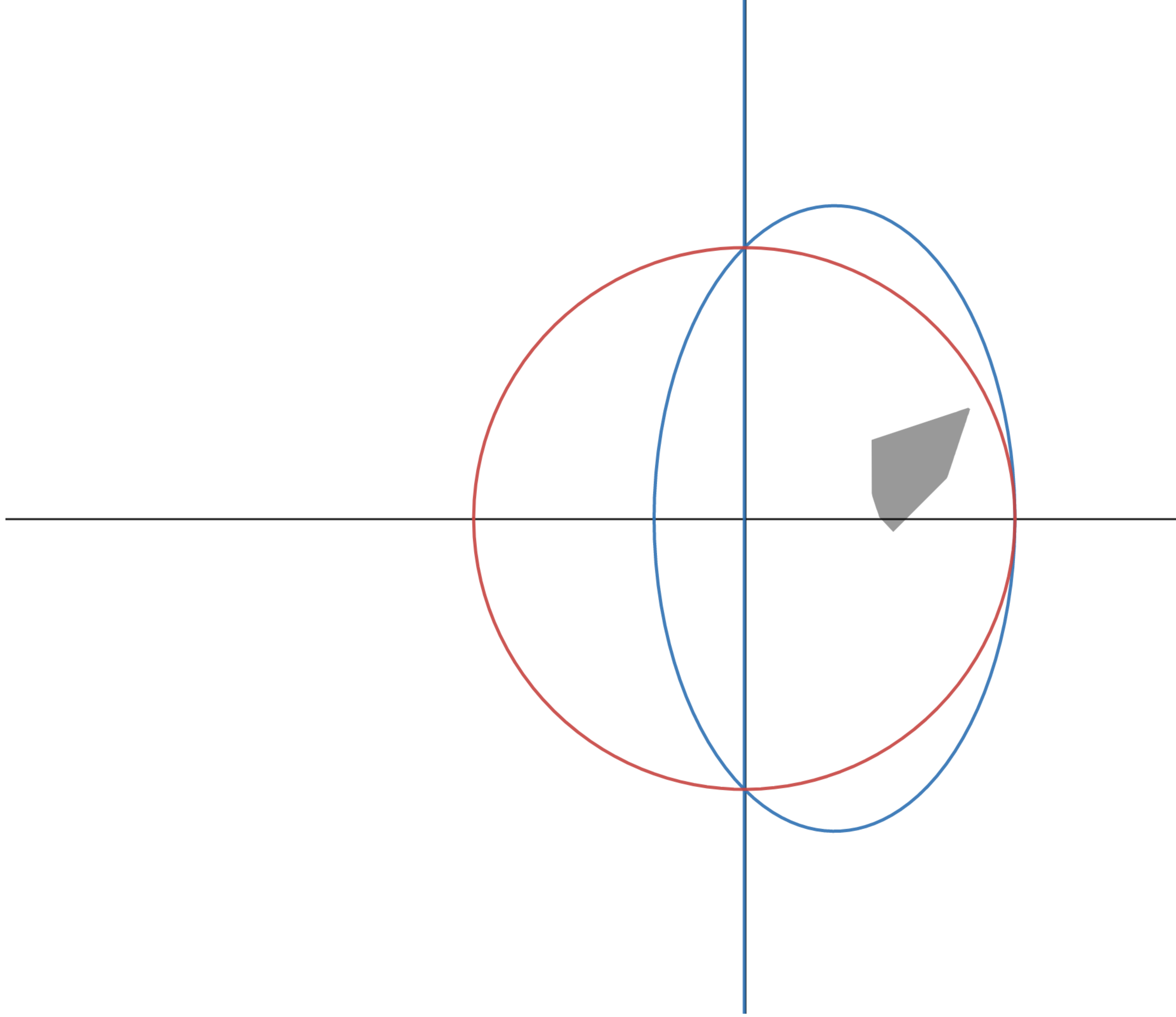
Key Lemma: vol(E')/vol$(E) \leq e^{\frac{-1}{2(n+1)}}$

Corollary: vol(P)/vol$(E') \geq e^{\frac{1}{2(n+1)}} \cdot$ vol(P)/vol(E)

Corollary: After t rounds,
vol(P)/vol$(E') \geq e^{\frac{t}{2(n+1)}} \cdot$ vol(P)/vol(E)

Corollary: The algorithm must terminate in poly(input length) steps.

