Arithmetic Circuits

$Perm(X) = \sum_{i=1}^{s} g_i(X)\, h_i(X)$, $\deg g_i \le 2n/3$

Thm: Every size s circuit computing a poly of deg d can be simulated by a circuit of size $poly(s,d)$, depth $O(\log s \cdot \log d)$.

⇓

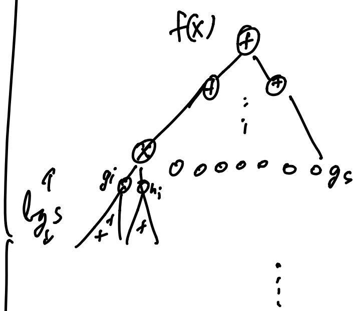
size $\le 2^{O(\log s \cdot \log d)} = s^{O(\log d)}$.

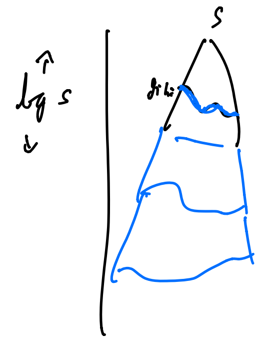
$x_1^2 + x_1 + x_2$ vs x_2

Suppose $f(X)$ can be computed in size s (deg d)

(*) $$\boxed{f(X) = \sum_{i=1}^{s} g_i(X) \cdot h_i(X) \quad \deg h_i \le 2d/3, \ \deg g_i \le 2d/3.}$$

g_i is a gate of circuit





(1) Suppose $f(X)$ is a homogenous polynomial of deg d

all monomials have same degree

$x_1^2 x_2 + x_3 x_4^2 + \cancel{x_5}$

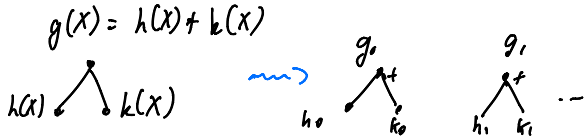
computed by size s circuit,

$\Rightarrow \exists$ size $O(sd^2)$ circuit computing $f(X)$

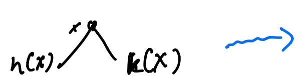
s.t every gate computes a ~~hom~~. poly.

Replace $g(X) = g_0(X) + g_1(X) + \dots + g_d(X) + \dots$

$g(X) = h(X) + k(X)$



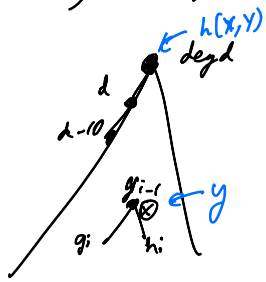
$g(X) = h(X) \cdot k(X)$



$g_i(X) = h_0(X) k_i(X) + h_1(X) k_{i-1}(X) + \dots + h_i(X) k_0(X)$

(2) Suppose f is hom, every gate is hom.

$f(X) =$



$f = g_0, g_1, g_2, \ldots$

$\deg(g_{i-1}) \geq 2d/3$

$n/3 \leq \deg g_i < 2d/3 \qquad \deg h_i < 2d/3.$

$f(X) = h(X, g_{i-1}(X))$

$h(X,y) = a(X) + b(X)\,y$

$h(X,0) = a(X) \leftarrow$ can be computed by size $s-1$.

$b(X)\, g_{i-1}(X) = g_i(X) \cdot \hat{h}_i(X)$

$\vdots$

$$f(X) = \sum_{i=1}^{s} g_i(X) \cdot h_i(X)$$

Alice $\quad X \rightsquigarrow G$

Bob $\quad Y \to H$