Complexity Hierarchies

Def: $\mathrm{DTIME}(t(n))$

$= \{ f : \{0,1\}^* \to \{0,1\} \mid f$ is computable in time $O(t(n)) \}$

Def: $\mathrm{DSPACE}(s(n))$

$= \{ f : \{0,1\}^* \to \{0,1\} \mid f$ is computable in space $O(s(n)) \}$

Efficiently Computable

① • should contain linear time $\mathrm{DTIME}(n)$

② • should remain same under composition.

Def: $P = \bigcup_{c \geq 1} \mathrm{DTIME}(n^c)$

P is smallest set satisfying ① & ②.

(polynomial time)

Def: $\mathrm{EXP} = \bigcup_{c \geq 1} \mathrm{DTIME}(2^{n^c})$

Def: $E = \bigcup_{c \geq 1} \mathrm{DTIME}(2^{cn})$

Def: $L = \mathrm{DSPACE}(\log n)$.

Def: $\mathrm{PSPACE} = \bigcup_{c \geq 1} \mathrm{DSPACE}(n^c)$

Def: $\mathrm{SIZE}(s(n))$

$= \{ f : \{0,1\}^* \to \{0,1\} \mid f$ is computable by circuits of size $s(n)$ $\}$

Thm: $\mathrm{DTIME}(t(n)) \subseteq \mathrm{SIZE}(t(n) \cdot \log t(n))$.

Def: $t : \mathbb{N} \to \mathbb{N}$ is <u>time constructible</u> if $t(n) \geq n$ and on input x $\exists$ M computing $t(|x|)$ in time $O(t(|x|))$.

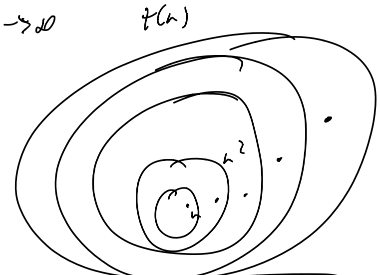
Thm: $\exists$ M --- $\forall$ α, M simulates

M_α $\quad T$ steps of M_α $\longrightarrow$ $O(T \log T)$ steps of M.

Thm: If r, t are time-constructible and $r(n) \log r(n) = o(t(n))$

then $\mathrm{DTIME}(r(n)) \neq \mathrm{DTIME}(t(n))$

$$\lim_{n \to \infty} \frac{r(n) \cdot \log(r(n))}{t(n)} = 0.$$



$$f(\alpha) = \begin{cases} 1 & \text{if } M_\alpha(\alpha) = 0 \\ 0 & \text{o.w.} \end{cases}$$

$$f(\alpha) = \begin{cases} 1 & \text{if } M_\alpha(\alpha) \text{ halts and outputs } 0 \text{ in } t(|\alpha|) \text{ steps} \\ 0 & \text{else.} \end{cases}$$

$f \in DTIME(t(n))$ ✓

$f \notin DTIME(r(n))$

Suppose $\exists$ code β for a machine computing f in $DTIME(r(n))$.

Consider: $M_\beta(\beta) = f(\beta)$ ✗

$\exists \beta'$ s.t $t(|\beta'|) \gg r(|\beta'|) \cdot \log r(|\beta'|)$

and $M_{\beta'}$ carries out same computation as M_β.

If $M_\beta(\beta') = 1 \Rightarrow M_{\beta'}(\beta') = 1 \Rightarrow f(\beta') = 0$

$\dots \Rightarrow M_\beta(\beta') \neq f(\beta')$

Def: $s : \mathbb{N} \to \mathbb{N}$ is space constructible if $s(n) \geq \log n$ and on input x there is a machine computing $s(|x|)$ is space $O(s(|x|))$.

Thm: $\exists$ universal simulator.

$S \geq \log |x|$

$$\underset{S \text{ space}}{M_\alpha(x)} \xrightarrow{\text{simulate}} O(s) \text{ space}$$

Thm: If q, s are space constructible $q(n) = o(s(n))$ then $DSPACE(q(n)) \neq Dspace(s(n))$

Hierarchy for circuits

Thm: $\exists c$ s.t if $2^n/n > s'(n) > c \cdot s(n) > n$ then $SIZE(s(n)) \neq SIZE(s'(n))$.

Pf: set l s.t $s'(n)$ enough to compute every function of first l bits

Thm: $\exists c$ s.t if $10 \cdot 2^n/n > c\, s(n)$

$SIZE(10 2^n/n) \neq SIZE(s(n))$