E: $\sum_i x_i^2 \leq 1$



E': ellipsoid containing right half-ball

$$\left(\frac{n+1}{n}\right)^2\left(x_1 - \frac{1}{n+1}\right)^2 + \frac{n^2-1}{n^2}\cdot\sum_{i>2} x_i^2 \leq 1$$

Claim: E' contains right half-ball.

If $x \in E, x_1 \geq 0$, then

$$\left(\frac{n+1}{n}\right)^2\left(x_1 - \frac{1}{n+1}\right)^2 + \frac{n^2-1}{n^2}\cdot\sum_{i>2} x_i^2$$

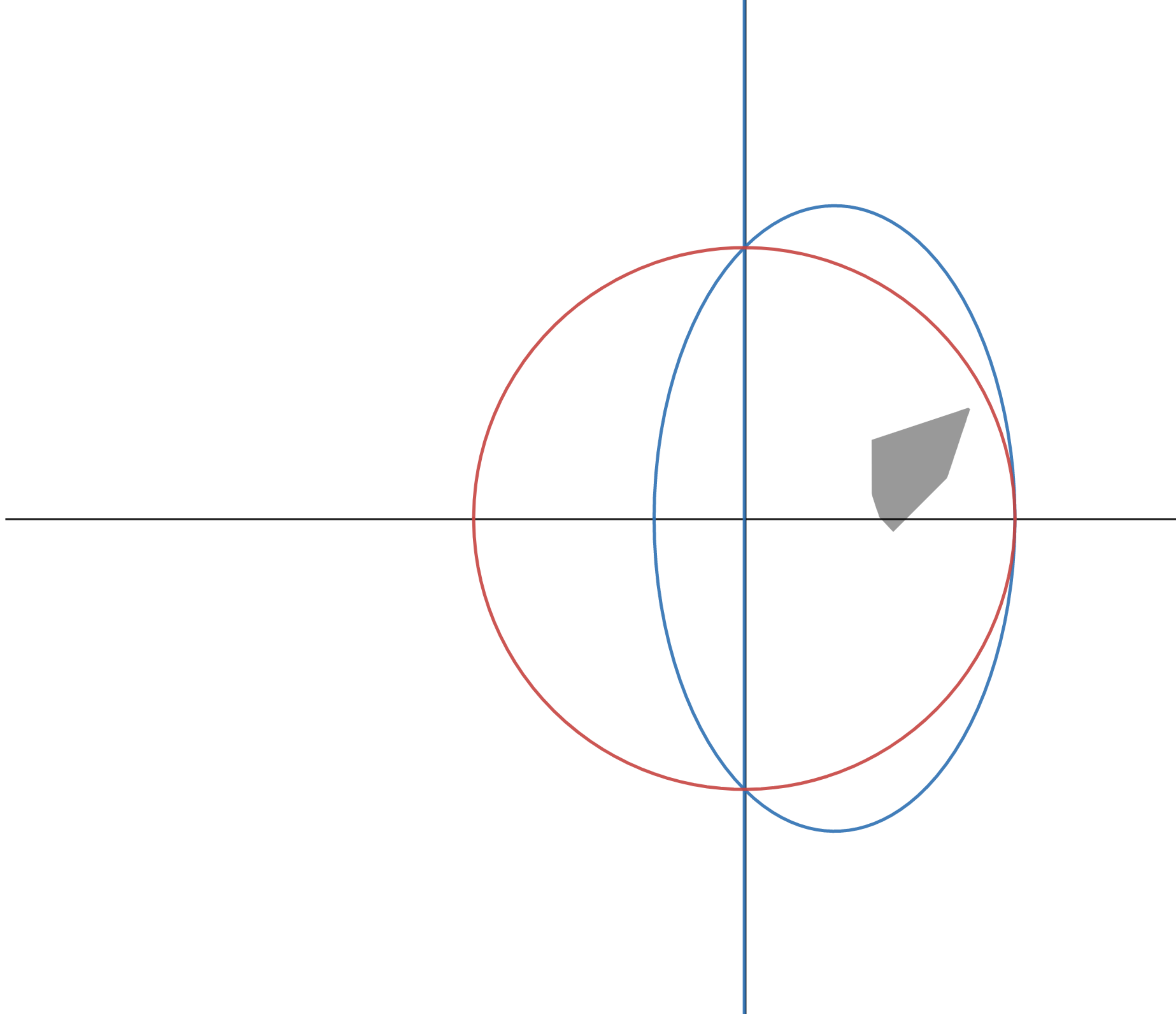
$$= \left(\frac{(n+1)x_1 - 1}{n}\right)^2 + \frac{n^2-1}{n^2}\cdot\sum_{i>2} x_i^2$$

