PSPACE & TQBF

$$A(u,v,i) = \begin{cases} 1 & \text{if } \exists \text{ path from } u \to v \text{ of length } \le 2^i \\ 0 & \text{o/w.} \end{cases}$$

$$A(u,v,i) = \exists z \ A(u,z,i-1) \wedge A(z,v,i-1)$$

$$\phi = \exists x_1 \forall x_2 \exists x_3 \cdots \forall x_m \ \psi(x_1, \ldots, x_m)$$

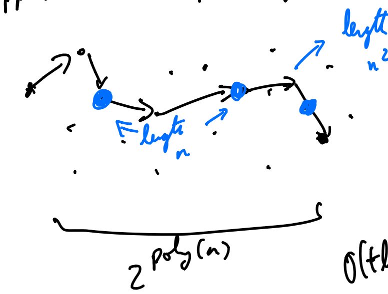
$$TQBF(\phi) = \begin{cases} 1 & \text{if } \phi \text{ is true} \\ 0 & \text{o/w} \end{cases}$$

Thm: Every $f \in$ PSPACE can be reduced to TQBF in poly time.

Thm: There is not linear time, log space algorithm for 3-SAT.

Suppose $f \in$ PSPACE via M

$G_{M,x}$



$O(t \log t)$

FACT: Can generate circuit C of size, simulating t steps of M in time $O(t \log t)$ and space $O(\log t)$.

$$\boxed{A(u,v,poly(n)) \overset{\text{WANT}}{=} \phi}$$

1. $A(u,v,0) \iff C(u) = v$

$\iff \exists z \ \psi(z)$

$\iff \exists z_1 \exists z_2 \cdots \exists z_k \ \psi(z)$

$\iff \exists z_1 \forall x_1 \exists z_2 \cdots \exists z_k \ \psi(z)$

2. $A(u,v,i) = \exists z \ A(u,z,i-1) \wedge A(z,v,i-1)$

$= \ \ldots - \ldots \ A(a,b,i-1) \ldots$

$= \exists z \ \forall a,b \ \big((u=a \wedge z=b) \vee (a=z \wedge b=v)\big) \Rightarrow A(a,b,i-1)$

$= \exists z \ \forall a \forall b \ \big(\neg(u=a \wedge z=b) \wedge \neg(a=z \wedge b=v)\big) \vee A(a,b,i-1)$

$\vdots$

poly(n) size formula.

$\exists z \forall y \ \psi(\,) \vee (\exists a \forall b \ldots)$

$\iff \exists z \forall y \exists a \forall b \ (\psi(\,) \vee \ldots)$

Thm: There is no linear time log space algorithm for 3-SAT.

Pf: Suppose not. $DTIME(n^2) \subseteq DTIME(n \cdot \text{polylog}(n))$.

Suppose $f \in DTIME(n^2)$ via M

Claim: f can be computed in time $t(n) = O(n^2 \log n)$, space $O(\log n)$.

M'
1. Compute C simulates n^2 steps of M. $\to$ time $O(n^2 \log n)$ space $O(\log n)$
2. Convert $C(x)^{=1} \Leftrightarrow \exists z\, \varphi(z, x)$ } - time $O(n^2 \log n)$ space $O(\log n)$
3. Check $\varphi(z, x)$ satisfiable. } "

$\forall z\ \varphi(z, x) \Leftrightarrow \neg \exists z\ \neg \varphi(z, x)$

$\Leftrightarrow \exists z'\ \varphi'(z', x)$

Claim: $M'(x) \xrightarrow{\text{linear time}} \exists a\ \forall b\ \varphi(a, b, x)$

$\exists \underbrace{c_1, c_2, \ldots, c_n}_{\text{config. of turing machine}}\ \forall i\ \left(\begin{array}{c} c_i \to c_{i+1} \\ \text{in } n \text{ steps} \end{array}\right) \wedge c_n \text{ accept state}$

$\exists C_1, \ldots, C_n\ \forall i\ \underbrace{\exists z\ \varphi(C_1, \ldots, C_n, i, z, x)}$

$\exists C_1 \cdots C_n\ \neg \underbrace{\exists i\ \neg A(C_1, \ldots, C_n, i, x)}$

$\exists C_1 \cdots C_n\ \neg B(C_1, \ldots, C_n, x)$

$\exists C_1 \cdots C_n, \exists z\ \gamma(C_1, \ldots, C_n, z)$

& solve.