Complexity Classes

P, NP, Exp, coNP, L, PSPACE, NL, coNL.

$$\begin{tikzcd}[column sep=small, row sep=small]
 & & & & & IP & \\
 & NL \arrow[r, phantom, "\subseteq"] & P \arrow[r, phantom, "\subseteq"] & NP \arrow[urr, phantom, "\subseteq"] \arrow[r, phantom, "\subseteq"] & & PSPACE \arrow[u, phantom, "="] \arrow[r, phantom, "\subseteq"] & EXP \\
L \arrow[ur, phantom, "\subseteq"] \arrow[dr, phantom, "\subseteq"] & & & & & & \\
 & coNL \arrow[uu, phantom, "="] \arrow[uur, phantom, "\subseteq"] \arrow[d, phantom, "\subseteq"] & & coNP \arrow[uu, phantom, "\subseteq"] \arrow[uurr, phantom, "\subseteq"] & & & \\
 & L^2 \arrow[uuurrrr, "\subseteq"] & & & & &
\end{tikzcd}$$

(Immerman)

<u>Thm</u>: $NL = coNL$

1. $coNL \subseteq NL$, 2. $NL \subseteq coNL$

<u>Core</u>:
Given a directed graph G
two vertices s, t
Is s connected to t?

<u>Easy</u>
Certify $s \to t$

1. Guess $s \to v_1 \to v_2 \to v_3 \to \dots \to t$
verify that these edges $\in G$.

<u>Hard</u>
certify no path $s \to t$ ($t \notin C_n$)

- If no path $\Rightarrow$ $\exists$ seq. of guesses s.t. alg outputs 1
- If is path $\Rightarrow$ $\forall$ seq. of guesses alg output 0.

<u>G</u>

C_i : set of vertices at distance $\leq i$ from s.

① v, i : verify $v \in C_i$.
Guess the path of length $\leq i$.

② $|C_i|$: enumerate elements of C_i
Guess elements $v_1 < v_2 < \dots < v_\ell \in C_i$
Check $\forall i$ $v_i < v_{i+1}$
$v_{i+1} \in C_i$
check $\ell = |C_i|$.

③ $|C_i| \to$ output $|C_{i+1}|$.

For $u = 1, \dots, n$
Is $u \in C_{i+1}$?
Enumerate over all $v \in C_i$
Check $v \to u$.

$|C_0| = 1$
$\vdots$
$|C_1|$
$\vdots$
$|C_2|$
$\vdots$
$|C_n|$

enumerate C_n, check that $t \notin C_n$.

NP

Verifier (x) Prover (x)

$V(x, w)$ $\xleftarrow{w}$

if $f(x)=1$ $\exists$ prover that can convince verifier. $f(x)=1$

if $f(x)=0$ no prover can " " $f(x)=1$

IP

Verifier (x) Prover (x)

V $\xrightarrow{r_1}$

$\xleftarrow{a_1}$

$\rightarrow$

$\vdots$

$\leftarrow$

$\vdots$ $Pr[V=1] \geq 2/3$ (1) $(1-2^{-n})$

if $f(x)=1$ $\exists$ prover that can convince verifier. $f(x)=1$

if $f(x)=0$ no prover can " " $f(x)=1$

$\rightarrow Pr[V=1] \leq 1/3$. (0)

$$\boxed{IP = PSPACE}$$

PSPACE complete problem

Totally Quantified Boolean Formulae (TQBF)

$\varphi = \exists x_1 \forall x_2 \exists x_3 \forall x_4 \cdots \exists x_m \; \phi(x_1, \ldots, x_m)$] true or false

$$TQBF(\varphi) = \begin{cases} 1 & \text{if } \varphi \text{ is true} \\ 0 & \text{if } \varphi \text{ is false.} \end{cases}$$

1. TQBF $\in$ PSPACE

Thm: $f \in$ PSPACE $\Rightarrow$ $\exists$ poly time computable g s.t

$\forall x$ $f(x) =$ TQBF$(g(x))$.

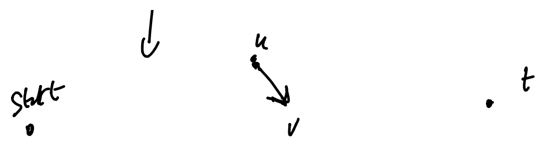
$A(u, v, i)$

$= \begin{cases} 1 & \text{if } u \to v \text{ of length} \leq 2^i \\ 0 & \text{o.w} \end{cases}$

$A(u, v, i)$

$= \exists z$ s.t

$A(u, z, i-1) \wedge A(z, v, i-1)$



$f(x)=1$ iff $\exists$ path from $s \to t$ in config. graph.

$A(u,v,0)$

Is there an edge from $u \to v$?
(Recall Ckt-SAT $\to$ 3-SAT)

$$A(u,v,0) = \exists z\ \psi(u,v,z)$$

$$A(u,v,i) = \exists a \text{ s.t. } A(u,a,i-1) \wedge A(z,v,i-1)$$

$\vdots$

$\exists a_1 \exists a_2 \cdots$ ______

$A(s,t,n)$