$$= \frac{(n^2+2n+1)x_1^2 - 2(n+1)x_1 + 1}{n^2} + \frac{n^2-1}{n^2}\cdot\sum_{i>2} x_i^2$$

using $0 \leq x_1 \leq 1$ and $\sum_i x_i^2 \leq 1$

$$= \frac{(2n+2)x_1^2 - (2n+2)x_1}{n^2} + \frac{1}{n^2} + \frac{n^2-1}{n^2}\cdot\sum_i x_i^2 = \frac{(2n+2)x_1(x_1-1)}{n^2} + \frac{1}{n^2} + \frac{n^2-1}{n^2}\cdot\sum_i x_i^2 \leq \frac{1}{n^2} + \frac{n^2-1}{n^2} \leq 1.$$

Claim: vol(E')/vol(E) $\leq e^{\frac{-1}{2(n+1)}}$



E: $\sum_i x_i^2 \leq 1$

E':

$$\left(\frac{n+1}{n}\right)^2\left(x_1 - \frac{1}{n+1}\right)^2 + \frac{n^2-1}{n^2} \cdot \sum_{i>2} x_i^2 \leq 1$$

vol(E')/vol(E)

$$= \frac{n}{n+1} \cdot \left(\sqrt{\frac{n^2}{n^2-1}}\right)^{n-1}$$

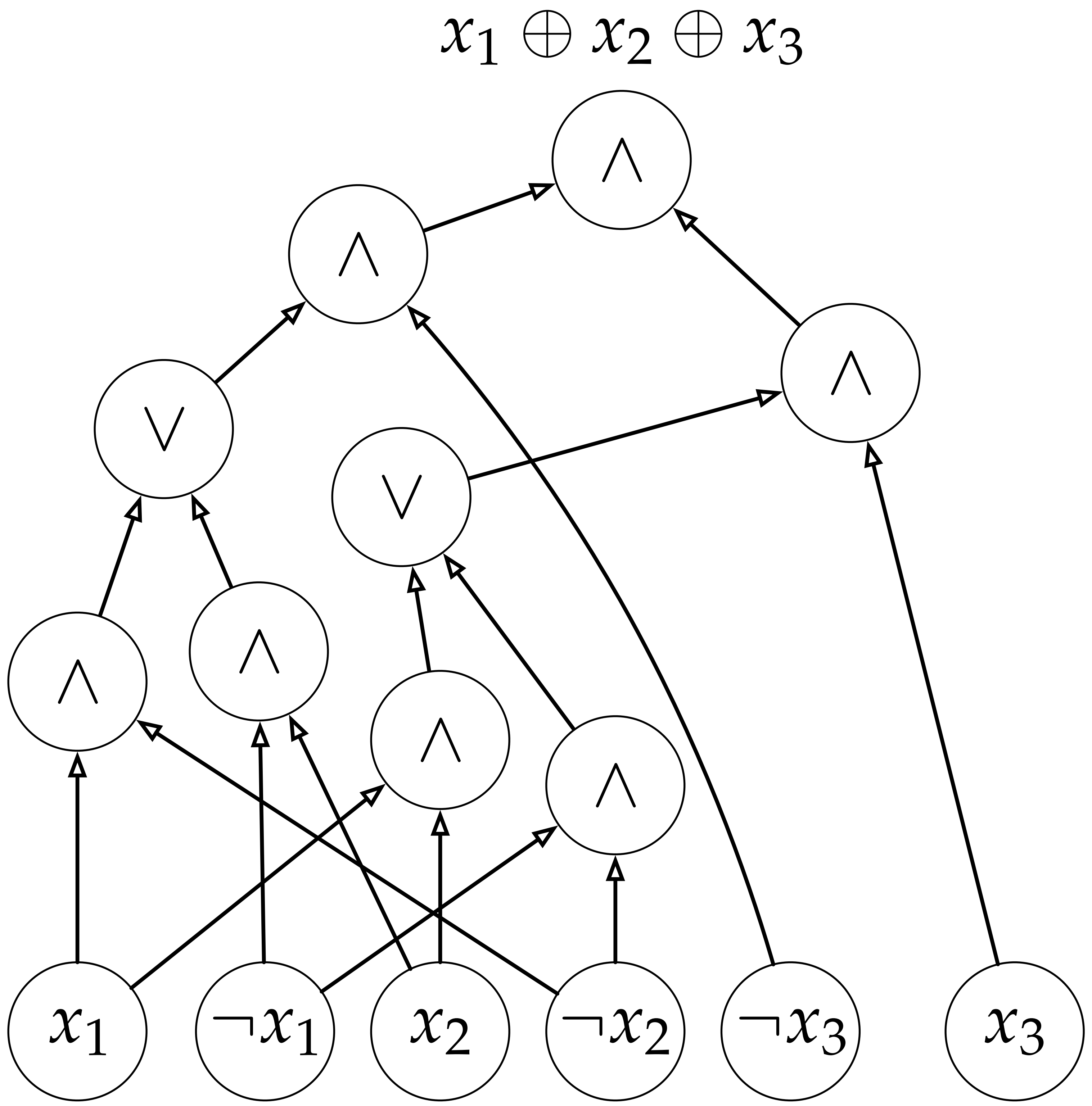
using $1 + z \leq e^z$

$$= \left(1 - \frac{1}{n+1}\right) \cdot \left(1 + \frac{1}{n^2-1}\right)^{(n-1)/2} \leq e^{-\frac{1}{n+1}} \cdot e^{\frac{(n-1)/2}{n^2-1}} = e^{-\frac{1}{n+1}} \cdot e^{\frac{1}{2(n+1)}} = e^{\frac{-1}{2(n+1)}}$$

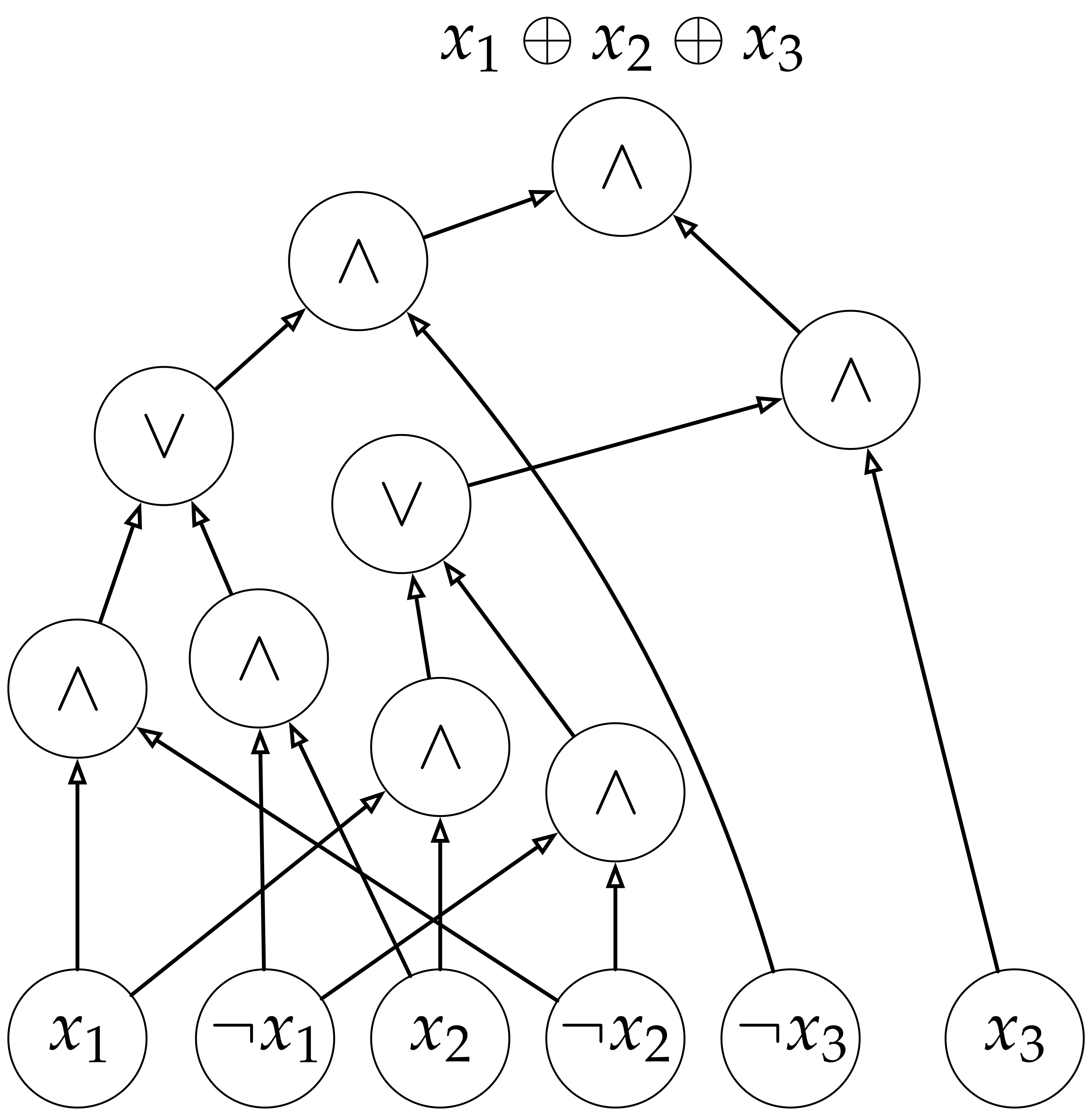
Why is linear programming so powerful?

In a sense, every algorithm can be expressed as linear program!

Boolean circuits

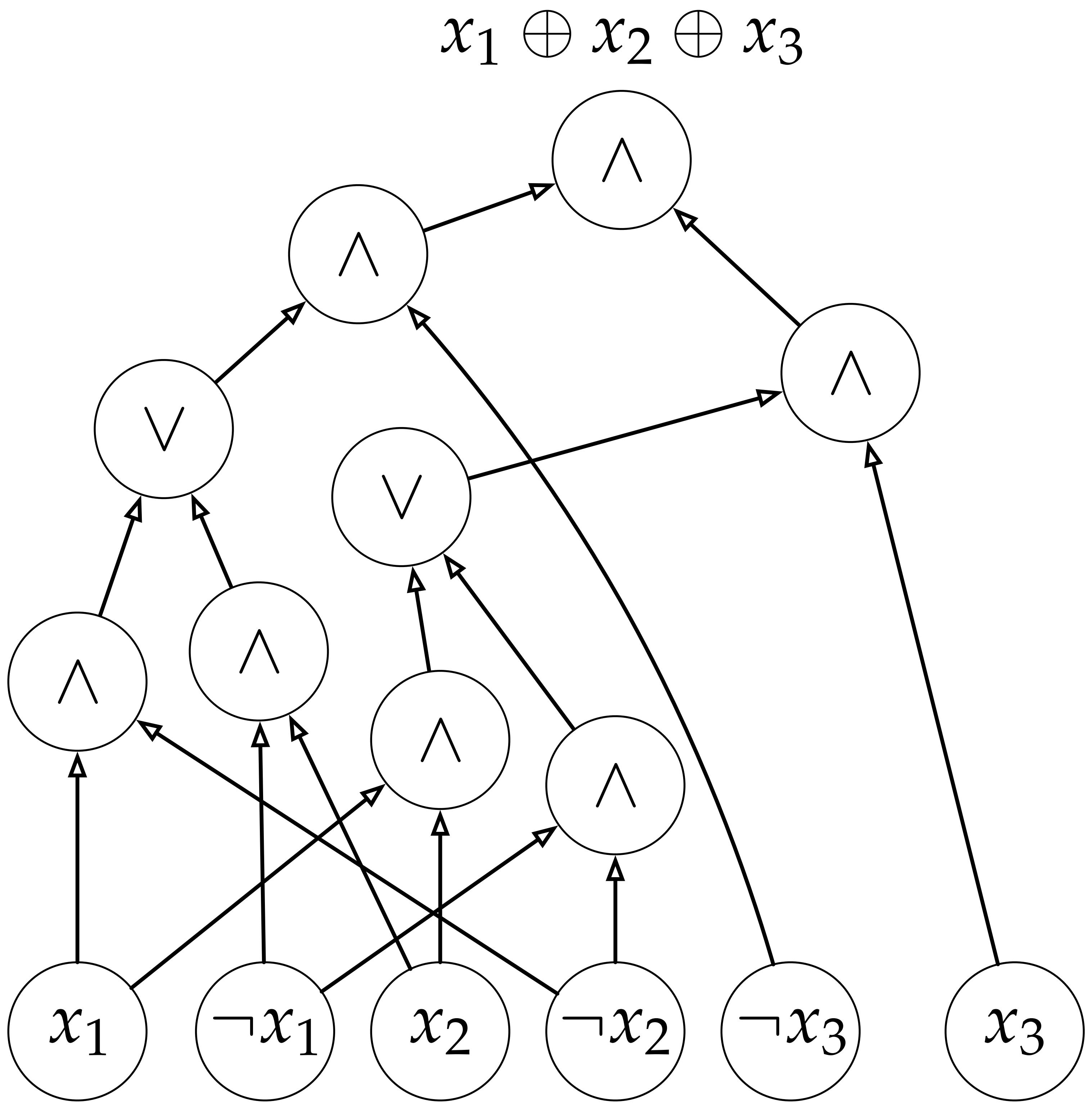


Boolean circuits



Fact: If $f : \{0,1\}^n \to \{0,1\}$ can be computed in time T, then it can be computed by a circuit of size $O(T \log T)$.

Boolean circuits



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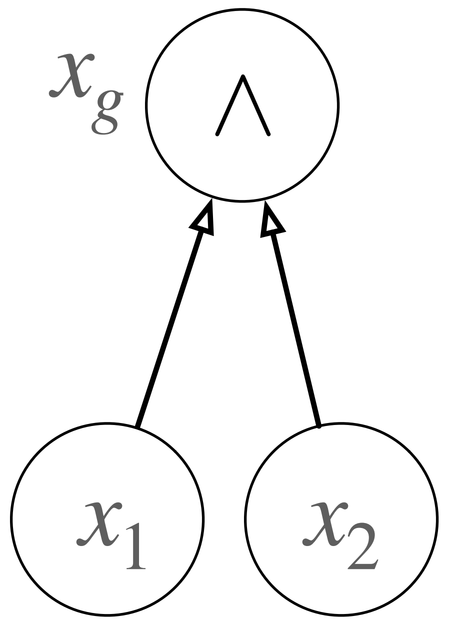
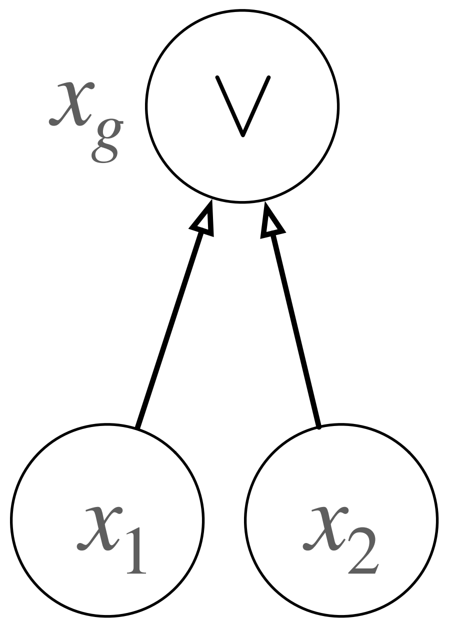


$$x_g \leq x_1$$
$$x_g \leq x_2$$
$$x_g \geq x_1 + x_2 - 1$$



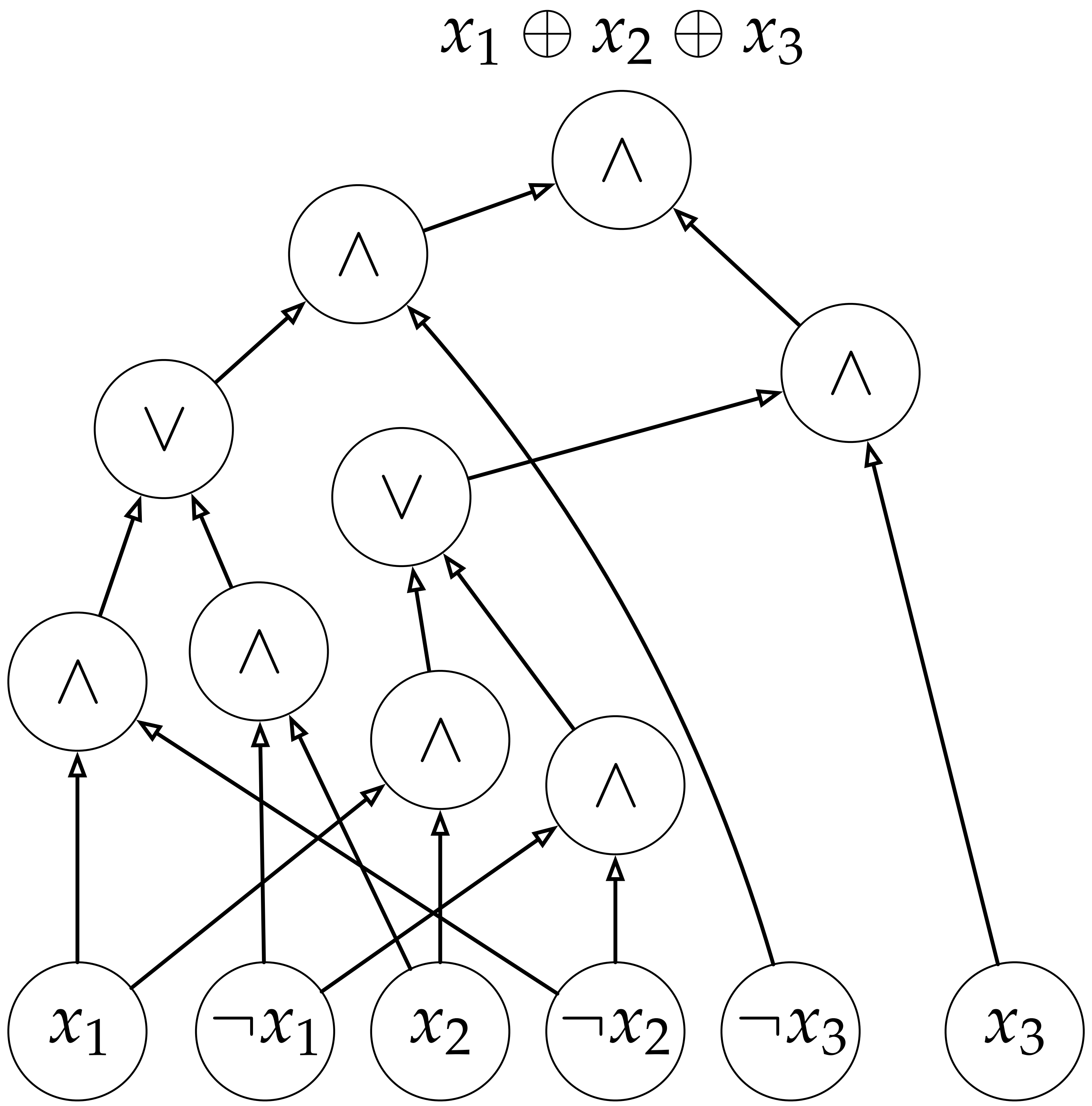
$$x_g \geq x_1$$
$$x_g \geq x_2$$
$$x_g \leq x_1 + x_2$$

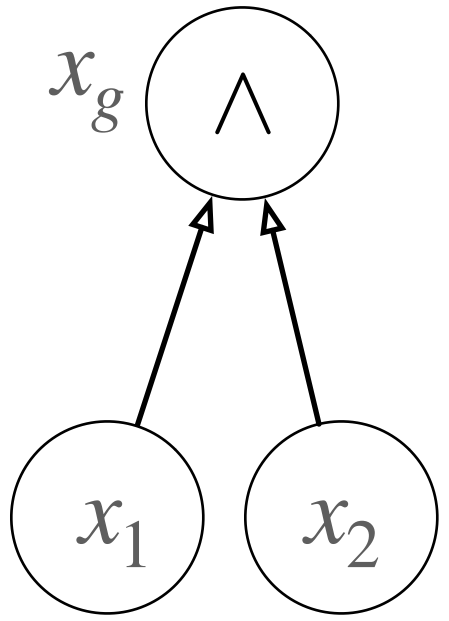
$$\neg x_g = 1 - x_g$$

$$0 \leq x \leq 1$$

Boolean circuits

Fact: If $f : \{0,1\}^n \to \{0,1\}$ can be computed in time T, then it can be computed by a circuit of size $O(T \log T)$.





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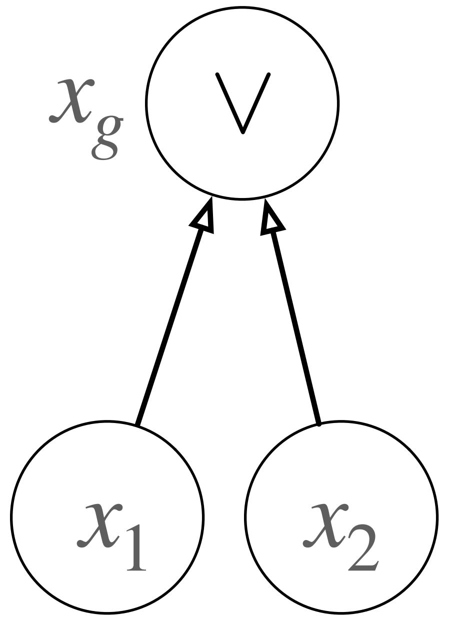

$$x_g \geq x_1$$
$$x_g \geq x_2$$
$$x_g \leq x_1 + x_2$$

$$\neg x_g = 1 - x_g$$

$$0 \leq x \leq 1$$

Computing f is equivalent to finding x satisfying these constraints!

Approximation Algorithms from Linear Programs

Vertex cover:

minimize $\sum_v x_v$

subject to

for all v,
$0 \leq x_v \leq 1$,

for all $e = \{u, v\}$
$x_u + x_v \geq 1$

Claim: Any feasible solution that is a vertex of the polytope must have $x_v \in \{0,1/2,1\}$.

Pf:
Consider the solutions:

$$y_v = \begin{cases} x_v & \text{if } x_v \in \{0,1/2,1\} \\ x_v + \epsilon & \text{if } x_v > 1/2 \\ x_v - \epsilon & \text{otherwise.} \end{cases}$$

$$z_v = \begin{cases} x_v & \text{if } x_v \in \{0,1/2,1\} \\ x_v - \epsilon & \text{if } x_v > 1/2 \\ x_v + \epsilon & \text{otherwise.} \end{cases}$$

y, z are valid solutions to the program. If $x_v \notin \{0,1/2,1\}$, then $y \neq z$, yet $x = (y+z)/2$, so x cannot be a vertex.

Approximation Algorithms from Linear Programs

Vertex cover:

minimize $\sum_v x_v$

subject to

for all v,

$0 \leq x_v \leq 1$,

for all $e = \{u, v\}$

$x_u + x_v \geq 1$

Claim: Any feasible solution that is a vertex of the polytope must have $x_v \in \{0,1/2,1\}$.

Consequence: Let x be a solution that is a vertex of the polytope. If we pick the set of vertices

Let $S = \{v : x_v > 0\}$, this is a valid vertex cover that is at most twice as large as the best one!

Approximation Algorithms from Linear Programs

Set cover:

minimize $\sum_S x_S$

subject to

for all S,
$0 \leq x_S$,

for all $i \in \{1, \ldots, n\}$
$\sum_{i \in S} x_S \geq 1$

Dual program:

maximize $\sum_i y_i$

subject to

for all i,
$0 \leq y_i$,

for all S
$\sum_{i \in S} y_i \leq 1$

Approximation Algorithms from Linear Programs

Dual program:

maximize $\sum_i y_i$

subject to

for all i,

$0 \leq y_i$,

for all S

$\sum_{i \in S} y_i \leq 1$

Recall greedy algorithm:

In each step, pick the set that covers the most remaining elements.

Let $z_i = 1/k$, if i was covered in a group of k elements.

Let $H_r = 1 + 1/2 + \ldots + 1/r$.

Claim: z/H_n is a valid solution to dual.

Pf:

Without loss of generality, suppose $S = \{1,2,\ldots,k\}$, and the elements are covered in order.

Then we see:

$\sum_{i \in S} z_i$

$\leq (1/k + 1/(k-1) + \ldots + 1)$

$\leq H_n$.

Approximation Algorithms from Linear Programs

Dual program:

maximize $\sum_i y_i$

subject to

for all i,
$0 \leq y_i$,

for all S
$\sum_{i \in S} y_i \leq 1$

Recall greedy algorithm:

In each step, pick the set that covers the most remaining elements.

Let $z_i = 1/k$, if i was covered in a group of k elements.
Let $H_r = 1 + 1/2 + \ldots + 1/r$.

Claim: z/H_n is a valid solution to dual.

Consequence:
The dual has value at least the size of greedy solution $/H_n$. Since $H_n \leq O(\log n)$, the greedy is within $O(\log n)$ of the optimal solution.